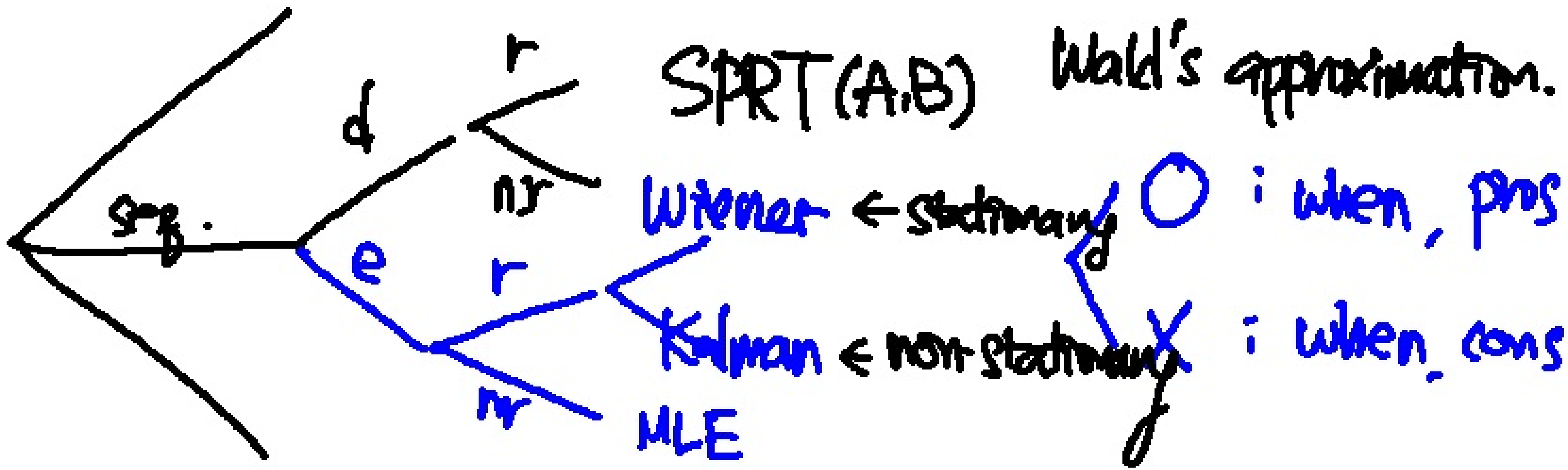
Lec. 31

Announcement 1. Quiz #5 this Friday.

2. Make-up next Monday 2pm.

Review



p2

Observation sequence $(Y[n])_n$

parameter sequence $(X[n])_n$

Sequence
- Vector
- one-sided
- two-sided (DT signal)

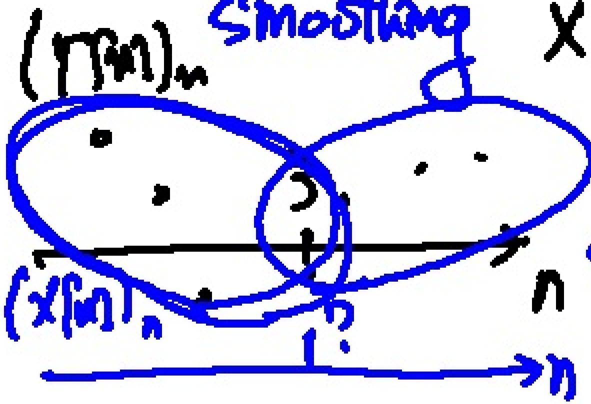
present present past (past of)

Filtering $X[n]$, $Y[n], Y[n-1], Y[n-2], \cdots$ $\hat{X}[n]$

Smoothing $X[n]$, $Y[n+m_1] \cdots, Y[n+1]$,

$Y[n], Y[n-1], \cdots Y[n-m_2]$ $\hat{X}[n]$

future present past



p3

$X[n] \triangleq$

prediction (extrapolation) $Y[n+L]$, $\underline{Y[n], Y[n-1] \cdots}$ $\hat{Y}[n+L]$

$L > 1$.

interpolation $Y[n]$. present

$Y[n-1], Y[n-2] \cdots$ past

$Y[n+1] \cdots$ future

Jointly WSS sequences $(X[n])_n$ & $(Y[n])_n$

zero mean

parameter observation

optimality criterion : LMMSE → The Wiener filtering

parameter to be estimated → $X[n]$.

