Time-Domain Analysis of Second-Order Cyclostationary (SOCS) Processes under Frequency Up- and Down-Conversions

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November 13, 2025

Abstract

This document presents a **time-domain analysis** of how frequency **up-conversion** (to a real passband signal $Y(t)$) and **down-conversion** (to a complex baseband signal $\hat{X}(t)$) affect the properties of a **Second-Order Cyclostationary (SOCS)** process $X(t)$. By inspecting the periodicity of the mean, autocorrelation (ACF), and complementary autocorrelation (CACF) functions, we derive the strict frequency conditions required to preserve cyclostationarity. We demonstrate that the ACF of the down-converted signal $\hat{X}(t)$ **always** maintains T-periodicity, regardless of the frequency offset Δf. However, the SOCS property is generally lost unless the synchronization condition is met. Specifically, for a non-zero mean process, the carrier f_c (for $Y(t)$) or frequency error Δf (for $\hat{X}(t)$) must be an integer multiple of the cyclic frequency ($1/T$). For a zero-mean, non-proper (improper) SOCS process, this condition is relaxed to $f_c = m/(2T)$ or $\Delta f = k'/(2T)$. We also note that for zero-mean, proper-complex processes, cyclostationarity is unconditionally preserved.

1. Cyclostationary Signal Definitions

1.1. Real Wide-Sense Cyclostationary (WSCS) Process $Y(t)$

A real random process $Y(t)$ is defined as Wide-Sense Cyclostationary (WSCS) with cyclic period T if:

- **Mean Function:** $\mu_Y(t) = E[Y(t)]$ is periodic with period T, i.e., $\mu_Y(t) = \mu_Y(t+T)$.
- **Autocorrelation Function (ACF):** $R_{YY}(t,s) = E[Y(t)Y(s)]$ satisfies $R_{YY}(t,s) = R_{YY}(t+T, s+T)$.

For a real process, the relationship between the Autocorrelation Function (ACF), Autocovariance Function (ACVF), and Mean Function is:

- **Autocovariance Function (ACVF):**

$$C_{YY}(t,s) = E[(Y(t) - \mu_Y(t))(Y(s) - \mu_Y(s))]$$

- **ACF and ACVF Relation:**

$$R_{YY}(t,s) = C_{YY}(t,s) + \mu_Y(t)\mu_Y(s)$$

1.2. Complex Wide-Sense Cyclostationary (WSCS) Process $X(t)$

A complex random process $X(t)$ is defined as Wide-Sense Cyclostationary (WSCS) with cyclic period T if:

- **Mean Function:** $\mu_X(t) = E[X(t)]$ is periodic with period T, i.e., $\mu_X(t) = \mu_X(t+T)$.
- **Autocorrelation Function:** $R_{XX}(t,s) = E[X(t)X^*(s)]$ satisfies $R_{XX}(t,s) = R_{XX}(t+T, s+T)$.

1.3. Second-Order Cyclostationary (SOCS) Process

$X(t)$ is an SOCS RP with cyclic period T if it is WSCS and satisfies the following **SOCS additional condition** for its Complementary Autocorrelation Function (CACF):

- **CACF Condition:** $\tilde{R}_{XX}(t,s) = E[X(t)X(s)]$ satisfies $\tilde{R}_{XX}(t,s) = \tilde{R}_{XX}(t+T, s+T)$.

1.4. Proper-Complex Random Process

A complex random process $X(t)$ is defined as a **proper-complex random process** (or circular symmetric process) if its Complementary Autocovariance Function (CACVF) is zero for all t and s:

$$\tilde{C}_{XX}(t,s) = E[(X(t) - \mu_X(t))(X(s) - \mu_X(s))] = 0 \tag{1}$$

For a proper-complex process, the CACF is solely determined by the mean function:

$$\tilde{R}_{XX}(t,s) = \tilde{C}_{XX}(t,s) + \mu_X(t)\mu_X(s) = \mu_X(t)\mu_X(s)$$

If a proper-complex process $X(t)$ is WSCS, then $\mu_X(t) = \mu_X(t+T)$. Consequently, $\tilde{R}_{XX}(t+T, s+T) = \mu_X(t+T)\mu_X(s+T) = \mu_X(t)\mu_X(s) = \tilde{R}_{XX}(t,s)$. Thus, for a proper-complex process, the **WSCS condition implies the SOCS condition**.

