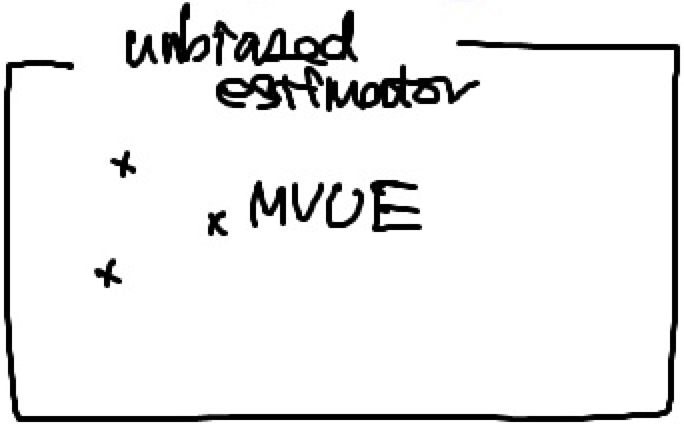
Lec. 26

Announcements

Review

- Midterm exam covers upto the last lecture.
- An efficient estimator

Def. An unbiased estimator that achieves the CRLB.



Given $\{f_{\underline{Y};\underline{\theta}}(\underline{y});\ \underline{\theta} \in A\}$.

p2.

- Scalar parameter θ.

[unit] & ☺

$$E_\theta\{(\hat{\theta}(\underline{Y})-\theta)^2\} \geq \frac{1}{I(\theta)}, \ \forall \theta, \ \forall \text{ unbiased estimate } \hat{\theta}(\underline{Y}) \text{ of } \theta.$$

where Fisher's information $I(\theta)$ is given by

$$I(\theta) \triangleq E_\theta\left\{\left(\frac{\partial}{\partial\theta} \ln f_{\underline{Y};\theta}(\underline{Y})\right)^2\right\}$$

$$= -E_\theta\left\{\frac{\partial^2}{\partial\theta^2} \ln f_{\underline{Y};\theta}(\underline{Y})\right\}$$

regularity conditions.

p.3 - scalar θ. but given $\{f_{\underline{Y};\phi}(\underline{y}) : \phi \in A\}$

$\theta = g(\phi)$.

$$E_{\phi}\left\{\left(\hat{\theta}(\underline{Y}) - \theta\right)^2\right\} \geq \frac{\left(\frac{\partial}{\partial\phi} g(\phi)\right)^2}{I(\phi)}$$

, $\forall \phi$, $\forall$ unbiased estimators $\hat{\theta}(\underline{Y})$.

where
$$I(\phi) = E_{\phi}\left\{\left(\frac{\partial}{\partial\phi} \ln f_{\underline{Y};\phi}(\underline{Y})\right)^2\right\}$$
$$= -E_{\phi}\left\{\frac{\partial^2}{\partial\phi^2} \ln f_{\underline{Y};\phi}(\underline{Y})\right\}$$

p4

Given an unbiased estimator $\hat{\underline{\theta}}(\underline{Y})$, find a lower bound on the covariance of $\hat{\underline{\theta}}(\underline{Y})$.

$\left\{ \begin{array}{l} \text{commonality} \\ \text{difference.} \end{array} \right.$

$$\underline{X}, \quad \mathrm{cov}(\underline{X}) \triangleq E\{(\underline{X}-\bar{\underline{X}})(\underline{X}-\bar{\underline{X}})^T\}$$

$$\hat{\underline{\theta}}(\underline{Y}), \quad \mathrm{cov}_{\underline{\theta}}(\hat{\underline{\theta}}(\underline{Y})) \triangleq E_{\underline{\theta}}\left[\left(\hat{\underline{\theta}}(\underline{Y}) - E_{\underline{\theta}}\{\hat{\underline{\theta}}(\underline{Y})\}\right)\left(\quad\right)^T\right]$$

$$\mathrm{Cov}_{\underline{\theta}}(\hat{\underline{\theta}}(\underline{Y})) \succeq$$

p.5

$$\mathrm{Var}_\theta(\hat\theta(\underline{Y})) \geq \frac{\left(\frac{\partial}{\partial\theta} E_\theta\{\hat\theta(\underline{Y})\}\right)^2}{I(\theta)}$$

