Lec. 30

Announcement : Quiz #5 next Friday

Corrections : Last lectures results holds only when C does not change as a function of the time index n.

Review

i.i.d. sequence observation : $Y_1, Y_2, Y_3, \cdots$

binary hypothesis testing : $\{\Theta=0\}, \{\Theta=1\}$

random : $\Pr(\Theta=1)=p$.

cost : $C_{0,0}, C_{1}$

objective: To find a sequential detector

$(\delta_n)_{n=0}^{\infty}$ such that

$$\delta_n = \delta_n(\underline{y}_1, \underline{y}_2, \cdots, \underline{y}_n)$$

stopping rule decision rule

$$= (\phi_n(\underline{y}_1, \underline{y}_2, \cdots, \underline{y}_n), \hat{\theta}_n(\underline{y}_1, \underline{y}_2, \cdots, \underline{y}_n))$$

where $\phi_n = 1 \iff$ Stop at time n

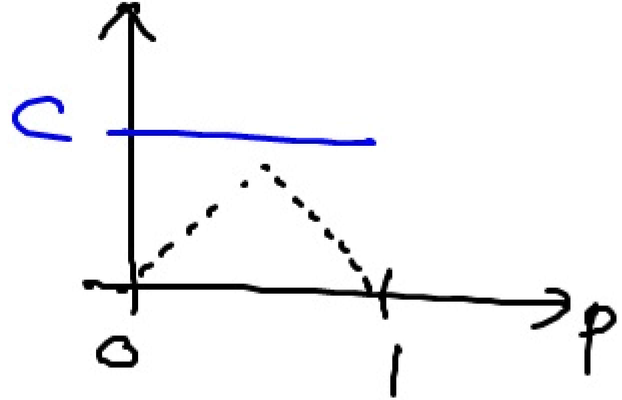
$\hat{\theta}_n = \begin{cases} 0 \\ 1 \end{cases} \iff \begin{matrix} \text{Say } H_0 \\ \text{say } H_1 \end{matrix}$

p.3

ex/ $C_{\hat{\theta},\theta} = 1 - \delta_{\theta,\hat{\theta}}$ uniform

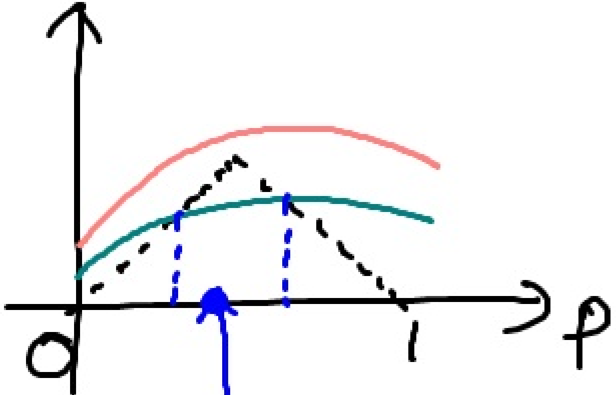
$C \geq \frac{1}{2}$

$\delta_0 = (\phi_0, \hat{\theta}_0) = \begin{cases} (1, 0) \\ (1, 1) \end{cases}$



ex/ $C_{\hat{\theta},\theta} = 1 - \delta_{\theta,\hat{\theta}}$

$C > 0$



$Pr(\Theta = 1)$. $Pr(\Theta = 1 \mid \underline{Y}_1 = \underline{y}_1)$

SPRT(A,B) is optimal for some A, B.

