Lec. 35

Announcement (Quiz on this Friday,
Covers mainly the Wiener filtering &
the Kalman filtering

Review

Theorem Given an energy signal $x(t)$, i.e.,

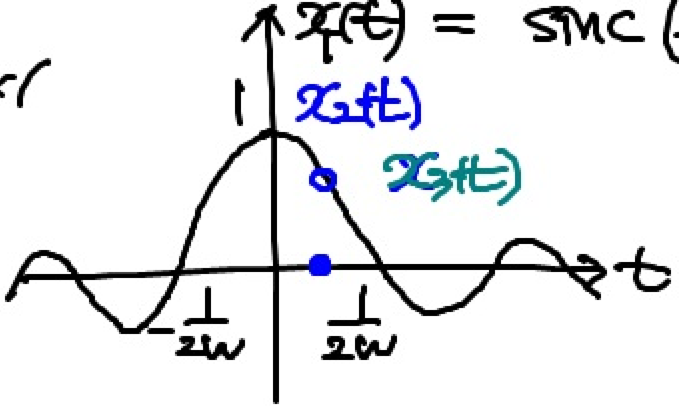
$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty,$$

if $$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-W}^{W} |X(f)|^2 df$$

then $$x(t) \overset{L^2}{=} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \operatorname{sinc}\left(2W\left(t-\frac{n}{2W}\right)\right)$$

p.2

ex)



periodic extension.

TL signal $\quad x(t),\ 0<t\leq T.$

energy-

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(t-nT) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi k t}{T}}$$

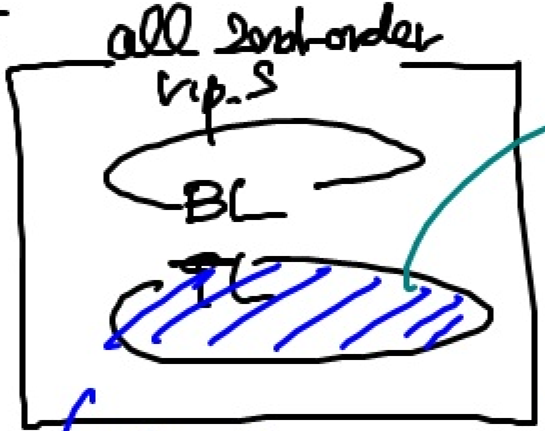
BL signal $\quad x(t),\ X(f)\ \ |f|\leq W.$

energy-

$$\tilde{X}(f) = \sum_{n=-\infty}^{\infty} X(f-kW) = \sum_{k=-\infty}^{\infty} b_k e^{j\frac{2\pi k f}{W}}$$

Nyquist Sampling th.		Shannon-Nyquist Sampling th.
band-limited	Common	band-limited
finite-energy deterministic	Diff.	finite-power random process
$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt < \infty$ "energy signal"		$E[X(t)X(s)] < \infty,\ \forall t, \forall s$ "2nd-order random process"
$\sum_{n=-N}^{N} x\left(\frac{n}{2W}\right) \operatorname{sinc}\left(2W\left(t-\frac{n}{2W}\right)\right) \longrightarrow x(t)$ in L^2-sense i.e., $\int_{-\infty}^{\infty} \left\lvert x(t) - \sum_{n=-N}^{N} \cdots \right\rvert^2 dt \to 0$		$\sum_{n=-N}^{N} X\left(\frac{n}{2W}\right) \operatorname{sinc}\left(2W\left(t-\frac{n}{2W}\right)\right) \to X(t)$ in L^2-sense i.e. $E\left\{\left\lvert X(t) - \sum_{n=-N}^{N} X\left(\frac{n}{2W}\right) \cdots \right\rvert^2\right\} \to 0$ $\forall t$, as $N \to \infty$.

p.4.



→ KL expansion

(literature survey)

simplify

visualize.

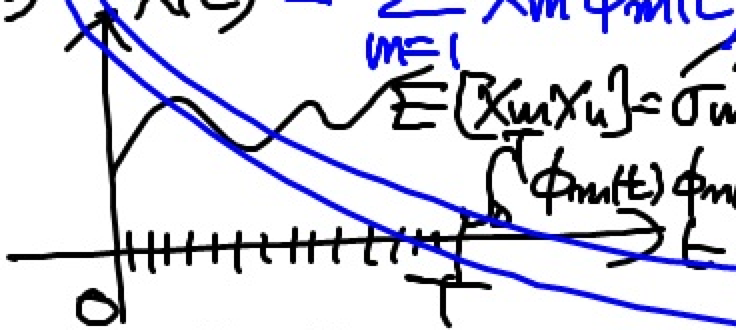
$X(t) \Leftarrow$ 2nd-order r.p.

$x(\omega, t)$

$\mathcal{F}\{x(\omega, t)\}$ does not exist!