p4

Wiener filter < causal / non-causal

Causal

$x[n]$

$$\hat{x}[n] = c_n y[n] + c_{n-1} y[n-1] + \cdots$$

$$= \sum_{k=-\infty}^{n} h[n-k]\, y[k] \qquad \text{LTI}$$

$$= \sum_{k=\cancel{-\infty}\ 0}^{n} h[n, n-k]\, y[k] \qquad \text{LTV}$$

$$\hat{x}[0] = h[0,0]\, y[0]$$

$$\hat{x}[1] = h[1,1]\, y[0] + h[1,0]\, y[1]$$

$$\hat{x}[n] = h[n,n]\, y[0] + \cdots + h[n,0]\, y[n]$$

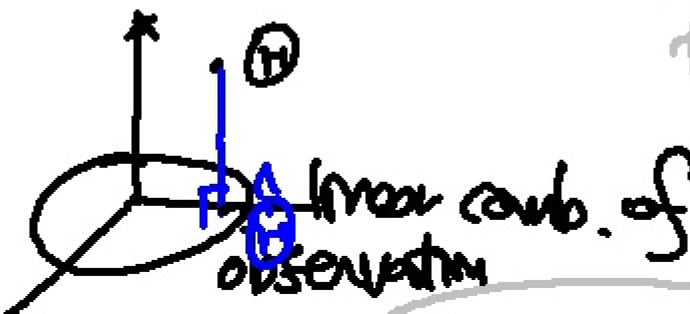
p5

$$\hat{X}[n] = \sum_{k=0}^{n} h[n,n-k]\, Y[k] = \underbrace{\begin{bmatrix} h[n,n] \\ h[n,n-1] \\ \vdots \\ h[n,0] \end{bmatrix}^T}_{\underline{w}} \underbrace{\begin{bmatrix} Y[0] \\ Y[1] \\ \vdots \\ Y[n] \end{bmatrix}}_{\underline{Y}}$$

$\hat{\Theta}$

$$\hat{\Theta}_{LMMSE}(\underline{y}) = \underline{w}^T \underline{y}$$

The Wiener-Hopf equation ($\Leftarrow$ Orthogonality principle)



for scalar parameter

$$R_{YY}\underline{w} = R_{Y\Theta}$$

estimation error $\perp$ observation.

$$E[(\underline{w}^T \underline{Y} - \Theta)\underline{Y}] = \underline{0}.$$

$$E[\underline{Y}(\underline{Y}^T \underline{w} - \Theta)] = \underline{0}$$

p6

Toeplitz matrix

$R_{rx}[n] = E[r[0]x[n]]$

$$\begin{bmatrix} R_{rr}[0] & R_{rr}[1] & & R_{rr}[n] \\ & \ddots & & \\ & & \ddots & \\ R_{rr}[n] & & & R_{rr}[0] \end{bmatrix} \begin{bmatrix} h[n,n] \\ \vdots \\ \vdots \\ h[n,0] \end{bmatrix} = \begin{bmatrix} R_{rx}[n] \\ R_{rx}[n-1] \\ \vdots \\ \vdots \\ R_{rx}[0] \end{bmatrix}$$

$\underline{R_{rr}}$ $\quad$ $\underline{w}$ $\quad$ $R_{r\theta}$ $\quad x[n]$

P7

$n \gg 1$. recursion algorithm → shortly for prediction

$n \gg 1$. LTV → LTI →

$$\hat{X}[n] = \sum_{k=-\infty}^{n} \cancel{h[n, n-k]}\, h[n-k]\, Y[k]$$

Causal LMMSE filter. Orthogonality principle.

$$E\left[\left(\sum_{k=-\infty}^{n} h[n-k]\,Y[k] - X[n]\right) Y[l]\right] = 0,$$

$$\sum_{k=-\infty}^{n} h[n-k]\, R_{YY}[l-k] = R_{YX}[n-l] \quad \forall\, l \le n.$$

p8

$$\sum_{k=0}^{\infty} R_{rr}[l-k]\, h[k] = R_{rx}[l], \quad l = 0, 1, 2, \cdots \quad \cdots (*)$$

Unilateral Z-transform!

$(x[n])_n$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$ ← bilateral ZTF

causal $1, z^{-1}, z^{-2}, \cdots$

$$\left[X(z)\right]_+ = \sum_{n=0}^{\infty} x[n] z^{-n}$$ ← unilateral ZTF

$$(*) \Rightarrow \left[R_{rr}(z) H(z)\right]_+ = \left[R_{rx}(z)\right]_+ \quad \cdots (**)$$

$R_{rr}(z) = \hat{R}_{rr}(z)\hat{R}_{rr}(z^{-1})$

rational approximation

$$\left[R_{YY}(z) H(z) - R_{YX}(z) \right]_f = 0$$

$$\left[\hat{R}_{YY}(z) H(z) - \frac{R_{YX}(z)}{R_{YY}(z^{-1})} \right]_f = 0$$

$$\Rightarrow \left[\underbrace{\hat{R}_{YY}(z)}_{1, z^{-1}, z^{-2}, \cdots} \hat{R}_{YY}(z^{-1}) \underbrace{H(z)}_{1, z^{-1}, z^{-2} \cdots} - R_{YX}(z) \right]_f = 0$$

$$\Rightarrow \left[\underbrace{\left(\hat{R}_{YY}(z) H(z) - \frac{R_{YX}(z)}{R_{YY}(z^{-1})} \right)}_{1, z^{-1}, z^{-2}, \cdots} \underbrace{\hat{R}_{YY}(z^{-1})}_{1, z, z^{2}, z^{3} \cdots} \right]_+ = 0$$

$$\cancel{\hat{R}_{YY}(z) H(z)} = \frac{1}{\hat{R}_{YY}(z)} \left[\frac{R_{YX}(z)}{R_{YY}(z^{-1})} \right]_f$$

$[A(z) B(z)]_+ = 0$. $B(z)$ anti causal. $\Rightarrow [A(z)]_f = 0$.