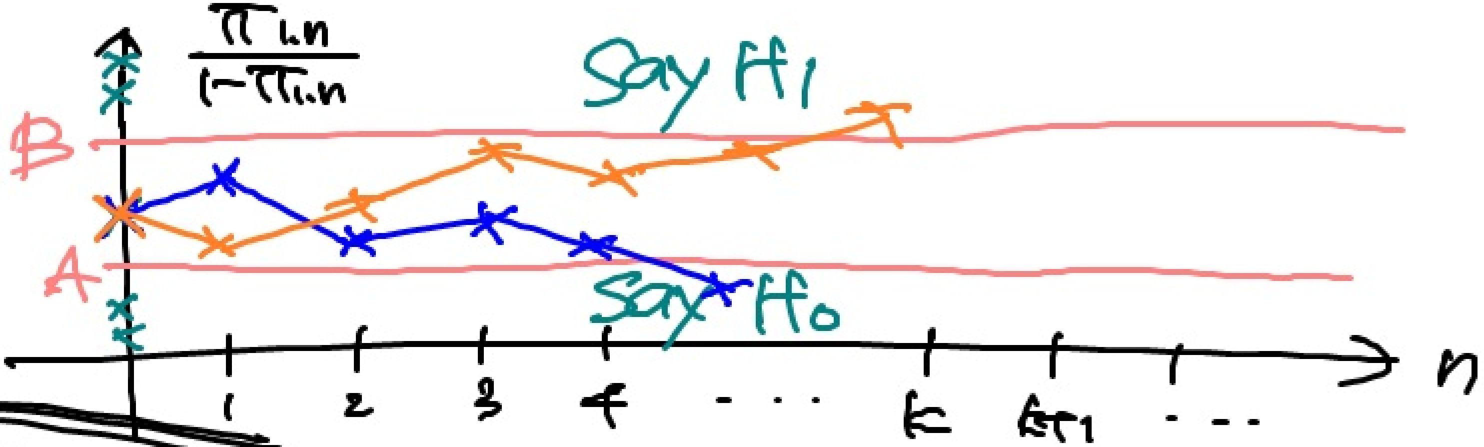
fixed!

$$\frac{\Pr(\Theta=1 \mid \underline{Y}_1=\underline{y}_1, \underline{Y}_2=\underline{y}_2, \dots, \underline{Y}_n=\underline{y}_n)}{\Pr(\Theta=0 \mid \underline{Y}_1=\underline{y}_1, \underline{Y}_2=\underline{y}_2, \dots, \underline{Y}_n=\underline{y}_n)}$$

$$\triangleq \frac{\pi_{1,n}}{\pi_{0,n}} = \frac{\pi_{1,n}}{1-\pi_{1,n}} = \prod_{k=0}^{n} \frac{f_{\underline{Y}_k \mid \Theta=1}(\underline{y}_k)}{f_{\underline{Y}_k \mid \Theta=0}(\underline{y}_k)}$$

$n=0 \quad \dfrac{\Pr(\Theta=1)}{\Pr(\Theta=0)}$

p5



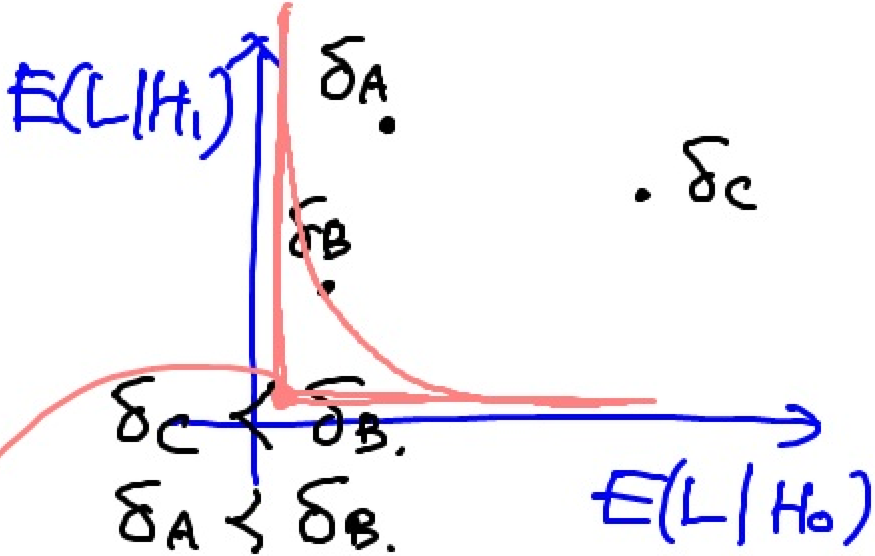
Today

θ is not random

No δ_0.

$\boxed{\text{Starts w/ } \delta_1}$

SPRT(A, B).



p6 $L \triangleq$ stopping time. Λ_n : prob-ratio @ time n.

$$P_F = Pr(\hat{\theta}_L(\underline{Y}_1, \cdots, \underline{Y}_L) = 1 \,;\, \theta = 0)$$

$$= Pr(\Lambda_L(\underline{Y}_1, \underline{Y}_2, \cdots, \underline{Y}_L) \geq B \,;\, \theta = 0)$$

$$= Pr\left(\frac{f_{\underline{Y}_n;\theta=1}(\underline{Y}_1)}{f_{\underline{Y}_n;\theta=0}(\underline{Y}_1)} \times \cdots \times \frac{f_{\underline{Y}_n;\theta=1}(\underline{Y}_L)}{f_{\underline{Y}_n;\theta=0}(\underline{Y}_L)} \geq B \,;\, \theta = 0\right)$$