$E_\theta\{\hat\theta(\underline{Y})\} = \theta, \forall\theta$

cf)

$$\mathrm{Cov}_{\underline{\theta}}\{\underline{\hat\theta}(\underline{Y})\} \geq \left(\frac{\partial}{\partial\underline{\theta}} E_{\underline{\theta}}\{\underline{\hat\theta}(\underline{Y})\}\right) I(\underline{\theta})^{-1} \left(\qquad \right)^T$$

p.6

CRLB

Schwarz' inequality.

$$\frac{\partial}{\partial \theta} \ln f_{\underline{Y};\theta}(\underline{y}) = h(\theta)\left(\hat{\theta}(\underline{y}) - \theta\right), \forall \theta, \forall \underline{y}.$$

$$\frac{\partial}{\partial \underline{\theta}} \ln f_{\underline{Y};\underline{\theta}}(\underline{y}) = H(\theta)\left(\underline{\hat{\theta}}(\underline{y}) - \underline{\theta}\right), \forall \underline{\theta}, \forall \underline{y},$$

Information inequality.

$$\frac{\partial}{\partial \theta} \ln f_{\underline{Y};\theta}(\underline{y}) = h(\theta)\left(\hat{\theta}(\underline{y}) - E_{\theta}\{\hat{\theta}(\underline{Y})\}\right) \forall \theta, \forall \underline{y}$$

$$\frac{\partial}{\partial \underline{\theta}} \ln f_{\underline{Y};\underline{\theta}}(\underline{y}) = H(\theta)\left(\underline{\hat{\theta}}(\underline{Y}) - E_{\underline{\theta}}\{\underline{\hat{\theta}}(\underline{Y})\}\right), \forall \underline{\theta}, \forall \underline{y}.$$

p.7 Def. $\{f_{\underline{Y};\underline{\theta}}(\underline{y}) ; \underline{\theta} \in A\}$ is an exponential family of

$f_{\underline{Y};\underline{\theta}}(\underline{y}) = (\underline{\theta}\text{의 함수}) \times (\underline{y}\text{의 함수}) \times$

$e(\quad " \quad \times \quad " \quad) \times$

$e(\quad " \quad \times \quad " \quad) \cdots$

$= c(\underline{\theta})\, h(\underline{y}) \exp\left(\sum_{m=1}^{M} Q_m(\underline{\theta}) T_m(\underline{y})\right)$

Lemma. $\underline{T}(\underline{Y})$ is a sufficient statistic for $\underline{\theta}$.

by N-F factorization theorem.

p.8 1-parameter exponential family

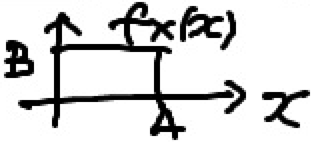
$\{f_{\underline{Y};\theta}(\underline{y}) ; \theta \in A\}$.

$\hat{\underline{\theta}}(\underline{Y})$ achieves the information lower bound.

$\Longleftrightarrow$

$\{f_{\underline{Y};\theta}(\underline{y}) : \theta \in A\}$ is a 1-parameter exponential family such that

$$f_{\underline{Y};\theta}(\underline{y}) = c(\theta) h(\underline{y}) \exp(g(\theta)\hat{\theta}(\underline{y}))$$



$$f_X(x) = B\{u(x) - u(x-A)\}$$

$$= \frac{h(\underline{y})\exp(g(\theta)\hat{\theta}(\underline{y}))}{\int \cdots \, d\underline{y}}$$