Lecture 31

Ch. 25 Discrete Optimization, Variational Problems

EECE 695H
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- discrete vs. continuous optimization
- variational problems

Discrete Optimization

In a discrete optimization problem, the constraint set Ω is discrete, $i.e.$, Ω is a countable set.

There are two kinds of sets:

- countable set
  - finite number of elements
  - countably infinite number of elements
- uncountable set
  - uncountably infinite number of elements

§19.1 Introduction

A representative example of discrete optimization is **integer linear programming (ILP)** or, in short, integer programming in Ch. 19 of the 4th edition.

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq \boldsymbol{0} \\ & \boldsymbol{x} \in \mathbb{Z}^n. \end{aligned}$$

§19.2 Unimodular Matrices

There is a class of ILP problems that can be solved using standard LP methods.

Def: An $m \times n$ integer matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}, m \leq n$, is called *unimodular* if all its nonzero m-th order minors are ± 1.

Let $\boldsymbol{B}$ be a corresponding basis matrix. Then, the unimodularity of $\boldsymbol{A}$ is equivalent to $|\det \boldsymbol{B}| = 1$ for any such $\boldsymbol{B}$.

It can be shown that unimodularity allows us to solve ILP prblems using the simplex method.

§25.1 Variational Problems

So far, the decision variable has been $\boldsymbol{x} \in \mathbb{R}^n$. In some problems, we want to find a function that minimizes a real-valued functional.
Examples:

- Brachistochrone by John Bernoulli
- Brachistochrone through Earth by Newton
- greatest area problem by Queen Dido of Carthage in about 850 B.C.
- greatest area in given time problem by Chaplygin
- minimum transit time problem of a boat crossing a river
- optimal control problems, etc.

§25.2 A Class of Variational Problems

Sometimes, the domain of an objective functional is a normed vector space $\mathcal{X}$. In such cases, we can easily generalize our study on the theory for the objective functions of $\boldsymbol{x} \in \mathbb{R}^n$.

Def: A Gâteaux variation of a functional J at $x \in D$ is defined as

$$\delta J(x; \Delta x) = \lim_{\varepsilon \to 0} \frac{J(x + \varepsilon \Delta x) - J(x)}{\varepsilon}$$

when the limit exists for all Δx in the vector space $\mathcal{X}$.

Euler introduced the name *Calculus of Variations* to describe Lagrange's method for variational problems.

§25.3 FONCs

Thm: If x^* is an interior point and a local extremum, then

$$\delta f_0(x^*; \Delta x) = 0 \text{ for all } \Delta x \in \mathcal{X}.$$

Thm: (Euler-Lagrange Multiplier Theorem) If x^* is a local minimizer and regular for an equality constrained problem, then there exists $\boldsymbol{\nu}^*$ such that

$$\delta f_0(x^*; \Delta x) + \sum_j \nu_j^* \delta h_j(x^*; \Delta x) = 0 \text{ for all } \Delta x \in \mathcal{X}.$$

Thm: (John Multiplier Theorem) If x^* is a local minimizer and regular for an inequality constrained problem, then there exists $\boldsymbol{\lambda}^* \succeq \mathbf{0}$ such that

$$\boldsymbol{\lambda}^{*\top} \boldsymbol{f}(x^*) = 0$$

and

$$\delta f_0(x^*; \Delta x) + \sum_i \lambda_i^* \delta f_i(x^*; \Delta x) = 0 \text{ for all } \Delta x \in \mathcal{X}.$$

etc.

§25.4 Example

The solution to the Brachistochrone problem is **a cycloid** such that

$$x = x_0 + \frac{A}{2}(\theta - \sin\theta)$$
$$y = y_0 - \frac{A}{2}(1 - \cos\theta)$$

for $\theta_0 \leq \theta \leq \theta_1$, where $\theta_0 = 0$ and θ_1 are the values of θ which corresponds to the start point $P_0 = (x_0, y_0)$ and the end point $P_1 = (x_1, y_1)$ with $x_1 > x_0$ and $y_1 \leq y_0$.

It can be shown that given $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$, $\theta_1 \in (0, 2\pi)$ and A are uniquely determined.

The Brachistochrone through Earth has the solution that is **a hypocycloid** that can be obtained by tracing out a fixed point on a smaller circle as smaller circle rolls along indside the larger circle.