The receiver consists of a linear time-invariant filter with the impulse response $w(-t)^*$ and a sampler with the sampling rate $1/T$ (samples/s). The decision statistic $z[l]$ for the detection of the lth symbol $b[l]$ is defined as

$$z[l] \triangleq w(-t)^* \otimes z(t)\Big|_{t=lT} = \int_{-\infty}^{\infty} w(t-lT)^* z(t)dt$$
$$= \sum_{k=-\infty}^{\infty} b[k]q((l-k)T) + n[l] \tag{4}$$

where the waveform $q(t)$ is defined as

$$q(t) \triangleq w(-t)^* \otimes p(t) = \int_{-\infty}^{\infty} w(\tau)^* p(t+\tau)d\tau \tag{5}$$

and the noise component $n[l]$ is defined as

$$n[l] \triangleq \int_{-\infty}^{\infty} w(t-lT)^* N(t)dt. \tag{6}$$

The noise variance can be written as

$$\mathbb{E}\{|n[0]|^2\} = \mathbb{E}\left\{\left(\int_{-\infty}^{\infty} w(t)^* N(t)dt\right)\left(\int_{-\infty}^{\infty} w(s)^* N(s)ds\right)^*\right\}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(t)^* r_N(t,s) w(s) dt ds. \tag{7}$$

Hence, the condition (3) leads to nonzero noise in the observation after filtering.

The performance metric used in this correspondence is the MSE defined as

$$\varepsilon \triangleq \mathbb{E}\{|b[k]-z[k]|^2\} \tag{8}$$
$$= \mathbb{E}\{|b[k]|^2 - 2\Re(b[k]^* z[k]) + |z[k]|^2\}$$
$$= \mathbb{E}\{|b[k]|^2\} \tag{9a}$$
$$- 2\Re(\mathbb{E}\{b[k]^* z[k]\}) \tag{9b}$$
$$+ \mathbb{E}\{|z[k]-n[k]|^2\} \tag{9c}$$
$$+ \mathbb{E}\{|n[k]|^2\} \tag{9d}$$

where $\Re$ denotes the real part, (9a) is the reference component, (9b) is the cross-correlation component between the reference signal and the output, (9c) is the signal component, and (9d) is the noise component.

We are going to find the optimum transmitter and receiver waveforms $s_{\text{opt}}(t)$ and $w_{\text{opt}}(t)$ that jointly minimize the MSE defined as above. Similar to [1], the MSE can be made arbitrarily small by spending more and more power in transmitting the signal. So, it is reasonable to impose a constraint on the average power of the transmitted signal. Hence, the problem is formally stated as

Proof: By definition, the inverse discrete-time Fourier transform of (1) is given by

$$m[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} M(f) e^{j2\pi fk} df.$$

Since $m[0] = \mathbb{E}\{|b[k]|^2\}$, we obtain

$$\mathbb{E}\{|b[k]|^2\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} M(f) df.$$

Now, a simple change of variable leads to the conclusion. □

It is worth noting that the power spectral density $M(f)$ is periodic with period 1 because it is the Fourier transform of a discrete-time signal $m[k]$. Consequently, $M(fT) = M((f - l/T)T), \forall l \in \mathbb{Z}$. We will use this property in the next two subsections.

B. Cross Term (9b)

Before we derive the expression for the cross term, we first define the excess bandwidth and the vectorized Fourier transform. The excess bandwidth is a parameter that is used to quantify the extra bandwidth, over the Nyquist bandwidth, spent in pulse shaping. It will be seen that the excess bandwidth determines the degrees of freedom in our optimal waveform design. The vectorized Fourier transform is a representation as a vector-valued function of the ordinary Fourier transform that is a scalar-valued function. This alternate expression of the Fourier transform facilitates our optimal system design in the frequency domain.

Definition 4: The excess bandwidth β is defined by the relation

$$BT = \frac{1+\beta}{2} \tag{12}$$

where B and $1/T$ are, respectively, the bandwidth and the transmission rate of the symbol waveform $s(t)$.

