LEC. 27

Announcements

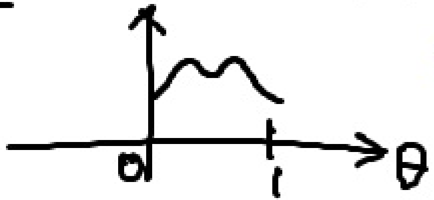
1. 2nd Midterm Exam Tonight.
2. Quiz tomorrow @5pm.

Def.

MLE

$$\boxed{\hat{\underline{\theta}}_{ML}(\underline{y}) \triangleq \arg\max_{\underline{\theta}} f_{\underline{Y};\underline{\theta}}(\underline{y}), \ \forall \underline{y}}$$

$f_{\underline{Y};\underline{\theta}}(\underline{y})$
- as a ft of $\underline{y}$: pdf of $\underline{Y}$
- as a ft of $\underline{\theta}$: likelihood ft.



Def
- existence $\triangle$:
- unique X

$$\hat{\underline{\theta}}_{ML}(\underline{y}) = \underset{\underline{\theta}}{\arg\max}\ f_{\underline{Y};\underline{\theta}}(\underline{y})$$

$$= \underset{\underline{\theta}}{\arg\max}\ \ln f_{\underline{Y};\underline{\theta}}(\underline{y})$$

→ log-likelihood function

FONC (first-order necessary cond.)

If $\hat{\underline{x}}$ is local maximum/minimum of $f(\underline{x})$,

then $\cancel{f'(\hat{x})} = 0$.

$\nabla f(\underline{x})\big|_{\underline{x}=\hat{\underline{x}}} = \underline{0}$.

If differentiable,

Global maximum ⇒ local maximum ⇒ FONC.

p.3

FONC

$$\frac{\partial}{\partial \underline{\theta}} f_{\underline{Y}}(\underline{y};\underline{\theta})\Big|_{\underline{\theta}=\hat{\underline{\theta}}_{ML}(\underline{y})} = \underline{0}, \quad \forall \underline{y}$$

likelihood equation

$$\frac{\partial}{\partial \underline{\theta}} \ln f_{\underline{Y}}(\underline{y};\underline{\theta})\Big|_{\underline{\theta}=\hat{\underline{\theta}}_{ML}(\underline{y})} = \underline{0}, \quad \forall \underline{y}$$

Score function!

log-likelihood equation.

p.4

(iii) $\{f_{\underline{Y};\underline{\phi}}(\underline{y}) ; \underline{\phi} \in A\}$, $\underline{\phi}_{ML}(\underline{y})$.

If $\underline{\theta} = \underline{g}(\underline{\phi})$ where $\underline{g}$ is an invertible function

then $\hat{\underline{\theta}}_{ML}(\underline{y}) = \underline{g}(\underline{\phi}_{ML}(\underline{y}))$.

invariance property

$$f_{Y;\underline{\phi}}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

where $\underline{\phi} = \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}$. $\mu \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}^+$

$$f_{\underline{Y};\underline{\theta}}(\underline{y}) = f_{\underline{Y};\underline{g}(\underline{\phi})}(\underline{y}) = f_{\underline{Y};\underline{\phi}}(\underline{y})$$

Q. $\underline{\theta} = \underline{g}(\underline{\phi}) \triangleq \begin{bmatrix} \phi_1 \\ \sqrt{\phi_2} \end{bmatrix} = \begin{bmatrix} \mu \\ \sigma \end{bmatrix}$ $f_{Y;\underline{\theta}}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$

pf.

$$\underline{\hat{\theta}}_{ML}(\underline{y}) = \arg\max_{\underline{\theta}} f_{\underline{Y};\underline{\theta}}(\underline{y})$$

$$= g\left(\arg\max_{\underline{\phi}} f_{\underline{Y};\underline{g}(\underline{\phi})}(\underline{y})\right)$$

$$= g\left(\underline{\hat{\phi}}_{ML}(\underline{y})\right).$$

(iv) $\boxed{\begin{array}{l}\text{Given } \{f_{\underline{Y};\underline{\theta}}(\underline{y});\ \underline{\theta}\in A\},\ \text{suppose that } \underline{T}(\underline{Y}) \text{ is a s.s.} \\ \exists\ \underline{\check{\theta}}(\underline{t}) \text{ s.t.} \\ \underline{\hat{\theta}}_{ML}(\underline{y}) = \underline{\check{\theta}}(\underline{T}(\underline{y})),\ \forall \underline{y}\end{array}}$

$\underline{\hat{\theta}}_{ML}(\underline{Y})$ = a function of a s.s.

p.6

$\boxed{\begin{array}{l}\text{By the N-F factorization,} \\ \quad \underline{T}(\underline{Y}) \text{ is sufficient} \\ \Leftrightarrow f_{\underline{Y};\underline{\theta}}(\underline{y}) = h(\underline{y})\, g(\underline{T}(\underline{y}), \underline{\theta}), \ \forall \underline{y}, \forall \underline{\theta}\end{array}}$

$$\hat{\underline{\theta}}_{ML}(\underline{y}) = \arg\max_{\underline{\theta}} f_{\underline{Y};\underline{\theta}}(\underline{y})$$

$$= \arg\max_{\underline{\theta}} h(\underline{y})\, g(\underline{T}(\underline{y}), \underline{\theta})$$

$$= \arg\max_{\underline{\theta}} g(\underline{T}(\underline{y}), \underline{\theta}) = \check{\underline{\theta}}(\underline{T}(\underline{y}))$$

p.7 (V) If the efficient estimator exists, then it is the MLE.

CRLB
unbiased.

(proof) $\frac{\partial}{\partial \underline{\theta}} \ln f_{\underline{Y};\underline{\theta}}(\underline{y}) = H(\underline{\theta})(\hat{\underline{\theta}}(\underline{y}) - \underline{\theta})$, $\forall \underline{y}, \forall \underline{\theta}$.

FONC $\Rightarrow$ $\frac{\partial}{\partial \underline{\theta}} \ln f_{\underline{Y};\underline{\theta}}(\underline{y}) \Big|_{\underline{\theta} = \hat{\underline{\theta}}_{ML}(\underline{y})} = \underline{0}$, $\forall \underline{y}$.

EM algorithm
expectation
maximization

$\Rightarrow$ $H(\hat{\underline{\theta}}_{ML}(\underline{y}))(\hat{\underline{\theta}}(\underline{y}) - \hat{\underline{\theta}}_{ML}(\underline{y})) = \underline{0}$, $\forall \underline{y}$

$\Rightarrow$ $\hat{\underline{\theta}}(\underline{y}) - \hat{\underline{\theta}}_{ML}(\underline{y}) = \underline{0}$, $\forall \underline{y}$

$\Rightarrow$ $\hat{\underline{\theta}}(\underline{y}) = \hat{\underline{\theta}}_{ML}(\underline{y})$, $\forall \underline{y}$.