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$\square$ $E\{|n[0]|^2\} = E\left\{\left|\int_{-\infty}^{\infty} N(t)\, w^*(t)\, dt\right|^2\right\} = E\left\{\left(\int_{-\infty}^{\infty} w^*(t) N(t) dt\right)\left(\int_{-\infty}^{\infty} w^*(s) N(s)\, ds\right)^*\right\}$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^*(t)\, R_{NN}(t,s)\, w(s)\, dt\, dt$ (cf) $\underline{w}^H R\, \underline{w}$

$= \cdots$

$= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left(\underline{w}^H(f)\, \underbrace{R_{NN}(f)}_{\text{matrix-valued PSD of } N(t)}\, \underline{w}(f)\right) df$ (cf) $\int_{-\infty}^{\infty} x(t)\, \delta(t-t_0)\, dt = x(t_0)$

$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f-f_0)$

Note that the double Fourier transform of $r(t,s)$ satisfying (21b) is given as

$$r(t,s) \overset{\mathcal{F}}{\longleftrightarrow} R(\xi,\tilde{\xi}) = \sum_{k=-\infty}^{\infty} R^{(k)}\left(\xi - \frac{k}{T}\right)\delta\left(\xi - \tilde{\xi} - \frac{k}{T}\right) \quad (22)$$

where $R^{(k)}(\xi)$ is the Fourier transform of $r^{(k)}(\tau)$. In this case, the double Fourier transform of $r(t,s)$ satisfying (21) consists of impulse fences on lines parallel to the $\xi = \tilde{\xi}$ diagonal and separated by $1/T$ [11]. Now, by applying this result to the double integration (20), we obtain an intermediate result as

$$\mathbb{E}\{|n[0]|^2\} = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} W(\xi)^* R_N^{(k)}\left(\xi - \frac{k}{T}\right) W\left(\xi - \frac{k}{T}\right) d\xi \quad (23)$$

where $R_N^{(k)}(\xi)$, $k \in \mathbb{Z}$, are the Fourier transforms of $r_N^{(k)}(\tau)$, $k \in \mathbb{Z}$, such that

$$r_N(t,s) = \sum_{k=-\infty}^{\infty} r_N^{(k)}(t-s)e^{j2\pi\frac{kt}{T}}. \quad (24)$$

Remark 2: If $N(t)$ is wide-sense stationary, i.e., $r_N(t,s) = r_N(t-s,0)$, then (23) becomes $\int_{-\infty}^{\infty} R_N^{(0)}(\xi)|W(\xi)|^2 d\xi$, which is a well-known result from the filtering theory of random processes. In other words, the double Fourier transform of the two-dimensional autocorrelation function consists of a single impulse fence on the line $\xi = \tilde{\xi}$, and the power spectral density (PSD) corresponds to the magnitude of the fence.

The expression (23) can be further simplified when the WSCS process $N(t)$ is band-limited, which is our case. Recall Definition 1 of a band-limited process. This definition implies that the autocorrelation function $r_N(t,s)$ is the same as the autocorrelation of the ideal low-pass filtered version of $N(t)$. Hence, using Lemma 1, we obtain

$$\mathbb{E}\{N(t)N(s)^*\} = \mathbb{E}\{L_B[N(t)]L_B[N(s)]^*\} \Longrightarrow$$
$$r_N(t,s) = \int_{-B}^{B}\int_{-B}^{B} R_N(\xi,\tilde{\xi})e^{+j2\pi(\xi t - \tilde{\xi}s)}d\xi d\tilde{\xi} \quad (25)$$

where $L_B[x(t)]$ denotes the response of the ideal low-pass filter with bandwidth B to the input $x(t)$. From the definition of the inverse Fourier transform of $R_N(\xi,\tilde{\xi})$ combined with (25), we find that $R_N(\xi,\tilde{\xi}) = 0$ outside the square area defined as