Definition 5: The vectorized Fourier transform $\boldsymbol{x}(f)$ with sampling period $1/T$ of the time function $x(t)$ having bandwidth B is defined as

$$\boldsymbol{x}(f) \triangleq \begin{bmatrix} X\left(-\frac{L}{T}+f\right) \\ X\left(-\frac{L-1}{T}+f\right) \\ \vdots \\ X\left(\frac{L}{T}+f\right) \end{bmatrix} \tag{13}$$

$s_{\rm opt}(t)$ and $w_{\rm opt}(t)$ that jointly minimize the MSE defined as above. Similar to [1], the MSE can be made arbitrarily small by spending more and more power in transmitting the signal. So, it is reasonable to impose a constraint on the average power of the transmitted signal. Hence, the problem is formally stated as

$$\begin{aligned}&\underset{s(t),w(t)}{\text{minimize}} \quad \varepsilon \\ &\text{subject to } \mathrm{E}\left\{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|\sum_{k=-\infty}^{\infty} b[k]s(t-kT)\right|^2 dt\right\} = A \quad (10)\end{aligned}$$

where $A > 0$ is the average transmitted power.

This problem is inherently a variational problem, where the MSE is a complicated objective functional of two time functions $s(t)$ and $w(t)$. In the next section, we convert this time-domain expression to a frequency-domain expression term by term, to solve the problem in the frequency domain.

III. Frequency-Domain Representation of MSE

A. Reference Component (9a)

Proposition 1: The reference component (9a) in the MSE can be rewritten as

$$\mathrm{E}\{|b[k]|^2\} = \mathrm{E}\{|b[0]|^2\} = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} TM(fT)df. \quad (11)$$

$b[k]\,\overline{b[k]}$

$$\boldsymbol{x}(f) \triangleq \begin{bmatrix} X\left(-\frac{L-1}{T}+f\right) \\ \vdots \\ X\left(\frac{L}{T}+f\right)\end{bmatrix} \quad (13)$$

for $f \in [-1/(2T), 1/(2T))$ with $L \triangleq \lceil \beta/2 \rceil$, where $X(\xi)$ is the Fourier transform[5] of $x(t)$ and β is given by the relation (12).

As shown in Fig. 2, the vectorized Fourier transform requires the sampling of an ordinary Fourier transform in the frequency domain. Note that the sampling rate is T (samples/Hz) and that the argument f serves as the offset of the uniform samples from the integer multiples of $1/T$. In the vectorization, the excess bandwidth determines the number of samples to be taken. As will be later seen, the dimension $2L+1$ is an upper bound on the degrees of freedom in the waveform design.

Proposition 2: The cross term (9b) in the MSE can be rewritten as

$$-2\Re(\mathrm{E}\{b[k]^* z[k]\}) = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} -2\Re\left(\boldsymbol{w}(f)^H\left(M(fT)\boldsymbol{p}(f)\right)\right)df \quad (14)$$

where $\boldsymbol{w}(f)$ and $\boldsymbol{p}(f)$ are, respectively, the vectorized Fourier transform of the receiver waveform $w(t)$ and that of the channel response $p(t)$ of the transmitter waveform.

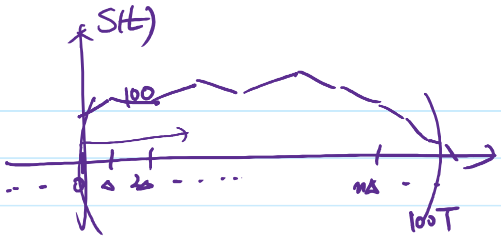
[5] Throughout this correspondence, a capitalized frequency function represents the Fourier transform of the corresponding time function.

$(b[k])_{k\in\mathbb{Z}}$ WSS. $E\{b[k]\}=0.$

$r_{bb}[m,n] \triangleq E\{b[m]\,b^*[n]\}$

$= r_{bb}[n-n]$

$r_{bb}[l] \leftrightarrow M(f) = \sum_{l=-\infty}^{\infty} r_{bb}[l]\cdot e^{-j2\pi fl}$



(i) $r_{bb}[0] = m[0]$

(ii) $= \int_{-\frac{1}{2}}^{\frac{1}{2}} M(f)e^{j2\pi fl}df\Big|_{l=0} = \int_{-\frac{1}{2}}^{\frac{1}{2}} M(f)df = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} T\,M(fT)df.$

□ $|z-a|^2 = (z-a)(z-a)^* = (z-a)(z^*-a^*) = zz^* - za^* - az^* + aa^*$

$= |z|^2 - (za^* + (za^*)^*) + |a|^2$ $\qquad \mathrm{Re}\{b\} = \frac{b+b^*}{2}$, $b = az^*$