$$= \sum_{l=1}^{\infty} Pr(\Lambda_l(\underline{Y}_1, \cdots, \underline{Y}_l) \geq B \,;\, \theta = 0)$$

$$= \sum_{l=1}^{\infty} Pr(\hat{\underline{\theta}}_l(\underline{Y}_1, \underline{Y}_2, \cdots, \underline{Y}_l) = 1 \,;\, \theta = 0)$$

$$P_F^{\ell} = P_F$$

$$= \Pr\left(\hat{\theta}_\ell(\underline{Y}_1, \cdots, \underline{Y}_\ell) = 1 \,;\, \theta = 0\right) \qquad \le \frac{1}{B} \prod_{n=1}^{\ell} f_{\underline{Y}_n;\theta=1}(\underline{y}_n)$$

$$= \int_{\underline{y}_1, \underline{y}_2 \cdots \underline{y}_\ell} \prod_{n=1}^{\ell} f_{\underline{Y}_n;\theta=0}(\underline{y}_n)\, \mathbb{1}_{\boxed{\Lambda_\ell \ge B}}(\underline{y}_1, \cdots, \underline{y}_\ell)\, d\underline{y}_1 \cdots d\underline{y}_\ell$$

$$\Lambda_\ell(\underline{y}_1, \cdots, \underline{y}_\ell) = \frac{\prod_{n=1}^{\ell} f_{\underline{Y}_n;\theta=1}(\underline{y}_n)}{\prod_{n=1}^{\ell} f_{\underline{Y}_n;\theta=0}(\underline{y}_n)} \ge B$$

$$\le \frac{1}{B} \int_{\underline{y}_1 \cdots \underline{y}_\ell} \prod_{n=1}^{\ell} f_{\underline{Y}_n;\theta=1}(\underline{y}_n)\, \mathbb{1}_{\Lambda_\ell \ge B}(\underline{y}_1, \cdots, \underline{y}_\ell)\, d\underline{y}_1 \cdots d\underline{y}_\ell$$

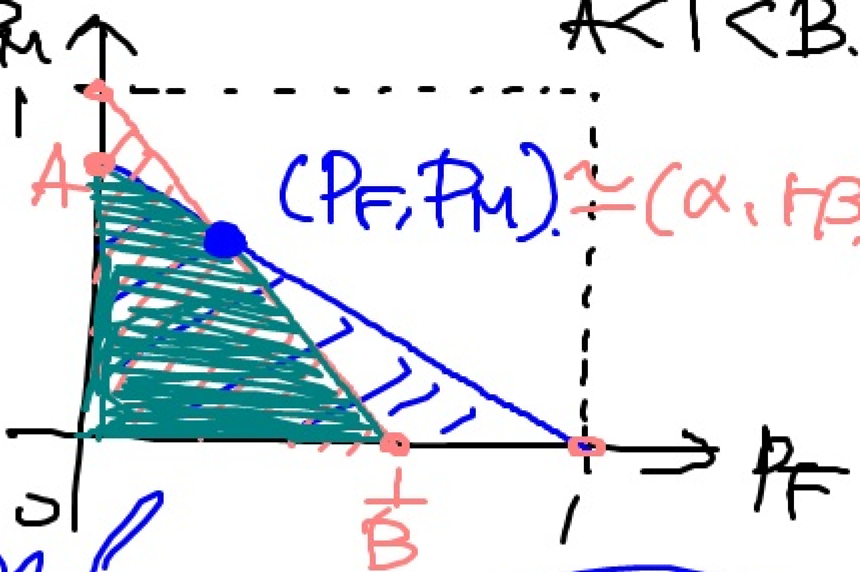
$$= \frac{1}{B} P_D = \frac{1}{B}(1 - P_M)$$

$$\boxed{\begin{aligned} P_F &\le \frac{1}{B}(1 - P_M) \\ P_M &\le A(1 - P_F) \end{aligned}}$$

p 8

$P_F \le \frac{1}{B}(1-P_M)$

$P_M \le A(1-P_F)$



$A < 1 < B.$

Wald's approximation !