$$\{(\xi,\tilde{\xi}) \,|\, |\xi| \le B, |\tilde{\xi}| \le B)\}.$$

Hence, the infinite sum (22) has at most $4L+1$ nonzero terms as

where the matrix-valued power spectral density $\boldsymbol{R}_N(f)$ of the cyclostationary noise $N(t)$ is defined as

$$[\boldsymbol{R}_N(f)]_{k,l} \triangleq \begin{cases} R_N^{(k-l)}\left(f + \frac{k-L-1}{T}\right), & \text{for } |k-l| \le 2L \\ 0, & \text{elsewhere} \end{cases} \quad (28)$$

with $k, l = 1, 2, \ldots, 2L+1$.

Proof: Using (26), we can rewrite (23) as

$$\mathbb{E}\{|n[0]|^2\} = \int_{-B}^{B} \sum_{k=-2L}^{2L} W(\xi)^* R_N^{(k)}(\xi - \frac{k}{T})W(\xi - \frac{k}{T})d\xi.$$

After dividing the integration interval into length-$1/T$ subintervals and properly rearranging terms, we obtain (27). $\square$

E. MSE

Now, we present the final result of this section. $\triangleq E\{|b[k] - z[k]|^2\}$

Theorem 1: The MSE (8) can be expressed as

$$\varepsilon = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} TM(fT) + \boldsymbol{w}(f)^H\boldsymbol{R}(f)\boldsymbol{w}(f) - 2\Re\left(\boldsymbol{w}(f)^H(M(fT)\boldsymbol{p}(f))\right)df \quad (29)$$

where

$$\boldsymbol{R}(f) \triangleq \boldsymbol{R}_N(f) + \frac{1}{T}M(fT)\boldsymbol{p}(f)\boldsymbol{p}(f)^H. \quad (30)$$

Proof: By combining Propositions 1 to 4, we obtain (29). $\square$

Since $p(t) = h(t) \otimes s(t)$, we can replace $\boldsymbol{p}(f)$ in (30) with $\boldsymbol{H}(f)\boldsymbol{s}(f)$, where $\boldsymbol{s}(f)$ is the vectorized Fourier transform of $s(t)$ and $\boldsymbol{H}(f)$ is a $(2L+1)$-dimensional, diagonal, matrix-valued frequency function defined as

$$\boldsymbol{H}(f) \triangleq \mathrm{diag}\left\{H\left(-\frac{L}{T}+f\right), \ldots, H\left(\frac{L}{T}+f\right)\right\} \quad (31)$$

for $f \in [-1/(2T), 1/(2T))$.

IV. Optimization of Waveforms

In this section, we find the jointly optimum transmitter and receiver waveforms using the objective functional (29) derived in the previous section. In order to conform to the frequency-domain expression of the objective functional, we rewrite the average power constraint in (10) as

$$A = \mathbb{E}\left\{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|\sum_{}^{\infty} b[k]s(t-kT)\right|^2 dt\right\}$$

(cf) Given $\underline{y} = b\underline{p} + \underline{N}$

$E\{b\} = 0$, $E\{|b|^2\} = M$.

$E\{\underline{N}\} = \underline{0}$, $E\{\underline{N}\,\underline{N}^H\} = R_N$,

find $\underline{w}$ that minimizes the MSE

$\varepsilon \triangleq E\{|b - \underline{w}^H\underline{y}|^2\}$

$= E\{|b|^2 - 2\mathrm{Re}\{b^*\underline{w}^H\underline{y}\}$

$+ |\underline{w}^H\underline{y}|^2\}$

$\frac{d^m}{dx^m}x^n = n(n-1)\cdots$

$\frac{d^m}{dz^m}z^n = n(n-1)\cdots$

$\frac{d}{dz}\ln(1+z) = \frac{1}{1+z}$

$\frac{d}{dx}\ln(1+x) = \frac{1}{\ldots}$

Fourier transform of $R_N(\xi,\tilde{\xi})$ combined with (25), we find that $R_N(\xi,\tilde{\xi}) = 0$ outside the square area defined as