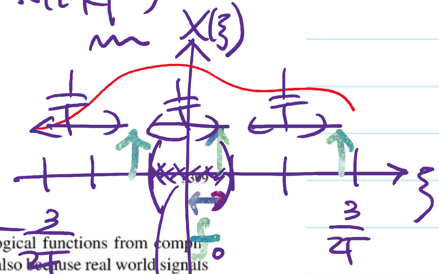
$= |z|^2 - 2\,\mathrm{Re}\{za^*\} + |a|^2$

$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$

$X(\xi) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi \xi t} d\xi$

$\mathbf{x}(f)$

Q. $\sum b[k] s(t-kT)$



$\mathbf{x}(f) = \begin{bmatrix} X(f_0 - \frac{1}{T}) \\ X(f_0) \\ X(f_0 + \frac{1}{T}) \end{bmatrix}$

$|f| \le \frac{1}{2T}$.

Nyquist zone

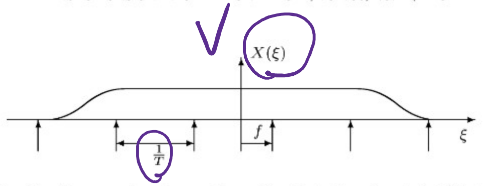


Fig. 2. Frequency-domain sampling with offset f and period $1/T$ to construct the vectorized Fourier transform $\boldsymbol{x}(f)$ of $x(t)$ from the Fourier transform $X(\xi)$.

Proof: Using Parseval's relation and the definition of $M(f)$, we obtain

$(\sum b[k] s(t-kT)) * h(t) + N(t)$

$$\mathbb{E}\{b[0]^* z[0]\} = \mathbb{E}\left\{ b[0]^* \int_{-\infty}^{\infty} w(t)^* \boxed{z(t)} dt \right\} \tag{15a}$$

$$= \sum_{k=-\infty}^{\infty} m[k] \int_{-\infty}^{\infty} w(t)^* p(t-kT) dt \tag{15b}$$

$$= \sum_{k=-\infty}^{\infty} m[k] \int_{-\infty}^{\infty} W(\xi)^* P(\xi) e^{-j2\pi\xi kT} d\xi \tag{15c}$$

$$= \int_{-\infty}^{\infty} W(\xi)^* P(\xi) M(\xi T) d\xi. \tag{15d}$$

$= \sum \int_{-\frac{1}{2T}}^{\frac{1}{2T}} df$

$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df$.

By dividing the integration interval into length-$1/T$ subintervals and properly rearranging terms, we obtain

$$\mathbb{E}\{b[0]^* z[0]\} = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \boldsymbol{w}(f)^H \left(\boxed{M(fT)} \boldsymbol{p}(f) \right) df \tag{16}$$

where we used the periodicity of $M(\xi T)$ with period $1/T$. From (16), the conclusion follows immediately. □

C. Signal Component (9c)

Proposition 3: The signal component (9c) in the MSE can be rewritten as

$$\mathbb{E}\{|z[0] - n[0]|^2\} = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \boldsymbol{w}(f)^H \left(\frac{M(fT)}{T} \boldsymbol{p}(f) \boldsymbol{p}(f)^H \right) \boldsymbol{w}(f) df. \tag{17}$$

$w^H \square w$

rank-one Hermitian symmetric mat

Proof: Note that

$$\mathbb{E}\{|z[0] - n[0]|^2\}$$

$$= \mathbb{E}\left\{ \left| \sum_{l=-\infty}^{\infty} b[l] q(-lT) \right|^2 \right\} \tag{18a}$$

$$= \sum_{k=-\infty}^{\infty} m[-k] \boxed{\left(\sum_{l=-\infty}^{\infty} q(lT) q((l-k)T)^* \right)} \tag{18b}$$

$$= \sum_{k=-\infty}^{\infty} m[-k] \left(\frac{1}{T} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} Q(\xi) Q\left(\xi - \frac{l}{T}\right)^*$$

$\sum x[l] y^*[l]$

not only because it prevents all pathological functions from complicating the derivation of the solution but also because real world signals are accurately modeled as signals having piecewise smooth spectra. The piecewise or sectional smoothness is also a standard assumption in the calculus of variations [14]. We follow this convention and, hence, all frequency functions in this correspondence are assumed piecewise smooth.

D. Noise Component (9d)

An autocorrelation function of a cyclostationary random process is a two-dimensional signal. In order to handle this signal in the frequency domain, we need the Fourier transform and the inverse Fourier transform defined for general two-dimensional signals.