$P_F \lesssim \frac{1}{B}(1-P_M)$ $\quad P_F \le \alpha$

$P_M \lesssim A(1-P_F)$ $\quad P_M \le 1-\beta.$

$A \simeq \frac{1-\beta}{1-\alpha}$

$B \simeq \frac{\beta}{\alpha}$

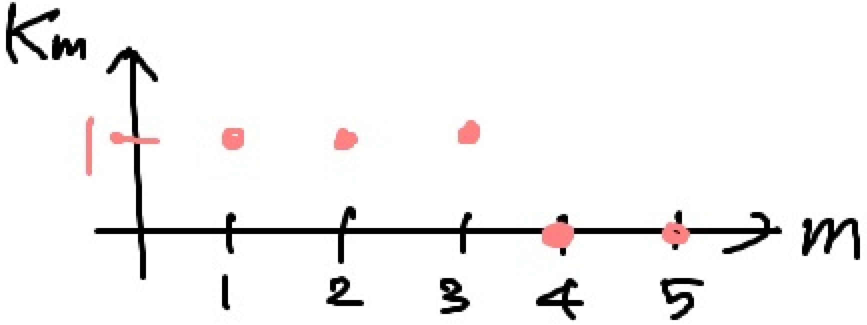
p? $E(L|H_1)$, $E(L|H_0)$

L: stopping time.

K_m : "SPRT(A,B) observes $\underline{Y}_m$"

$$K_m = \begin{cases} 1. \\ 0. \end{cases}$$

ex/ $L = 3$



"K_1 is not a ft of observation. $K=1$."

"K_2 is a ft only of $\underline{y}_1$".

$$\sum_{m=1}^{\infty} K_m = L.$$

$K_m(\underline{y}_1, \cdots, \underline{y}_{m-1})$.

$$E(L;\ \theta = i)$$

$$E\left(\ln \Lambda_L(\underline{Y}_1, \cdots, \underline{Y}_L);\ \theta = i\right)$$

$$= E\left(\ln \frac{\prod_{l=1}^{L} f_{\underline{Y}_n;\theta=1}(\underline{Y}_l)}{\prod_{l=1}^{L} f_{\underline{Y}_n;\theta=0}(\underline{Y}_l)};\ \theta = i\right)$$

$$= E\left(\sum_{l=1}^{L} \ln \frac{f_{\underline{Y}_n;\theta=1}(\underline{Y}_l)}{f_{\underline{Y}_n;\theta=0}(\underline{Y}_l)};\ \theta = i\right)$$

$$= E\left(\sum_{m=1}^{\infty} K_m(\underline{Y}_1, \underline{Y}_2, \cdots, \underline{Y}_{m-1}) \ln \frac{f_{\underline{Y}_n;\theta=1}(\underline{Y}_m)}{f_{\underline{Y}_n;\theta=0}(\underline{Y}_m)};\ \theta = i\right)$$

p11

$$= \sum_{m=1}^{\infty} E\left[K_m(\underline{Y}_1, \cdots, \underline{Y}_{m-1}) \ln \frac{f_{\underline{Y}_m;\theta=1}(\underline{Y}_m)}{f_{\underline{Y}_m;\theta=0}(\underline{Y}_m)} ; \theta = i \right]$$

statistically independent

$$= \left(\sum_{m=1}^{\infty} E\left[K_m(\underline{Y}_1, \cdots, \underline{Y}_{m-1}) ; \theta = i \right] \right) \times E\left[\ln \frac{f_{\underline{Y}_m;\theta=1}(\underline{Y}_m)}{f_{\underline{Y}_m;\theta=0}(\underline{Y}_m)} ; \theta = i \right]$$

$$= E\left[L ; \theta = i \right] \times E\left[\ln \frac{f_{\underline{Y}_n;\theta=1}(\underline{Y}_n)}{f_{\underline{Y}_n;\theta=0}(\underline{Y}_n)} ; \theta = i \right]$$

$$= E\left[\ln \Lambda_L(\underline{Y}_1, \cdots, \underline{Y}_L) ; \theta = i \right]$$

Wald's approximation.

p.12

$$E[\ln \Lambda_L(\underline{y}_1, \cdots, \underline{y}_L);\theta=0]$$

$$\cong \ln A(1-P_F) + \ln B \cdot P_F \simeq \ln A(1-\alpha) + \ln B \cdot \alpha$$

$$E[\ln \Lambda_L(\underline{y}_1, \cdots, \underline{y}_L);\theta=1]$$

$$\cong \ln A \cdot P_M + \ln B(1-P_M)$$

$$\simeq \ln A(1-\beta) + \ln B \cdot \beta$$

* SPRT(A,B) stops almost surely.

p.13

We take another sample if

$$A < \Lambda_\ell(\underline{Y}_1, \cdots, \underline{Y}_\ell) < B$$

$$|\Lambda_\ell(\underline{Y}_1, \cdots, \underline{Y}_\ell)|$$