→ $X(f)$: CTFT of $X(t)$.

p5 $X(t) = \sum_{m=1}^{\infty} X_m \phi_m(t)$



$E[X_m X_n] = \sigma_m^2 \delta_{m,n}$

$\int_0^T \phi_m(t)\phi_n(t)\,dt = \delta_{m,n}$

$\underline{X} = \sum_{m=1}^{M} X_m \underline{\Phi}_m \in \mathbb{R}^M$

$E[X_m X_n] = \sigma_m^2 \delta_{m,n}$ $\quad X_m \neq [\underline{X}]_m$

$\underline{\Phi}_m^T \underline{\Phi}_n = \delta_{m,n}$

$\mu_X(t) \triangleq E[X(t)] = 0, \ \forall t \in [0,T]$

$\underline{\mu}_X \triangleq E[\underline{X}] = \underline{0}$

$R_{XX}(t,s) \triangleq E[X(t)X(s)] < \infty$

$\forall t, \forall s \in [0,T]$

$\underline{C}_{XX} \triangleq E[\underline{X}\,\underline{X}^T] = \begin{bmatrix} & & \\ & R_{XX}(t_i,t_j) & \\ & & \end{bmatrix}$

$\succeq 0$

$$\left(\sum_i \sum_j w(t_i) R_{XX}(t_i,t_j) w(t_j) \ge 0 \right.$$
$$\left. \Leftrightarrow \int_0^T\!\!\int_0^T w(t) R_{XX}(t,s) w(s)\,dt\,ds \ge 0 \right)$$

$\succeq 0$

$$\left(\underline{w}^T \underline{C}_{XX} \underline{w} \ge 0, \ \forall \underline{w} \right)$$

p6

$$E[\underline{X}]=0$$

$$E[\underline{X}\,\underline{X}^T]=C \succeq 0$$

$$\underline{X}=\sum_{m=1}^{M} X_m \underline{\phi}_m$$

$$C = E\left[\sum_{m=1}^{M} X_m\underline{\phi}_m \left(\sum_{m'=1}^{M} X_{m'}\underline{\phi}_{m'}\right)^T\right]$$

$$= \sum_{m'=1}^{M} \sigma_{m'}^2\, \underline{\phi}_{m'}\underline{\phi}_{m'}^T$$

$$C\underline{\phi}_m = \sigma_m^2\, \underline{\phi}_m.$$

$$\lambda_m=\sigma_m^2$$

$$\underline{v}_m=\underline{\phi}_m.$$

$$\underline{X} \to \otimes \to X_m, \quad \underline{\phi}_m \uparrow$$

$$\underline{X}=\sum_{m=1}^{M} X_m\underline{\phi}_m$$

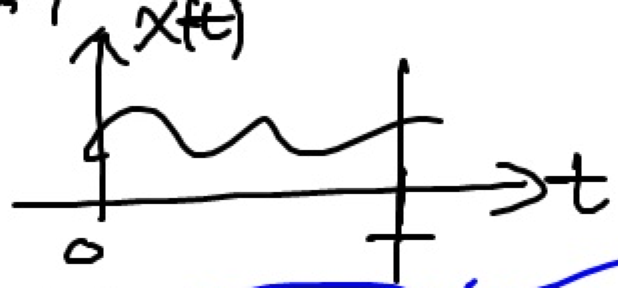
$$C\underline{v}_m=\lambda_m \underline{v}_m \qquad (\lambda_m, \underline{v}_m).$$

$$C = V\Lambda V^{-1}$$

$$= V\Lambda V^T = [\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_M]\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_M\end{bmatrix}\begin{bmatrix}\underline{v}_1^T \\ \vdots \\ \underline{v}_M^T\end{bmatrix}$$

$$= \sum_{m=1}^{M} \lambda_m\, \underline{v}_m\underline{v}_m^T.$$

P.7



Given

$$\int_0^T R_{xx}(t,s)\, v_m(s)\, ds = \lambda_m v_m(t), \quad \forall t \in [0,T]$$

$$\left(C \underline{v}_m = \lambda_m \underline{v}_m\right)$$

$$\downarrow$$

$$C = \sum_{m=1}^{M} \lambda_m \underline{v}_m \underline{v}_m^T$$

$$\left(\lambda_m, v_m(t)\right)_{m=1}^{\infty}$$ to find.

Fredholm integral equation.

$$\downarrow$$

$$R_{xx}(t,s) = \sum_{m=1}^{\infty} \lambda_m v_m(t) v_m(s), \quad \forall t, \forall s \in [0,T].$$

Mercer's theorem.

p 8

$$\boxed{X(t) = \sum_{m=1}^{\infty} X_m \phi_m(t), \ \forall t \in [0,T]}$$

where

$$\int_0^T \phi_m(t)\phi_n(t)\,dt = \delta_{m,n}$$

$$\Rightarrow X_m = \int_0^T \phi_m(t) X(t)\,dt$$

$$\Rightarrow E[X_m] = 0, \ \forall m$$

$$E[X_m X_n] = \cdots$$

$$= \sigma_m^2 \delta_{m,n}$$

$$\sigma_m^2 = \lambda_m$$

$$\phi_m(t) = v_m(t).$$

$$\underline{X} = \sum_{m=1}^{M} X_m \underline{\phi}_m$$

~~where~~ we want

$$X_m = \underline{\phi}_m^T \underline{X} \Leftarrow, \quad \underline{\phi}_m^T \underline{\phi}_n = \delta_{m,n}$$

$$\Rightarrow E[X_m] = 0$$

$$E[X_m X_n] = \sigma_m^2 \delta_{m,n}.$$

It turns out that

$$\underline{\phi}_m = \underline{v}_m \ \&$$

$$\sigma_m^2 = \lambda_m.$$

Karhunen-Loève expansion.