2. WSCS Analysis of Up-converted Signal $Y(t)$

We analyze the conditions under which the up-converted real signal $Y(t)$, which is related to the complex baseband signal $X(t)$ via the carrier frequency f_c, remains WSCS with period T.

The relationship is defined as:

$$Y(t) = \mathrm{Re}\left\{X(t)e^{j2\pi f_c t}\right\} = \frac{1}{2}\left(X(t)e^{j2\pi f_c t} + X^*(t)e^{-j2\pi f_c t}\right) \tag{2}$$

2.1. Mean Function Analysis of $Y(t)$

The periodic mean function $\mu_X(t)$ can be expanded using a Fourier series:

$$\mu_X(t) = \sum_k c_k e^{j2\pi(k/T)t} \tag{3}$$

The periodicity of the mean function $\mu_Y(t)$ requires the carrier frequency f_c to be synchronized with the cyclic frequency $1/T$.

$$\mu_Y(t) = \mathrm{Re}\left\{\mu_X(t)e^{j2\pi f_c t}\right\} = \mathrm{Re}\left\{\sum_k c_k e^{j2\pi(f_c + k/T)t}\right\} \tag{4}$$

For $\mu_Y(t)$ to be periodic with period T (assuming $\mu_X(t) \neq 0$), we require f_c to be an integer multiple of $1/T$:

$$f_c = \frac{k'}{T}, \quad k' \in \mathbb{Z} \tag{5}$$

2.2. Autocorrelation Function Analysis of $Y(t)$ (Detailed Derivation)

For $Y(t)$ to be WSCS, its autocorrelation function $R_{YY}(t,s) = E[Y(t)Y(s)]$ must satisfy $R_{YY}(t,s) = R_{YY}(t+T, s+T)$.

Using the expansion of $Y(t)$ from Eq. (2), $R_{YY}(t,s)$ is derived as follows:

$$\begin{aligned} R_{YY}(t,s) &= E\left[\frac{1}{2}\left(X(t)e^{j2\pi f_c t} + X^*(t)e^{-j2\pi f_c t}\right) \cdot \frac{1}{2}\left(X(s)e^{j2\pi f_c s} + X^*(s)e^{-j2\pi f_c s}\right)\right] \\ &= \frac{1}{4}E\Big[X(t)X(s)e^{j2\pi f_c(t+s)} + X(t)X^*(s)e^{j2\pi f_c(t-s)} \\ &\quad + X^*(t)X(s)e^{-j2\pi f_c(t-s)} + X^*(t)X^*(s)e^{-j2\pi f_c(t+s)}\Big] \end{aligned}$$

Applying the expectation operator $E[\cdot]$ and using the definitions $R_{XX}(t,s) = E[X(t)X^*(s)]$ and $\tilde{R}_{XX}(t,s) = E[X(t)X(s)]$:

$$R_{YY}(t,s) = \frac{1}{4}\Big[\tilde{R}_{XX}(t,s)e^{j2\pi f_c(t+s)} + R_{XX}(t,s)e^{j2\pi f_c(t-s)} \\ + R^*_{XX}(s,t)e^{-j2\pi f_c(t-s)} + \tilde{R}^*_{XX}(s,t)e^{-j2\pi f_c(t+s)}\Big] \quad (6)$$

Using the Hermitian property $R^*_{XX}(s,t) = R_{XX}(t,s)$ and the symmetry of $\tilde{R}_{XX}$ for proper-complex signals, $R_{YY}(t,s)$ can be rewritten as:

$$R_{YY}(t,s) = \frac{1}{2}\mathrm{Re}\left\{R_{XX}(t,s)e^{j2\pi f_c(t-s)} + \tilde{R}_{XX}(t,s)e^{j2\pi f_c(t+s)}\right\} \quad (7)$$

2.2.1 WSCS Periodicity Condition for $R_{YY}(t,s)$

For $R_{YY}(t,s)$ to be periodic with period T, we require $R_{YY}(t+T, s+T) = R_{YY}(t,s)$.