Definition 6: The double Fourier transform of $r(t, s)$ is defined as

$$R(\xi, \tilde{\xi}) \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(t, s) e^{-j2\pi(\xi t - \tilde{\xi} s)} dt ds$$

and the inverse transform as

$$r(t, s) \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(\xi, \tilde{\xi}) e^{+j2\pi(\xi t - \tilde{\xi} s)} d\xi d\tilde{\xi}.$$

Notice that, unlike the conventional definition of the two-dimensional Fourier transform, we correlate $r(t, s)$ with $e^{-j2\pi(\xi t - \tilde{\xi} s)}$ instead of $e^{-j2\pi(\xi t + \tilde{\xi} s)}$ to obtain $R(\xi, \tilde{\xi})$.

By combining this definition and the definition of Fourier transform pair for one-dimensional signals, we obtain the following lemma, which is a simple extension of Parseval's relation.

Lemma 1:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(t)^* r(t, s) v(s) dt ds \tag{19a}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(\xi)^* R(\xi, \tilde{\xi}) V(\tilde{\xi}) d\xi d\tilde{\xi}. \tag{19b}$$

Proof: Substituting the inverse Fourier transforms of $u(t)^*$ and $v(s)$ into (19a) and then using the definition of double Fourier transform, we obtain (19b). □

$\int\int_{-\infty}^{\infty} w(t) r_{NN}(t,s) w^*(s) dt ds$

By using Lemma 1, we can rewrite the noise component (7) as

$$\mathbb{E}\{|n[0]|^2\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\xi)^* R_N(\xi, \tilde{\xi}) W(\tilde{\xi}) d\xi d\tilde{\xi} \tag{20}$$

which we need to convert to a single integration with a finite integration interval, in order to conform to the frequency-domain expressions for other components in the MSE.

We start by observing the following lemma from [15].

Lemma 2: [15, p. 242] If $r(t, s)$ satisfies

$$\boxed{r(t, s) = r(t + kT, s + kT)}, \quad \forall k \in \mathbb{Z}, \tag{21a}$$

□ Def.

$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

□ Def.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(t,s) e^{-j2\pi f_1 t} e^{-j2\pi f_2 s} dt ds$

□ Def Simplectic

$= E\{N(t) N^*(s)\}$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(t,s) e^{-j2\pi f_1 t} e^{+j2\pi f_2 s} dt ds$

$\sum x[l] y^*[l]$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} X(e^{j2\pi\xi}) Y^*(e^{j2\pi\xi}) d\xi$

$= \frac{1}{T}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(e^{j2\pi\xi T}) Y^*(e^{j2\pi\xi T}) d\xi$

$$= \sum_{k=-\infty}^{\infty} m[-k] \boxed{\left(\sum_{l=-\infty}^{\infty} q(lT) q((l-k)T)^* \right)} \qquad (18b)$$

$$= \sum_{k=-\infty}^{\infty} m[-k] \left(\frac{1}{T}\sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} Q(\xi) Q\left(\xi - \frac{l}{T}\right)^* \cdot e^{j2\pi(\xi - \frac{l}{T})kT} d\xi \right) \qquad (18c)$$

$f(t) = p(t) * p^*(-t)$

$$= \boxed{\frac{1}{T}\sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\xi)Q\left(\xi-\frac{l}{T}\right)^* M(\xi T) d\xi} \qquad (18d)$$

where the Poisson summation formula [21, p. 577] leads to (18c) with $Q(\xi)$ being the Fourier transform of $q(t)$ defined in (5). After dividing the integration interval into length-$1/T$ subintervals and properly rearranging the terms, we obtain (17). □

A sufficient condition on $q(t)$ for the applicability of the Poisson summation formula is the piecewise smoothness of $Q(\xi)$.[6] Since the piecewise smoothness of $W(\xi)$, $H(\xi)$, and $S(\xi)$ leads to the piecewise smoothness of $Q(\xi)$, we assume as such. This assumption is reasonable

[6]For details, see [13, Proposition 1 and the related footnote].

We start by observing the following lemma from [15].

