Lec. 23

Announcements

1. Midterm exam #2 on next Thursday.
2. Two make-ups on tomorrow.

Review

complete

family of ⋯ $\{f_{\underline{Y};\underline{\theta}}(\underline{y}) : \underline{\theta} \in A\}$

$E_{\underline{\theta}}[g(\underline{Y})] = 0, \forall \underline{\theta} \Rightarrow Pr_{\underline{\theta}}(g(\underline{Y}) = 0) = 1, \forall \underline{\theta}$

sufficient statistic $\underline{T} = \underline{T}(\underline{Y})$

$\{f_{\underline{T};\underline{\theta}}(\underline{t}) : \underline{\theta} \in A\}$.

$E_{\underline{\theta}}\{h(\underline{T})\} = 0, \forall \underline{\theta} \Rightarrow$

$Pr_{\underline{\theta}}(h(\underline{T}) = 0) = 1, \forall \underline{\theta}.$

p.2

$\underline{g} \in \mathbb{R}^{N\times 1}$

$\{\underline{f_1}, \underline{f_2}, \cdots, \underline{f_M}\}$ is complete iff

$$\underbrace{\underline{g}^T \underline{f_\theta} = 0, \ \forall \theta = 1, 2, \cdots, M}_{\underline{f_\theta}^T \underline{g} = 0, \ \forall \theta = 1, 2, \cdots, M} \Rightarrow \underline{g} = \underline{0}.$$

$$\begin{bmatrix} \underline{f_1}^T \\ \underline{f_2}^T \\ \vdots \\ \underline{f_M}^T \end{bmatrix} \underline{g} = \begin{bmatrix} \underline{f_1} & \underline{f_2} & \cdots & \underline{f_M} \end{bmatrix}^T \underline{g} = 0.$$

p3

$$\min_{\underline{\hat{\theta}}(\underline{y})} E_{\underline{\theta}}\{\|\underline{\hat{\theta}}(\underline{Y})-\underline{\theta}\|^2\}$$

$$\text{s.t.} \quad E_{\underline{\theta}}\{\underline{\hat{\theta}}(\underline{Y})\}=\underline{\theta}, \forall \underline{\theta}$$

constraint set

Given $\{f_{\underline{Y};\underline{\theta}}(\underline{y}) ; \underline{\theta} \in A\}$,

the family " turns out to be complete, &

somehow we have found an unbiased estimator $\underline{\hat{\theta}}_1(\underline{y})$.

exist

$\Rightarrow \underline{\hat{\theta}}_1(\underline{y})$ is the unique unbiased estimator pf

$$\boxed{\underline{\theta} \in A} \xrightarrow{\underline{Y}} \boxed{\hat{\underline{\theta}}_1(\underline{y})} \xrightarrow{\hat{\underline{\theta}}_1(\underline{Y})}$$

complete

an unbiased estimator of $\underline{\theta}$.

$\hat{\underline{\theta}}_1(\underline{y})$ is the UMVU estimator.

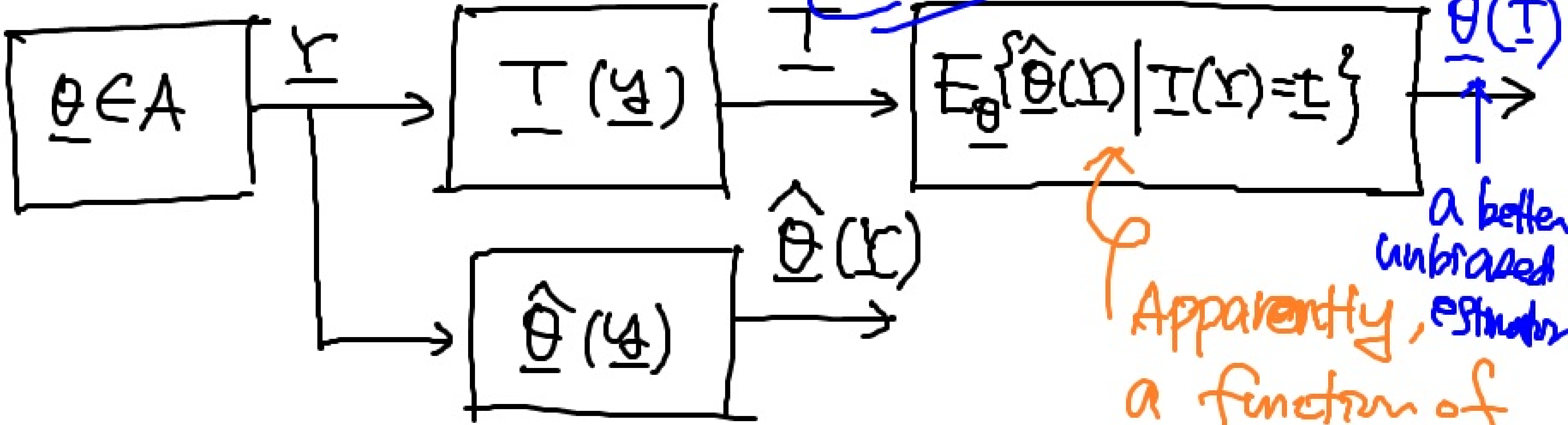
Rao-Blackwell Theorem.