$$\{(\xi,\tilde{\xi}) \| |\xi| \le B, |\tilde{\xi}| \le B)\}.$$

Hence, the infinite sum (22) has at most $4L+1$ nonzero terms as

$$R_N(\xi,\tilde{\xi}) = \sum_{k=-2L}^{2L} R_N^{(k)}\left(\xi - \frac{k}{T}\right)\delta\left(\xi - \tilde{\xi} - \frac{k}{T}\right) \quad (26)$$

where $R_N^{(k)}(\xi)$ has an additional restriction that it can be nonzero only on a finite interval

$$\xi \in \left[\max\left(-B, -B - \frac{k}{T}\right), \min\left(B, B - \frac{k}{T}\right)\right]$$

for $k = -2L, -2L+1, \ldots, 2L$, i.e., $R_N^{(k)}(\xi - \frac{k}{T}) = 0$ outside the square area.

Proposition 4: The noise component in the MSE can be rewritten as

2D PSD of N(t)

$$\boxed{\mathrm{E}\{|n[0]|^2\} = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \boldsymbol{w}(f)^H \boldsymbol{R}_N(f)\boldsymbol{w}(f)df} \quad (27)$$

section. In order to conform to the frequency-domain expression of the objective functional, we rewrite the average power constraint in (10) as

$$A = \mathbb{E}\left\{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|\sum_{k=-\infty}^{\infty} b[k]s(t-kT)\right|^2 dt\right\}$$
$$= \frac{1}{T}\int_{-\infty}^{\infty} M(\xi T)|S(\xi)|^2 d\xi$$
$$= \frac{1}{T}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} M(fT)\boldsymbol{s}(f)^H\boldsymbol{s}(f)df. \quad (32)$$

Our approach is to divide the problem into a cascade of individual receiver and transmitter optimizations as

$$\boxed{\begin{aligned} &\underset{\boldsymbol{s}(f)}{\text{minimize}}\ \boxed{\underset{\boldsymbol{w}(f)}{\text{minimize}}\ \varepsilon} \\ &\text{subject to} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{M(fT)}{T}a(f)df = A \end{aligned}} \quad (33)$$

← unconstrained!

where the energy density $a(f)$ of $\boldsymbol{s}(f)$ is defined as

$$a(f) \triangleq \boldsymbol{s}(f)^H\boldsymbol{s}(f) \quad (34)$$

for $f \in [-1/(2T), 1/(2T))$.

$= E\{|b|^2 - 2\mathrm{Re}\{b^* \underline{w}^H \underline{y}\} + |\underline{w}^H \underline{y}|^2\}$

(28) $\varepsilon = M - 2\mathrm{Re}\{\underline{w}^H M \underline{p}\} + \underline{w}^H(M\underline{p}\underline{p}^H + R_N)\underline{w}$

min (w.r.t. s(f))

s.t. (32)

$\frac{\partial \varepsilon}{\partial \underline{w}^*} = 2(M\underline{p}\underline{p}^H + R_N)\underline{w} - 2M\underline{p}.$