Lemma 2: [15, p. 242] If $r(t,s)$ satisfies

$$\boxed{r(t,s) = r(t+kT, s+kT)}, \quad \forall k \in \mathbb{Z}, \qquad (21a)$$

there exists $r^{(k)}(\tau)$, $k \in \mathbb{Z}$ such that

$$r(t,s) = \sum_{k=-\infty}^{\infty} r^{(k)}(t-s) e^{j2\pi\frac{kt}{T}}. \qquad (21b)$$

Proof: Let $\tilde{r}(t,\tau) = r(t, t-\tau)$, then $\tilde{r}(t,\tau)$ is periodic in t with period T. If we expand $\tilde{r}(t,\tau)$ using the Fourier series, we obtain (21b). □

$r(t+\tau, t)$

Remark 1: If $r(t,s)$ satisfies $r(t,s) = r(t-s, 0)$, every $r^{(k)}(t-s)$ vanishes except $r^{(0)}(t-s) = r(t-s)$ because

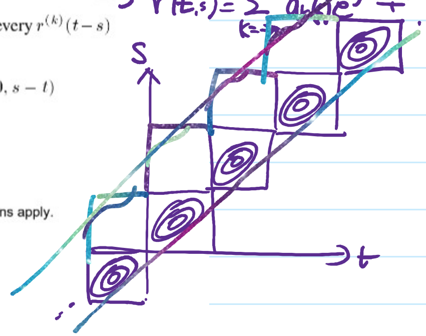
$$r(t-s,0) = \sum_{k=-\infty}^{\infty} r^{(k)}(t-s)e^{j2\pi\frac{kt}{T}} = r(0, s-t)$$
$$= \sum_{k=-\infty}^{\infty} r^{(k)}(t-s)$$

for all t and s.

cf) $r(t) = r(t+kT)$, $\forall t$, $\forall k \in \mathbb{Z}$

$= \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi k t}{T}}$

$\rightarrow r(t,s) = \sum_{k=-\infty}^{\infty} a_k(s) e^{j\frac{2\pi k t}{T}}$



□ Lemma. Given a CT signal $g(t) \xleftrightarrow{\text{CTFT}} Q(\xi)$

$x[l] \triangleq g(lT)$, $\forall l \in \mathbb{Z}$

$X(e^{j2\pi\xi T}) = \boxed{\frac{1}{T}\sum_{l=-\infty}^{\infty} Q\left(\xi - \frac{l}{T}\right)}$

□ $E\{|n[0]|^2\} = E\left\{\left|\int_{-\infty}^{\infty} N(t) w^*(t) dt\right|^2\right\} = E\left\{\left(\int_{-\infty}^{\infty} w^*(t) N(t) dt\right)\left(\int_{-\infty}^{\infty} w^*(s) N(s) ds\right)^*\right\}$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^*(t) R_{NN}(t,s) w(s) dt ds$ cf) $w^H R w$

$= \cdots$

$= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left(w^H(f) R_N(f) w(f) \right) df$

matrix-valued PSD of $N(t)$

cf) $\int_{-\infty}^{\infty} x(t)\delta(t-t_0) dt = x(t_0)$

$e^{j2\pi f_0 t} \xleftrightarrow{\text{FT}} \delta(f - f_0)$

Note that the double Fourier transform of $r(t,s)$ satisfying (21b) is given as

$$r(t,s) \xleftrightarrow{\text{FT}} R(\xi, \tilde{\xi}) = \sum_{k=-\infty}^{\infty} R^{(k)}\left(\xi - \frac{k}{T}\right)\delta\left(\xi - \tilde{\xi} - \frac{k}{T}\right) \qquad (22)$$

where the matrix-valued power spectral density $\boldsymbol{R}_N(f)$ of the cyclostationary noise $N(t)$ is defined as

$$[\boldsymbol{R}_N(f)]_{k,l} \triangleq \begin{cases} R_N^{(k-l)}\left(f + \frac{k-L-1}{T}\right), & \text{for } |k-l| \le 2L \\ 0, & \text{elsewhere} \end{cases} \qquad (28)$$

