Lec. 22

Announcements

1. Two make-ups on this Friday
2. 2nd midterm exam on Next Friday

Review

$$\min_{\underline{\hat{\theta}}(\underline{y})} E_{\underline{\theta}}\{\|\underline{\hat{\theta}}(\underline{y})-\underline{\theta}\|^2\}, \forall \underline{\theta}.$$

$$\text{s.t.} \quad E_{\underline{\theta}}\{\underline{\hat{\theta}}(\underline{y})-\underline{\theta}\}=\underline{0}, \forall \underline{\theta}$$

MVUE
UMVU estimator

$$\underline{\hat{\theta}}(\underline{y}) = W\underline{y} \longleftarrow \text{BLUE}$$

PR

$f_{Y;\theta_0}(y)$

$f_{Y;\theta_1}(y)$

$f_{Y;\theta}(y)$ not a ft of θ, $\forall y$.

$f_{\underline{Y}|\underline{T}(\underline{Y});\theta_0}(\underline{y}|\underline{t})$ $f_{\underline{Y}|\underline{T}(\underline{Y});\theta_1}(\underline{y}|\underline{t})$ not a ft of θ, $\forall \underline{y}, \forall \underline{t}$.

- a statistic
- a sufficient statistic

$\underline{\Theta}$ random $\Longrightarrow$ $f_{\underline{\Theta}|\underline{Y}}(\underline{\theta}|\underline{y})$ is a ft only of $\underline{\theta}$ and $\underline{T}(\underline{y})$.

$\underline{T}(\underline{Y})$ sufficient.

Neyman-Fischer Factorization Theorem.

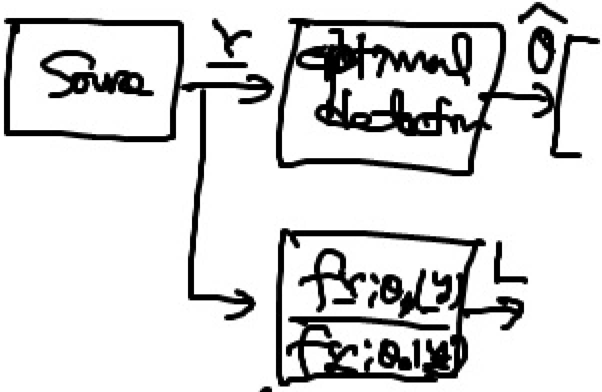
Given a family $\{f_{\underline{Y};\underline{\theta}}(\underline{y}) ; \underline{\theta} \in A\}$ of distribution functions,

$\underline{T}(\underline{Y})$ is sufficient for $\underline{\theta}$

$\Longleftrightarrow$ $\exists$ $h(\underline{y})$, $g(\underline{t}, \underline{\theta})$, and $\underline{T}(\underline{y})$ such that

$$f_{\underline{Y};\underline{\theta}}(\underline{y}) = h(\underline{y})\, g(\underline{T}(\underline{y}), \underline{\theta}), \quad \forall \underline{y}\, \forall \underline{\theta} \in A.$$

p.4 $\{ f_{\underline{Y};\theta_0}(\underline{y}),\ f_{\underline{Y};\theta_1}(\underline{y}) \}$



$\{ f_{L;\theta_1}(\ell),\ f_{L;\theta_0}(\ell) \}$

$$f_{\underline{Y};\theta}(\underline{y}) = h(\underline{y})\, g(T(\underline{y}), \theta)$$

$$\left(T(\underline{y}) = \frac{f_{\underline{Y};\theta_1}(\underline{y})}{f_{\underline{Y};\theta_0}(\underline{y})} \right)$$

$$= f_{\underline{Y};\theta_0}(\underline{y})\, g\left(\frac{f_{\underline{Y};\theta_1}(\underline{y})}{f_{\underline{Y};\theta_0}(\underline{y})}, \theta \right)$$

where $g(t,\theta) = \begin{cases} t, & \theta = 1 \\ 1, & \theta = 0 \end{cases}$

p.5.

$$\left(\boxed{E_{\underline{\theta}}\{g(\underline{Y})\}=0,\ \forall\underline{\theta}}\right.$$

$$\left.\Leftrightarrow \int_{\underline{y}} g(\underline{y})\, f_{\underline{Y};\underline{\theta}}(\underline{y})\, d\underline{y} = 0,\ \forall\theta\right)$$

$$\Rightarrow \left(g(\underline{Y})=0 \quad \text{a.s.}\ \forall\theta\right.$$

$$\left.\Leftrightarrow \boxed{\Pr(g(\underline{Y})=0\ ;\ \underline{\theta}) = 1,\ \forall\underline{\theta}}\right)$$

P.6

$$E_\theta\{g(Y)\} = 0, \ \forall\theta$$

$$\Leftrightarrow \int_{-\infty}^{\infty} g(y) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-\theta)^2}{2}\right) dy = 0, \ \forall\theta$$

$$\Rightarrow \int_{-\infty}^{\infty} g(y) \cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{y^2}{2}} \cdot e^{y\theta} \cdot \cancel{e^{-\frac{\theta^2}{2}}} \, dy = 0, \ \forall\theta$$

$\Rightarrow$ Laplace transform of $g(y) e^{-\frac{y^2}{2}}$

$$\int f(x) e^{-sx} dx. \qquad \therefore g(y) \cancel{e^{-\frac{y^2}{2}}} = 0, \ \forall y,$$

p.7 Given $\{\underline{f}_1, \underline{f}_2, \cdots, \underline{f}_N\}$,

$$\underline{g}^T \underline{f}_n = 0, \ \forall n \Rightarrow \underline{g} = \underline{0}.$$

(ex) $\underline{g} \in \mathbb{R}^{N \times 1}$ 1. $\{\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_N\}$ ← complete family

2. $\{\underline{e}_1, \underline{e}_2\}$

Exponential Family.

$$\{ \sim \exp(\quad) \} = f_{\underline{Y}; \theta}(\underline{y})$$

p. 8

$$f_{Y;\underline{\theta}}(\underline{y}) = h(\underline{y}) \left[c(\underline{\theta})\, e^{Q_1(\underline{\theta})T_1(\underline{y})} \cdot e^{Q_2(\underline{\theta})T_2(\underline{y})} \cdots\cdots e^{Q_M(\underline{\theta})T_M(\underline{y})} \right]$$

$$= h(\underline{y})\, c(\underline{\theta})\, e^{\theta_1 \cdot 0}\, e^{\theta_2 \cdot 0}\, e^{\theta_3 \cdot 0} \cdots e^{\theta_N \cdot 0} \cdot e^{Q_1(\underline{\theta})T_1(\underline{y})}\, e^{Q_2(\underline{\theta})T_2(\underline{y})} \cdots e^{Q_M(\underline{\theta})T_M(\underline{y})}$$

$$\underline{\theta}' = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \\ Q_1(\underline{\theta}) \\ \vdots \\ Q_M(\underline{\theta}) \end{bmatrix} \qquad = f_{Y;\underline{\theta}'}(\underline{y}). \qquad \underline{T}'(\underline{y}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ T_1(\underline{y}) \\ \vdots \\ T_M(\underline{y}) \end{bmatrix}$$

Reparameterize