Corollary

Definition
Lemma
Proposition
$\boxed{\text{Theorem}}$
Corollary

p5 The R-B Theorem

complete S.S. ← corollary theorem



a better unbiased estimator

Apparently, a function of $\underline{t}$ & $\underline{\theta}$

Actually, a f.t. of $\underline{t}$ only.

(i) $\underline{T}$ is sufficient

&

(ii) $\underline{\hat{\theta}}(\underline{Y})$ is unbiased p-6

$$E_{\underline{\theta}}\left\{\hat{\underline{\theta}}(\underline{Y}) \,\middle|\, \underline{T}(\underline{Y})=\underline{t}\right\}$$

$$= \int_{\underline{y}} \hat{\underline{\theta}}(\underline{y})\, f_{\underline{Y}|\underline{T}(\underline{Y});\underline{\theta}}(\underline{y}|\underline{t})\, d\underline{y} \quad \leftarrow \text{In general, a function of } \underline{\theta} \,\&\, \underline{t}$$

If $\underline{T}(\underline{Y})$ is sufficient,

then a ~~a~~ ft only of $\underline{t}$. ☺

So, we have $\underline{t} \longrightarrow \boxed{E_{\underline{\theta}}\{\hat{\underline{\theta}}(\underline{Y}) | \underline{T}(\underline{Y})=\underline{t}\}} \longrightarrow \check{\underline{\theta}}(\underline{t})$.

R7

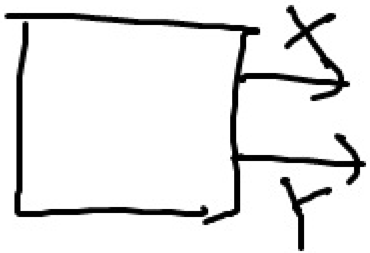
$$\boxed{\underline{\hat{\theta}} \in A} \xrightarrow{\underline{Y}} \boxed{\underline{T}(\underline{y})} \xrightarrow{\underline{T}} \boxed{\underline{\check{\theta}}(\underline{t})} \longrightarrow \underline{\check{\theta}}(\underline{T})$$

(i) $\underline{\check{\theta}}(\underline{T})$ is an unbiased estimator for $\underline{\theta}$.

$$E_{\underline{\theta}}\{\underline{\check{\theta}}(\underline{T})\} = \underline{\theta}, \quad \forall \underline{\theta}.$$

where $\underline{\check{\theta}}(\underline{t}) = E_{\underline{\theta}}\{\underline{\hat{\theta}}(\underline{Y}) \mid \underline{T}(\underline{Y}) = \underline{t}\}$

Example. (Simplify!)



$E[Y] = E\{E[Y|X]\}$

① $E[Y|X=x] = g(x)$

② $X \longrightarrow \boxed{g(x)} \xrightarrow{Z}$ $E[Z] = E[g(X)] = E[E[Y|X]]$

p.8

$E[Y|X=x]$ ← a function of x.

$E[Y|X]$ ← a random variable.

☆ $X \rightarrow \boxed{E[Y|X=x]} \rightarrow E[Y|X]$

학습 學習.

Duality
쌍대성
雙對性.

p. 9

$$E_{\underline{\theta}}\{\underline{\check{\theta}}(\underline{T})\} = E_{\underline{\theta}}\left\{ E_{\underline{\theta}}\{\underline{\hat{\theta}}(\underline{Y}) \mid \underline{T}(\underline{Y})\} \right\}$$

where

$$\underline{\check{\theta}}(\underline{t}) = E_{\underline{\theta}}\{\underline{\hat{\theta}}(\underline{Y}) \mid \underline{T}(\underline{Y}) = \underline{t}\}$$

$$= E_{\underline{\theta}}\{\underline{\hat{\theta}}(\underline{Y})\} = \underline{\theta}, \forall \underline{\theta}.$$

(ii) $\underline{\check{\theta}}(\underline{T})$ is a better unbiased estimator than $\underline{\hat{\theta}}(\underline{Y})$.

Originally variance but we handle cov. matrix.

p.10

Q1. minimize $E_{\underline{\theta}}\{\|\hat{\underline{\theta}}(\underline{Y})-\underline{\theta}\|^2\}$

min. $E_{\underline{\theta}}\{(\hat{\underline{\theta}}(\underline{Y})-\underline{\theta})^T C\,(\hat{\underline{\theta}}(\underline{Y})-\underline{\theta})\}$

$C \succeq 0.\ C = C^T.$

Q2. $A \succeq B \iff A - B \succeq 0$

$\iff \underline{x}^T(A-B)\underline{x} \geq 0, \forall \underline{x}.$

$A \succ B \iff A - B \succ 0$

$\iff \underline{x}^T(A-B)\,\underline{x} > 0, \forall \underline{x} \neq \underline{0}$

p.11

Orthonormal Eigen Decomposition.

$$C = C^T \succeq 0$$

$$C = U \Lambda U^T$$

$UU^T = U^T U = I$

$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$

w/ $\lambda_n \geq 0$.

$$= \sum_{n=1}^{N} \lambda_n \underline{u}_n \underline{u}_n^T.$$