Lec. 27th

Review

MVUE

1. $\underline{Y}$ is complete. $\hat{\underline{\theta}}(\underline{Y})$ is unbiased.
2. $\underline{T}(\underline{Y})$ is " . $\hat{\underline{\theta}}(\underline{Y})$ " " .

   $\check{\underline{\theta}}(\underline{T}(\underline{Y})) \triangleq E_{\underline{\theta}}\{\hat{\underline{\theta}}(\underline{Y}) \mid \underline{T}(\underline{Y})\}$ is the MVUE.
3. Use the Cramer-Rao Lower Bound on error covariance matrix of an unbiased estimator
4. (Later)

ex/ $\min.\ x^2-2x$

st. $-2 \le x \le 2$

tight

achievable lower bound.

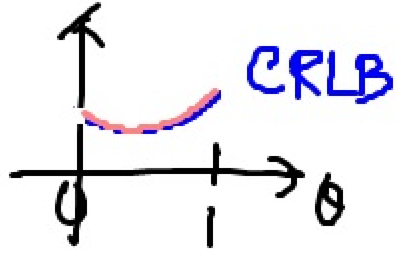
Hint. $x^2-2x \ge -10^5$, $\forall\, -2 \le x \le 2$.

p.2

$$E_{\underline{\theta}}\{(\hat{\underline{\theta}}(\underline{r})-\underline{\theta})(\hat{\underline{\theta}}(\underline{r})-\underline{\theta})^T\} \geq C(\underline{\theta}), \quad \forall \underline{\theta}$$

$\forall$ unbiased $\hat{\underline{\theta}}(\underline{r})$.

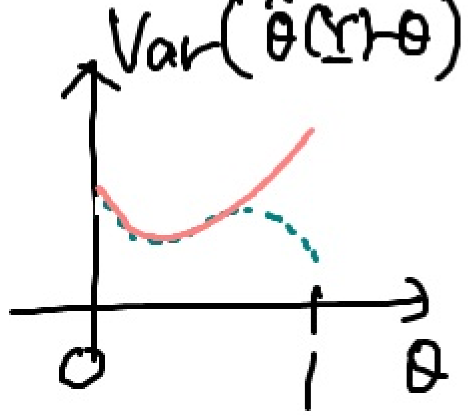
The Cramer–Rao lower bound may or may not be tight!



Simplify. θ is scalar

$$\underbrace{E_{\theta}\{(\hat{\theta}(\underline{r})-\theta)^2\}}_{\text{a function of } \theta} \geq \frac{1}{\underbrace{E_{\theta}\left\{\left(\frac{\partial}{\partial\theta}\ln f_{\underline{r};\theta}(\underline{r})\right)^2\right\}}_{\text{a function of } \theta.}}, \quad \forall\theta$$

p.3



$$I(\theta)$$

$$= J(\theta) \triangleq E_\theta\left\{\left(\overbrace{\frac{\partial}{\partial\theta}\ln f_{\underline{Y};\theta}(\underline{Y})}^{\text{score function.}}\right)^2\right\}$$

Fisher's information (matrix)

- Extension to Vector Parameter

$$E_{\underline{\theta}}\left\{(\underline{\hat{\theta}}(\underline{Y})-\underline{\theta})(\underline{\hat{\theta}}(\underline{Y})-\underline{\theta})^T\right\} \succeq I(\underline{\theta})^{-1}, \ \forall \underline{\theta}$$

where $$[I(\underline{\theta})]_{i,j} = E_{\underline{\theta}}\left\{\left(\frac{\partial}{\partial\theta_i}\ln f_{\underline{Y};\underline{\theta}}(\underline{Y})\right)\left(\frac{\partial}{\partial\theta_j}\ln f_{\underline{Y};\underline{\theta}}(\underline{Y})\right)\right\}$$

P.4 If $\frac{\partial^2}{\partial\theta_i^2} \ln f_{\underline{Y};\underline{\theta}}(\underline{y})$ exists $\forall \theta_i, \forall \underline{y}$ ... + ..
condition

then
$$I(\theta) = -E_\theta\left\{\frac{\partial^2}{\partial\theta^2} \ln f_{Y;\theta}(Y)\right\}$$

$$[I(\underline{\theta})]_{i,j} = -E_\theta\left\{\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} \ln f_{\underline{Y};\underline{\theta}}(\underline{Y})\right\}.$$

the Cramer-Rao Lower Bound (CRLB)

$I^{-1}(\underline{\theta})$. ← a function of $\underline{\theta}$ / matrix-valued p.15 Schwarz' inequality

高: $(x_1^2+x_2^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2)$

$$\geq (x_1y_1+x_2y_2+\cdots+x_ny_n)^2$$

w/ equality iff $\underline{x}=k\underline{y}$ for some k.

大: $\|\underline{x}\|^2\cdot\|\underline{y}\|^2 \geq |\underline{x}^H\underline{y}|^2$.

$$\|\underline{x}\|\cdot\|\underline{y}\| \geq |\underline{x}^H\underline{y}|$$

$$\left(\int|x(t)|^2dt\right)\left(\int|y(t)|^2dt\right) \geq \left|\int x(t)y(t)^*dt\right|^2$$

iff $\exists k$ s.t. $x(t)=k\,y(t),\ \forall t$

p6

Derivation w/ scalar θ.

$$f(x)=g(x),\ \forall x \Rightarrow f'(x)=g'(x),\ \forall x.$$

$$E_\theta\{\hat\theta(\underline{Y})-\theta\}=0,\ \forall\theta. \Leftrightarrow \int_{\underline{y}}(\hat\theta(\underline{y})-\theta)f_{\underline{Y};\theta}(\underline{y})\,d\underline{y}=0,\ \forall\theta.$$

$f(x)=x^2-x$
$g(x)=x$
$f(x)=g(x)$

$x^2-x=x$. $\exists x$.