Note that the double Fourier transform of $r(t,s)$ satisfying (21b) is given as

$$r(t,s) \xrightarrow{\text{FT}} R(\xi,\tilde{\xi}) = \sum_{k=-\infty}^{\infty} R^{(k)}\left(\xi - \frac{k}{T}\right)\delta\left(\xi - \tilde{\xi} - \frac{k}{T}\right) \tag{22}$$

where $R^{(k)}(\xi)$ is the Fourier transform of $r^{(k)}(\tau)$. In this case, the double Fourier transform of $r(t,s)$ satisfying (21) consists of impulse fences on lines parallel to the $\xi = \tilde{\xi}$ diagonal and separated by $1/T$ [11]. Now, by applying this result to the double integration (20), we obtain an intermediate result as

$$\mathbb{E}\{|n[0]|^2\} = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} W(\xi)^* R_N^{(k)}\left(\xi - \frac{k}{T}\right) W\left(\xi - \frac{k}{T}\right) d\xi \tag{23}$$

where $R_N^{(k)}(\xi)$, $k \in \mathbb{Z}$, are the Fourier transforms of $r_N^{(k)}(\tau)$, $k \in \mathbb{Z}$, such that

$$r_N(t,s) = \sum_{k=-\infty}^{\infty} r_N^{(k)}(t-s)e^{j2\pi\frac{ks}{T}}. \tag{24}$$

Remark 2: If $N(t)$ is wide-sense stationary, i.e., $r_N(t,s) = r_N(t-s,0)$, then (23) becomes $\int_{-\infty}^{\infty} R_N^{(0)}(\xi)|W(\xi)|^2 d\xi$, which is a well-known result from the filtering theory of random processes. In other words, the double Fourier transform of the two-dimensional autocorrelation function consists of a single impulse fence on the line $\xi = \tilde{\xi}$, and the power spectral density (PSD) corresponds to the magnitude of the fence.

The expression (23) can be further simplified when the WSCS process $N(t)$ is band-limited, which is our case. Recall Definition 1 of a band-limited process. This definition implies that the autocorrelation function $r_N(t,s)$ is the same as the autocorrelation of the ideal low-pass filtered version of $N(t)$. Hence, using Lemma 1, we obtain

$$\begin{aligned} \mathbb{E}\{N(t)N(s)^*\} &= \mathbb{E}\{L_B[N(t)]L_B[N(s)]^*\} \Longrightarrow \\ r_N(t,s) &= \int_{-B}^{B}\int_{-B}^{B} R_N(\xi,\tilde{\xi})e^{+j2\pi(\xi t - \tilde{\xi}s)} d\xi d\tilde{\xi} \end{aligned} \tag{25}$$

where $L_B[x(t)]$ denotes the response of the ideal low-pass filter with bandwidth B to the input $x(t)$. From the definition of the inverse Fourier transform of $R_N(\xi,\tilde{\xi})$ combined with (25), we find that $R_N(\xi,\tilde{\xi}) = 0$ outside the square area defined as

$$\{(\xi,\tilde{\xi}) \,|\, |\xi| \le B, |\tilde{\xi}| \le B)\}.$$

Hence, the infinite sum (22) has at most $4L+1$ nonzero terms as

$$R_N(\xi,\tilde{\xi}) = \sum_{k=-2L}^{2L} R_N^{(k)}\left(\xi - \frac{k}{T}\right)\delta\left(\xi - \tilde{\xi} - \frac{k}{T}\right) \tag{26}$$

where $R_N^{(k)}(\xi)$ has an additional restriction that it can be nonzero only on a finite interval

$$\xi \in \left[\max\left(-B, -B - \frac{k}{T}\right), \min\left(B, B - \frac{k}{T}\right)\right]$$

for $k = -2L, -2L+1, \ldots, 2L$, i.e., $R_N^{(k)}(\xi - \frac{k}{T}) = 0$ outside the square area.

Proposition 4: The noise component in the MSE can be rewritten as

2D PSD of N(t)

$$\boxed{\mathbb{E}\{|n[0]|^2\} = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \boldsymbol{w}(f)^H \boldsymbol{R}_N(f)\boldsymbol{w}(f)df} \tag{27}$$

where the matrix-valued power spectral density $\boldsymbol{R}_N(f)$ of the cyclostationary noise $N(t)$ is defined as

$$[\boldsymbol{R}_N(f)]_{k,l} \triangleq \begin{cases} R_N^{(k-l)}\left(f + \frac{k-L-1}{T}\right), & \text{for } |k-l| \le 2L \\ 0, & \text{elsewhere} \end{cases} \tag{28}$$

with $k, l = 1, 2, \ldots, 2L+1$.

Proof: Using (26), we can rewrite (23) as

$$\mathbb{E}\{|n[0]|^2\} = \int_{-B}^{B} \sum_{k=-2L}^{2L} W(\xi)^* R_N^{(k)}(\xi - \frac{k}{T})W(\xi - \frac{k}{T})d\xi.$$

After dividing the integration interval into length-$1/T$ subintervals and properly rearranging terms, we obtain (27). □

E. MSE

Now, we present the final result of this section.

