Lec. 37

Review

the KL expansion

- analogy : $\underline{Y} = \sum_{k=1}^{N} Y_k \underline{\psi}_k$
- deterministic counterpart : Fourier basis

Preview

d

e

d

e

d

Detection in AWGN

A GN

$H_i : Y(t) = \underbrace{s_i(t)}_{\text{known deterministic}} + \underbrace{N(t)}_{\text{AWGN}}$

$0 \le t < T$

e

Estimation in AWGN

AGN

$Y(t) = s_\theta(t) + N(t), \; 0 \le t < T$

p.2

Q. Given $\{r(t)=y(t)\}$ for $0\le t\le T$, find the likelihood ratio of the statistical inference problem

$$H_0:\ r(t)=N(t)$$
vs
$$H_1:\ r(t)=s(t)+N(t)$$
for $0\le t\le T$.

where $N(t)$ is AWGN w/ $R_{NN}(\tau)=\frac{N_0}{2}\delta(\tau)$.

A.
$$H_0:\ \underline{r}=\underline{N}$$
vs
$$H_1:\ \underline{r}=\underline{s}+\underline{N}$$

$\underline{N}\sim N(\underline{0},\sigma^2 I)$

$$\Lambda(\underline{y})=\frac{\exp\left(-\frac{\|\underline{y}-\underline{s}\|^2}{2\sigma^2}\right)}{\exp\left(-\frac{\|\underline{y}\|^2}{2\sigma^2}\right)}$$

p.3 If there exists $\{\phi_i(t)\}_{i=1}^{\infty}$ s.t.

(i) $\{\phi_i(t)\}_{i=1}^{\infty}$ is an orthonormal basis of $L_2([0,T))$;

(ii) $\phi_1(t) = \dfrac{s(t)}{\|s(t)\|}$, $0 \le t < T$.

i.i.d. zero-mean Gaussian w/ $\mathrm{Var} = \frac{N_0}{2}$,

then $H_0:\ Y(t) = \sum_{i=1}^{\infty} N_i \phi_i(t)$, where $N_i \triangleq \int_0^T N(t)\phi_i(t)\,dt$

vs

$s_1 = \|s(t)\|$

$s_i = 0$, $\forall i \neq 1$.

$$H_1:\ Y(t) = \sum_{i=1}^{\infty} \left(s_i \phi_i(t) + N_i \phi_i(t) \right)$$

$$= \sum_{i=1}^{\infty} (s_i + N_i)\,\phi_i(t)$$

pf Note that

$$E\{N_i\} = E\left\{\int_0^T N(t)\phi_i(t)\,dt\right\} = 0.$$

$$E\{N_i N_j\} = E\left\{\int_0^T N(t)\phi_i(t)\,dt \int_0^T N(s)\phi_j(s)\,ds\right\}$$

$$= \int_0^T\int_0^T \underbrace{E\{N(t)N(s)\}}_{\frac{N_0}{2}\delta(t-s)} \phi_i(t)\phi_j(s)\,dt\,ds$$