$R_{XX}(t,s)$ has period T when $X(t)$ is **WSCS**, but $\tilde{R}_{XX}(t,s)$ requires $X(t)$ to be **SOCS** to have period T. This analysis proceeds by assuming $X(t)$ satisfies the **SOCS** condition, so both functions $(R_{XX}, \tilde{R}_{XX})$ are T-periodic. We examine the phase terms in Eq. (7):

1. **Term 1: Autocorrelation R_{XX}** The phase term is $e^{j2\pi f_c(t-s)}$. $t-s$ is the time difference τ, which is independent of the absolute time t. Since $R_{XX}(t,s) = R_{XX}(t+T, s+T)$, this term **always** satisfies the periodicity condition, regardless of f_c.
2. **Term 2: Complementary Autocorrelation $\tilde{R}_{XX}$** The phase term is $e^{j2\pi f_c(t+s)}$. When shifting $t \to t+T$ and $s \to s+T$, the new phase is:

$$e^{j2\pi f_c((t+T)+(s+T))} = e^{j2\pi f_c(t+s)} \cdot e^{j2\pi f_c(2T)}$$

For this term to be periodic with period T, the factor $e^{j2\pi f_c(2T)}$ must equal 1:

$$e^{j2\pi(2f_c)T} = 1$$

This requires the doubled carrier frequency $2f_c$ to be an integer multiple of the cyclic frequency $1/T$:

$$2f_c = \frac{m}{T}, \quad m \in \mathbb{Z} \quad \text{or} \quad f_c = \frac{m}{2T} \quad (8)$$

2.2.2 Overall Conclusion for $Y(t)$ WSCS

- **If $X(t)$ is general SOCS with $\mu_X(t) \neq 0$:** Both mean function (μ_Y) and autocorrelation (R_{YY}) must be T-periodic. The dominant condition is $f_c = k'/T$. **Result:** $Y(t)$ is WSCS **only if** $f_c = k'/T$.
- **If $X(t)$ is general SOCS with $\mu_X(t) = 0$:** Only the autocorrelation condition is required. Since $\tilde{R}_{XX}(t,s) \neq 0$ (for general SOCS), Term 2 must be periodic. **Result:** $Y(t)$ is WSCS **only if** $f_c = m/2T$.
- **If $X(t)$ is Proper-Complex:** If $X(t)$ is Proper-Complex and $\mu_X(t) \neq 0$, the mean condition $f_c = k'/T$ dominates. If $X(t)$ is Proper-Complex and $\mu_X(t) = 0$, then $\tilde{R}_{XX}(t,s) = 0$. Term 2 is zero, and R_{YY} is always T-periodic regardless of f_c. **Result:** $Y(t)$ is WSCS **only if** $f_c = k'/T$ (if $\mu_X(t) \neq 0$) or **always** (if $\mu_X(t) = 0$).

3. Analysis of Down-Conversion with Incorrect Frequency

We analyze the down-converted complex signal $\hat{X}(t)$, defined by:

$$\hat{X}(t) = X(t)e^{j2\pi(f_c - f_0)t} \quad (9)$$

Let $\Delta f = f_c - f_0$. The question is: Is $\hat{X}(t)$ SOCS with the period T?

3.1. Mean Function (WSCS Condition)

The mean function $\mu_{\hat{X}}(t) = \mu_X(t)e^{j2\pi\Delta ft}$ is periodic with period T (WSCS condition) **only if**:

$$\Delta f = f_c - f_0 = \frac{k}{T}, \quad k \in \mathbb{Z}$$

3.2. Autocorrelation Function (ACF) Analysis

For $\hat{X}(t)$ to be WSCS, its Autocorrelation Function $R_{\hat{X}\hat{X}}(t,s) = E[\hat{X}(t)\hat{X}^*(s)]$ must be periodic with period T.

