Lec. 21

Proof of the N-F factorization theorem.

$\underline{T}(\underline{Y})$ is sufficient. $\Longleftrightarrow$ $\exists$ $h(\underline{y})$ and $g(\underline{t},\underline{\theta})$ s.t.

$= Pr_{\underline{Y}|\underline{T}(\underline{Y});\underline{\theta}}(\underline{y}|\underline{t}) = \begin{cases} 0, & \underline{t} \neq \underline{T}(\underline{y}) \\ \dots \end{cases}$

is not a ft. of $\underline{\theta}$

$p_{\underline{Y};\underline{\theta}}(\underline{y}) = h(\underline{y})\, g(\underline{T}(\underline{y}),\underline{\theta})$, $\forall \underline{y}, \forall \underline{\theta}$

"Simplify"

(M)

$\underline{Y}$ is discrete

1) ($\Leftarrow$)

$$Pr(\underline{Y}=\underline{y} \mid \underline{T}(\underline{Y})=\underline{t}\,;\underline{\theta})$$

$$= \frac{Pr(\underline{Y}=\underline{y}, \underline{T}(\underline{Y})=\underline{t}\,;\underline{\theta})}{Pr(\underline{T}(\underline{Y})=\underline{t}\,;\underline{\theta})} = \frac{Pr(\underline{Y}=\underline{y};\underline{\theta})\,\delta(\underline{t},\underline{T}(\underline{y}))}{\sum_{\underline{y}} Pr(\underline{Y}=\underline{y};\underline{\theta})\,\delta(\underline{t},\underline{T}(\underline{y}))}$$

p.2

$$\Pr(\underline{Y}=\underline{y} \mid \underline{T}(\underline{y})=\underline{t};\underline{\theta}) = \frac{\Pr(\underline{Y}=\underline{y};\underline{\theta})\,\delta(\underline{t},\underline{T}(\underline{y}))}{\sum_{\underline{y}} \Pr(\underline{Y}=\underline{y};\underline{\theta})\,\delta(\underline{t},\underline{T}(\underline{y}))}$$

$$= \frac{h(\underline{y})\,g(\underbrace{\underline{T}(\underline{y})}_{\underline{t}},\underline{\theta})\,\delta(\underline{t},\underline{T}(\underline{y}))}{\sum_{\underline{y}} h(\underline{y})\,g(\underbrace{\underline{T}(\underline{y})}_{\underline{t}},\underline{\theta})\,\delta(\underline{t},\underline{T}(\underline{y}))}$$

$$\int_{-\infty}^{\infty} f(a)\,\delta(x-a)\,dx = \int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx$$

$$= \frac{h(\underline{y})\,\delta(\underline{t},\underline{T}(\underline{y}))}{\sum_{\underline{y}} h(\underline{y})\,\delta(\underline{t},\underline{T}(\underline{y}))}$$

Done!

p.3

$(\Longrightarrow)$ $\Pr(\underline{Y}=\underline{y} \mid T(\underline{Y})=\underline{t} ; \underline{\theta})$ is not a function of θ. $\Rightarrow$ $\exists\ h(\underline{y})$ & $g(\underline{t}, \underline{\theta})$ s.t.

$$\Pr(\underline{Y}=\underline{y} ; \underline{\theta}) = h(\underline{y})\, g(T(\underline{y}), \underline{\theta}), \quad \forall \underline{y}, \forall \underline{\theta}$$

$$\underbrace{\Pr(\underline{Y}=\underline{y} \mid T(\underline{Y})=\underline{t} ; \underline{\theta})}_{\text{not a function } \underline{\theta}} = \frac{\Pr(\underline{Y}=\underline{y},\ T(\underline{Y})=\underline{t} ; \underline{\theta})}{\Pr(T(\underline{Y})=\underline{t} ; \underline{\theta})}$$

$$= \underbrace{\frac{\Pr(\underline{Y}=\underline{y} ; \theta)\, \delta(\underline{t}, T(\underline{y}))}{\Pr(T(\underline{Y})=T(\underline{y}) ; \underline{\theta})}}_{\text{not a function of } \underline{\theta}} \triangleq w(\underline{y})$$

Pf

$$\frac{\Pr(\underline{Y}=\underline{y};\underline{\theta})}{\Pr(\underline{T}(\underline{Y})=\underline{T}(\underline{y});\underline{\theta})} \triangleq w(\underline{y}), \quad \forall\underline{y}, \forall\underline{\theta}$$

$$\Rightarrow \Pr(\underline{Y}=\underline{y};\underline{\theta}) = w(\underline{y}) \Pr(\underline{T}(\underline{Y})=\underline{T}(\underline{y});\underline{\theta}), \quad \forall\underline{y}, \forall\underline{\theta}$$

Done!

Example 3. Binary detection problem

w/ $f_{\underline{Y};\theta=0}(\underline{y})$, $f_{\underline{Y};\theta=1}(\underline{y})$.

scalar

(Sol) $f_{\underline{Y};\underline{\theta}}(\underline{y}) = h(\underline{y})\, g(\underline{T}(\underline{y}), \underline{\theta}) = f_{\underline{Y};\theta=1}(\underline{y})\, g\left(\dfrac{f_{\underline{Y};\theta=1}(\underline{y})}{f_{\underline{Y};\theta=0}(\underline{y})}, \theta\right)$

$g(t,\theta) = \begin{cases} 1/t, & \theta=0 \\ 1, & \theta=1 \end{cases}$, $h(\underline{y}) = f_{\underline{Y};\theta=1}(\underline{y})$.