$2x-1=1$

$2x=2$

$x=1$.

$$\Rightarrow \frac{\partial}{\partial\theta}\int_{\underline{y}}(\hat\theta(\underline{y})-\theta)f_{\underline{Y};\theta}(\underline{y})\,d\underline{y}=0,\ \forall\theta.$$

$$\Rightarrow \int_{\underline{y}}\hat\theta(\underline{y})\frac{\partial}{\partial\theta}f_{\underline{Y};\theta}(\underline{y})\,d\underline{y} = \int_{\underline{y}}\frac{\partial}{\partial\theta}\left(\theta f_{\underline{Y};\theta}(\underline{y})\right)d\underline{y},\ \forall\theta.$$

$$\text{RHS}=\underbrace{\int_{\underline{y}}f_{\underline{Y};\theta}(\underline{y})\,d\underline{y}}_{1}+\int_{\underline{y}}\theta\frac{\partial}{\partial\theta}f_{\underline{Y};\theta}(\underline{y})\,d\underline{y}.$$

$$\text{P.7} \Leftrightarrow \int_{\underline{y}} (\hat{\theta}(\underline{y})-\theta) \frac{\partial}{\partial\theta} f_{\underline{Y};\theta}(\underline{y})\, d\underline{y} = 1, \quad \forall\theta.$$

$$\frac{f_{\underline{Y};\theta}(\underline{y})}{f_{\underline{Y};\theta}(\underline{y})}$$

cf) $\frac{d}{dx}\ln x = \frac{1}{x}$

$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}.$

$$\Leftrightarrow \int_{\underline{y}} (\hat{\theta}(\underline{y})-\theta)\underbrace{\left(\frac{\partial}{\partial\theta}\ln f_{\underline{Y};\theta}(\underline{y})\right)}_{\text{score function}} f_{\underline{Y};\theta}(\underline{y})\, d\underline{y} = 1, \forall\theta.$$

$$\Rightarrow \int_{y}\left((\hat{\theta}(\underline{y})-\theta)\sqrt{f_{\underline{Y};\theta}}\right)^2 dy \int_{y}\left(\frac{\partial}{\partial\theta}\ln f_{\underline{Y};\theta}(\underline{y})\sqrt{}\right)^2 dy \geq 1, \forall\theta.$$

$$\Leftrightarrow E_\theta\{(\hat{\theta}(\underline{Y})-\theta)^2\}\cdot E\left\{\left(\frac{\partial}{\partial\theta}\ln f_{\underline{Y};\theta}(\underline{Y})\right)^2\right\} \geq 1, \forall\theta.$$

p8 Fix θ to θ_0.

Equality holds iff $\exists\, h$ s.t.

$$\frac{\partial}{\partial\theta} \ln f_{\underline{r};\theta}(\underline{y}) \cancel{\sqrt{f_{\underline{r};\theta}(\underline{y})}} \Big|_{\theta=\theta_0} = h \cdot (\hat{\theta}(\underline{y}) - \theta) \cancel{\sqrt{f_{\underline{r};\theta}(\underline{y})}} \Big|_{\theta=\theta_0}$$

$$\frac{\partial}{\partial\theta} \ln f_{\underline{r};\theta}(\underline{y}) \Big|_{\theta=\theta_0} = h \cdot (\hat{\theta}(\underline{y}) - \theta) \Big|_{\theta=\theta_0}$$

$$\boxed{\frac{\partial}{\partial\theta} \ln f_{\underline{r};\theta}(\underline{y}) = h(\theta)(\hat{\theta}(\underline{y}) - \theta), \ \forall\theta.}$$

We factor the score function!

P.9 $$E_\theta\left\{\left(\frac{\partial}{\partial\theta}\ln f_{\underline{Y};\theta}(\underline{Y})\right)^2\right\} = h(\theta)^2 E_\theta\left\{(\hat{\theta}(\underline{Y})-\theta)^2\right\}, \forall\theta$$

$$\Rightarrow E_\theta\left\{(\hat{\underline{\theta}}(\underline{Y})-\theta)^2\right\} = \frac{1}{h(\theta)^2 E_\theta\left\{(\hat{\underline{\theta}}(\underline{Y})-\theta)^2\right\}}$$

$$\Rightarrow E_\theta\left\{(\hat{\underline{\theta}}(\underline{Y})-\theta)^2\right\} = \frac{1}{|h(\theta)|}, \forall\theta.$$

& $|h(\theta)|$ is the Fisher's Information.

$$\cancel{E_\theta\left\{\hat{\theta}(\underline{Y})-\theta\right\}=0, \forall\theta \Rightarrow \int_{\underline{y}} (\hat{\theta}(\underline{y})-\theta)\frac{\partial}{\partial\theta} f_{\underline{Y};\theta}(\underline{y})\, d\underline{y} = 1, \forall\theta,}$$

$$\frac{\partial}{\partial\theta}\int f_{\underline{Y};\theta}(\underline{y})\, d\underline{y} = 0, \forall\theta \Rightarrow \frac{\partial^2}{\partial\theta^2}\int f_{\underline{Y};\theta}(\underline{y})\, d\underline{y} = 0, \forall\theta.$$