Theorem 1: The MSE (8) can be expressed as

$$\begin{aligned} \varepsilon = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} & TM(fT) + \boldsymbol{w}(f)^H\boldsymbol{R}(f)\boldsymbol{w}(f) \\ & -2\Re\Big(\boldsymbol{w}(f)^H(M(fT)\boldsymbol{p}(f))\Big)df \end{aligned} \tag{29}$$

where

$$\boldsymbol{R}(f) \triangleq \boldsymbol{R}_N(f) + \frac{1}{T}M(fT)\boldsymbol{p}(f)\boldsymbol{p}(f)^H. \tag{30}$$

Proof: By combining Propositions 1 to 4, we obtain (29). □

Since $p(t) = h(t) \otimes s(t)$, we can replace $\boldsymbol{p}(f)$ in (30) with $\boldsymbol{H}(f)\boldsymbol{s}(f)$, where $\boldsymbol{s}(f)$ is the vectorized Fourier transform of $s(t)$ and $\boldsymbol{H}(f)$ is a $(2L+1)$-dimensional, diagonal, matrix-valued frequency function defined as

$$\boldsymbol{H}(f) \triangleq \operatorname{diag}\left\{H\left(-\frac{L}{T}+f\right), \ldots, H\left(\frac{L}{T}+f\right)\right\} \tag{31}$$

for $f \in [-1/(2T), 1/(2T))$.

IV. Optimization of Waveforms

In this section, we find the jointly optimum transmitter and receiver waveforms using the objective functional (29) derived in the previous section. In order to conform to the frequency-domain expression of the objective functional, we rewrite the average power constraint in (10) as

$$\begin{aligned} A &= \mathbb{E}\left\{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left|\sum_{k=-\infty}^{\infty} b[k]s(t-kT)\right|^2 dt\right\} \\ &= \frac{1}{T}\int_{-\infty}^{\infty} M(\xi T)|S(\xi)|^2 d\xi \\ &= \frac{1}{T}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} M(fT)\boldsymbol{s}(f)^H\boldsymbol{s}(f)df. \end{aligned} \tag{32}$$

Our approach is to divide the problem into a cascade of individual receiver and transmitter optimizations as

$$\begin{aligned} &\underset{\boldsymbol{s}(f)}{\text{minimize}} \quad \underset{\boldsymbol{w}(f)}{\text{minimize}}\ \varepsilon \\ &\text{subject to} \quad \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{M(fT)}{T}a(f)df = A \end{aligned} \tag{33}$$

where the energy density $a(f)$ of $\boldsymbol{s}(f)$ is defined as

$$a(f) \triangleq \boldsymbol{s}(f)^H\boldsymbol{s}(f) \tag{34}$$

for $f \in [-1/(2T), 1/(2T))$.

$$\boxed{E\{|n[0]|^2\} = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} w(f)^H R_N(f) w(f) df} \quad (27)$$

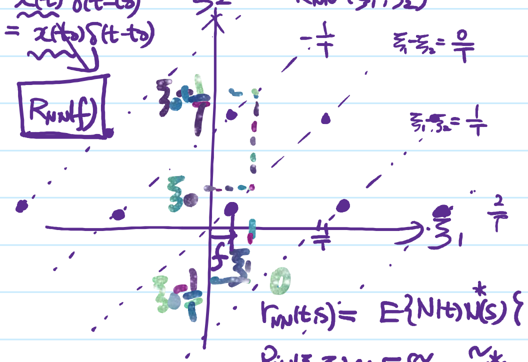
$$a(f) \doteq s(f)^H s(f) \quad (34)$$

for $f \in [-1/(2T), 1/(2T))$.

$r_{NN}(t,s) \overset{\mathcal{F}}{\longleftrightarrow} R_{NN}(\xi_1,\xi_2)$

$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

$\boxed{R_{NN}(f)}$



$r_{NN}(t,s) = E\{N(t)N^*(s)\}$

$R_{NN}(\xi_1,\xi_2) \simeq E\{\tilde{N}(\xi_1)\tilde{N}^*(\xi_2)\}$

$\tilde{N}(\xi)$

$\tilde{N}(\xi) \overset{?}{=} \int_{-\infty}^{\infty} N(t) e^{-j2\pi\xi t} dt$