$$= \int_0^T \frac{N_0}{2} \phi_i(s)\phi_j(s)\,ds$$

by the sifting property

$$= \frac{N_0}{2}\delta_{i,j}$$

*

$$\boxed{\delta(t-s) = \sum_{i=1}^{\infty} \phi_i(t)\phi_i(s)}$$

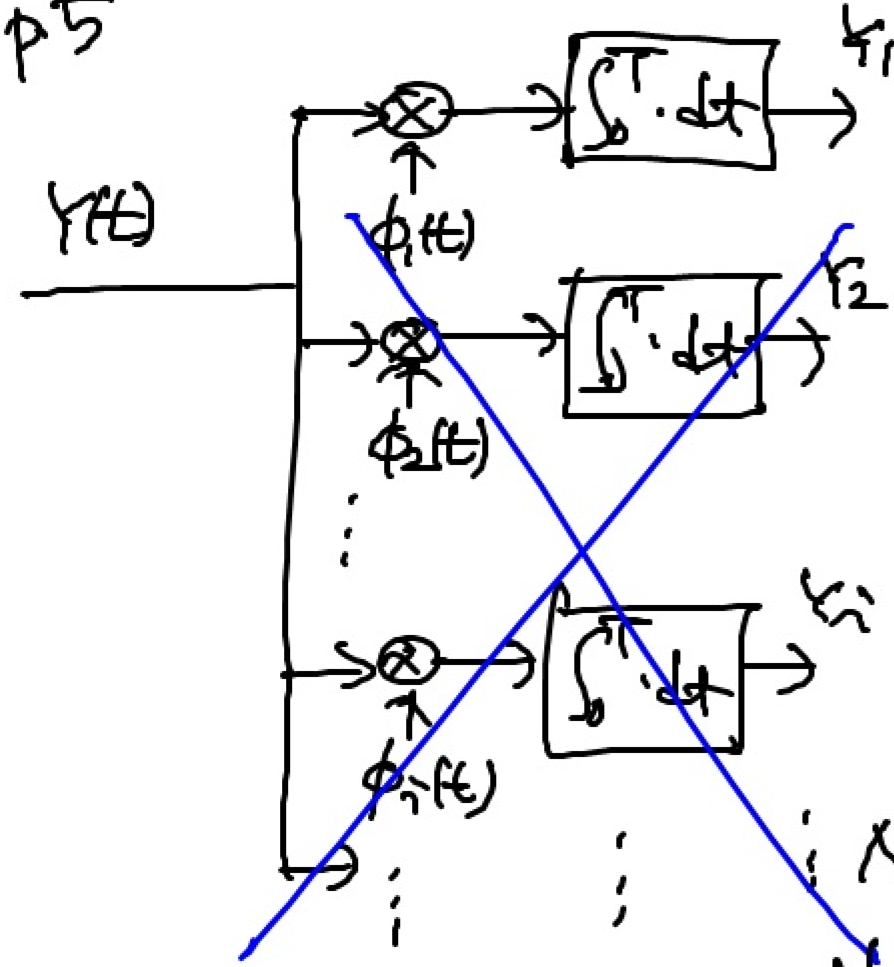
where $\{\phi_i(t)\}_{i=1}^{\infty}$ is an orthonormal basis for $L_2([0,T))$.

$U = [\underline{u}_1 \cdots \underline{u}_n]$

$UU^T = I$.

$\sum \underline{u}_i \underline{u}_i^T = I$.

p5



$H_0: Y_\infty^1 = N_\infty^1$

vs

$H_1: Y_\infty^1 = S_\infty^1 + N_\infty^1$

$H_0: Y_1 \sim N_1$

vs

$H_1: Y_1 \sim \|S(t)\| + N_1$

$N_1 \sim N(0, \frac{N_0}{2})$

$$\Lambda(y_1) = \exp\left(\quad \right) = \frac{\exp\left(-\frac{(y_1 - s_1)^2}{2 \cdot \frac{N_0}{2}}\right)}{\exp\left(-\frac{y_1^2}{2 \cdot \frac{N_0}{2}}\right)}$$

where $y_1 = \int_0^T y(t) \frac{s(t)}{\|s(t)\|} dt$

p6

Analogy.

$\underline{p}$

$$\frac{\underline{p}\underline{p}^T}{\underline{p}^T\underline{p}}\underline{b} \qquad \left(\underline{\tilde{p}}\underline{\tilde{p}}^T \underline{b}\right) = (\underline{\tilde{p}}^T \underline{b})\,\underline{\tilde{p}}$$

Gram-Schmidt procedure

$S = [\underline{s}_1\ \underline{s}_2 \cdots \underline{s}_M]$

QR factorization

$$Y(t) = \underbrace{s_{\underline{\theta}}(t)}_{\text{known deterministic function of unknown } \underline{\theta}} + \underbrace{N(t)}_{\text{AWGN}}, \quad 0 \le t \le T$$

$\frac{N_0}{2}\delta(\tau)$.

p7

$$\arg\min_a \int_0^T \left(y(t) - s(t-a) \right)^2 dt$$

FONC $\cancel{2\int_0^T (y(t)-s(t-a))\, s'(t-a)\, dt}$

$$\int_0^T (y(t))^2 dt - 2\int_0^T y(t)\, s(t-a)\, dt + \int_0^T (s(t-a))^2 dt$$

$$\hat{A}_{ML} = \arg\max_a \int_0^T y(t)\, s(t-a)\, dt$$