The ACF is given by:

$$\begin{aligned} R_{\hat{X}\hat{X}}(t,s) &= E[X(t)e^{j2\pi\Delta ft} \cdot X^*(s)e^{-j2\pi\Delta fs}] \\ &= E[X(t)X^*(s)]e^{j2\pi\Delta f(t-s)} \\ &= R_{XX}(t,s)e^{j2\pi\Delta f(t-s)} \end{aligned} \tag{10}$$

We check the periodicity condition $R_{\hat{X}\hat{X}}(t+T, s+T) = R_{\hat{X}\hat{X}}(t,s)$:

$$\begin{aligned} R_{\hat{X}\hat{X}}(t+T, s+T) &= R_{XX}(t+T, s+T)e^{j2\pi\Delta f((t+T)-(s+T))} \\ &= R_{XX}(t,s)e^{j2\pi\Delta f(t-s)} \\ &= R_{\hat{X}\hat{X}}(t,s) \end{aligned}$$

Since $R_{XX}(t,s)$ is T-periodic (due to $X(t)$ being WSCS), and the phase term $e^{j2\pi\Delta f(t-s)}$ depends only on the time difference $\tau = t - s$ (which is independent of the absolute time shift T), the ACF is **always** T-periodic, regardless of the frequency offset Δf.

Conclusion (ACF): The autocorrelation condition for $\hat{X}(t)$ to be WSCS is **always** satisfied if $X(t)$ is WSCS.

3.3. Complementary Autocorrelation (SOCS Additional Condition)

For $\hat{X}(t)$ to be SOCS, its CACF $\tilde{R}_{\hat{X}\hat{X}}(t,s) = E[\hat{X}(t)\hat{X}(s)]$ must be periodic with period T.

$$\tilde{R}_{\hat{X}\hat{X}}(t,s) = E[X(t)X(s)]e^{j2\pi\Delta f(t+s)} = \tilde{R}_{XX}(t,s)e^{j2\pi\Delta f(t+s)} \tag{11}$$

The periodicity condition $\tilde{R}_{\hat{X}\hat{X}}(t+T, s+T) = \tilde{R}_{\hat{X}\hat{X}}(t,s)$ requires the phase factor to satisfy:

$$e^{j2\pi\Delta f(2T)} = 1 \quad \Longrightarrow \quad \Delta f = \frac{k'}{2T}, \quad k' \in \mathbb{Z}$$

Conclusion (SOCS/CACF):

- **If $X(t)$ is general SOCS with $\mu_X(t) \neq 0$:** $\hat{X}(t)$ is SOCS **only if** $\Delta f = k/T$.
- **If $X(t)$ is general SOCS with $\mu_X(t) = 0$:** $\hat{X}(t)$ is SOCS **only if** $\Delta f = k'/2T$.
- **If $X(t)$ is Proper-Complex with $\mu_X(t) = 0$:** $\tilde{R}_{XX}(t,s) = 0$, so $\tilde{R}_{\hat{X}\hat{X}}(t,s) = 0$. The SOCS condition is trivially satisfied. $\hat{X}(t)$ is SOCS **always**.

4. Summary of Cyclostationary Preservation under Frequency Conversion

The preservation of the cyclostationary property depends on precise frequency synchronization.

- **Up-conversion $X(t) \to Y(t)$ (WSCS preservation):**
  - If $X(t)$ is general SOCS and $\mu_X(t) \neq 0$: $Y(t)$ is WSCS **only if** $f_c = k'/T$.
  - If $X(t)$ is general SOCS and $\mu_X(t) = 0$: $Y(t)$ is WSCS **only if** $f_c = m/2T$.
  - If $X(t)$ is Proper-Complex and $\mu_X(t) = 0$: $Y(t)$ is WSCS **always**.

- **Down-conversion** $X(t) \to \hat{X}(t)$ **(SOCS preservation):**
  - **ACF condition (WSCS/SOCS part 2): Always** satisfied regardless of Δf.
  - **Mean condition (WSCS/SOCS part 1):** If $\mu_X(t) \neq 0$, $\Delta f = k/T$.
  - **CACF condition (SOCS additional):** If $\tilde{R}_{XX} \neq 0$ (general SOCS, $\mu_X(t) = 0$), $\Delta f = k'/2T$.
  - **Proper-Complex case:** If $X(t)$ is Proper-Complex and $\mu_X(t) = 0$, $\hat{X}(t)$ is SOCS **always**.