The Wirtinger Calculus

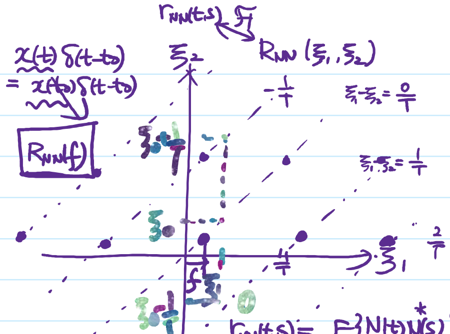
$\frac{d}{dz}\ln(1+z) = \frac{1}{1+z}$

$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$

$r_{NN}(t,s) \xleftrightarrow{\mathcal{F}} R_{NN}(\xi_1,\xi_2)$

$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

$\boxed{R_{NN}(f)}$

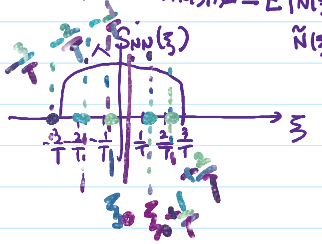


$r_{NN}(t,s) = E\{N(t)N^*(s)\}$

$R_{NN}(\xi_1,\xi_2) \simeq E\{\tilde{N}(\xi_1)\tilde{N}^*(\xi_2)\}$

$\tilde{N}(\xi)$

$\tilde{N}(\xi) \overset{?}{=} \int_{-\infty}^{\infty} N(t)e^{-j2\pi\xi t}dt$

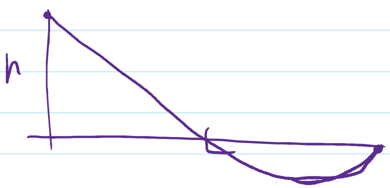


uniqueness

$f: \mathbb{R} \to \mathbb{R} \qquad f(x) \simeq f(x_0) + a(x - x_0)$

$f: \mathbb{C} \to \mathbb{C} \qquad f(z) \simeq f(z_0) + a(z - z_0)$

Def. f is Analytic if ...



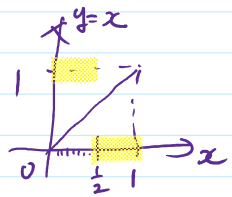
Given $f: [0,1] \to [0,1]$

find f that maximizes

$\int_0^1 |f(x) - x|\,dx$

$= \sum_{k=0}^{N-1}\int_{\frac{k}{N}}^{\frac{k+1}{N}} |f(x) - x|\,dx$

$\simeq \frac{1}{N}\sum_{k=0}^{N-1} |f(x_k) - x_k|$

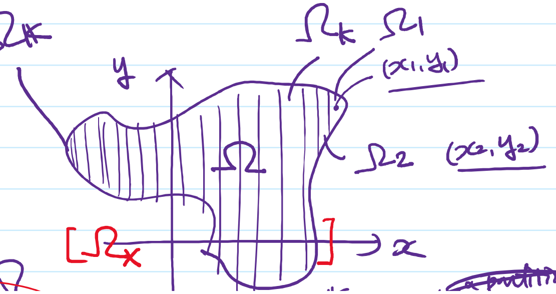


$f(x) = \begin{cases} 1, & [0, \frac{1}{2}) \\ 0, & [\frac{1}{2}, 1] \end{cases}$

□ $f: \mathbb{R}^2 \to \mathbb{R}$

$\min_{x,y} \ f(x,y)$

s.t. $(x,y) \in \Omega$



$\min_{x,y} f(x,y)$
s.t. $(x,y) \in \Omega$

$[\Omega_x] \to x$

$\Omega = \bigcup_{k=1}^{K} \Omega_k$ ← partition

$\min_x \; \min_y f(x,y)$
s.t. $y \in \Omega_x$
s.t. $x \in \Omega_x$

inner opt
out opt.

$\Omega_k \cap \Omega_{k'} = \phi, \; \forall k \neq k'$

$\{\Omega_k\}_{k=1}^{K}$

$\{0, (\tfrac{1}{2}, 1]$

$0 \quad \tfrac{1}{2} \quad 1 \to x$

$\min_y \; \min_x f(x,y)$
s.t. $x \in \Omega_y$
s.t. $y \in \Omega_Y$

A. Step 1: Receiver Optimization

In this subsection, we find the vectorized Fourier transform $\boldsymbol{w}_{\text{lmmse}}(f)$ of the optimum receiver waveform given an arbitrary transmitter waveform $s(t)$, and derive the receiver-only minimized MSE $\varepsilon_{\text{lmmse}}$ as a functional of $\boldsymbol{s}(f)$.

Although, apparently, the vectorized Fourier transforms $\boldsymbol{s}(f)$, $\boldsymbol{p}(f)$, and $\boldsymbol{w}(f)$ are defined as $(2L+1)$-dimensional vectors for $f \in [-1/(2T), 1/(2T))$, some of the entries must take zero values on some intervals of f because the ordinary Fourier transforms $S(\xi)$, $P(\xi)$, and $W(\xi)$ can be nonzero only on $|\xi| \le B$. Since $S(\xi) = 0$ for $\xi < -B = -(1+\beta)/(2T)$

$$S\left(-\frac{L}{T} + f\right) = 0, \qquad \text{for } -\frac{1}{2T} \le f < -\frac{1+\beta}{2T} + \frac{L}{T}. \tag{35}$$

In addition, since $S(\xi) = 0$ for $\xi > B = (1+\beta)/(2T)$

$$S\left(\frac{L}{T} + f\right) = 0, \qquad \text{for } \frac{1+\beta}{2T} - \frac{L}{T} < f < \frac{1}{2T}. \tag{36}$$

Note that $S(-L/T+f)$ and $S(L/T+f)$ correspond to the first and the last entries of $\boldsymbol{s}(f)$, respectively. This also applies to $\boldsymbol{p}(f)$ and $\boldsymbol{w}(f)$. Similarly, the matrix-valued functions $\boldsymbol{H}(f)$, $\boldsymbol{R}_N(f)$, and $\boldsymbol{R}(f)$ have all zeros in the first and/or last rows and columns, depending on β and f.

This observation says that, when we design transmitter and receiver waveforms in the frequency domain, the degrees of freedom are not always the same as the dimension $2L+1$. It can be easily verified that they are either $2L+1$ or $2L$ for even numbers of $\lceil \beta \rceil$, while either $2L$ or $2L-1$ for odd numbers of $\lceil \beta \rceil$, depending on the value of f. However, for integer values of β, the degrees of freedom do not vary and are the same as $1+\beta$ for all $f \in [-1/(2T), 1/(2T))$.

Definition 7: The effective vectorized Fourier transform of $s(t)$ is defined as a variable-dimensional function of f that is obtained after removing the first and/or last entries of $\boldsymbol{s}(f)$ according to the conditions described by (35) and (36). The effective matrix-valued PSD of $N(t)$ is also similarly defined.

For simplicity, we do not introduce new notations for effective vectorized Fourier transforms, effective matrix-valued PSDs, etc. However, all the vector-valued and matrix-valued functions in the remainder of this correspondence are effective ones, unless otherwise specified.

Lemma 3: If a second-order random process $N(t)$ is WSCS with period T, band-limited with bandwidth B, and positive definite, then its matrix-valued PSD is also positive definite for all $f \in [-1/(2T), 1/(2T))$, i.e., $\boldsymbol{R}_N(f) > 0 \; \forall f$.

for $f \in [-\frac{1}{2T}, \frac{1}{2T})$, and the MSE $\varepsilon_{\text{lmmse}}$ evaluated with $\boldsymbol{w}_{\text{lmmse}}(f)$ is given by

$$\boxed{\varepsilon_{\text{lmmse}}} = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} TM(fT) - |M(fT)|^2 \boldsymbol{p}(f)^H \boldsymbol{R}(f)^{-1} \boldsymbol{p}(f)\, df \tag{38a}$$

← $\underline{w}_{opt}(f)$

$$\boxed{= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{TM(fT)}{1 + \frac{M(fT)}{T} \boldsymbol{s}(f)^H \boldsymbol{C}(f) \boldsymbol{s}(f)} df} \tag{38b}$$

where

$$\boldsymbol{C}(f) \triangleq \boldsymbol{H}(f)^H \boldsymbol{R}_N(f)^{-1} \underline{\underline{\boldsymbol{H}(f)}}. \tag{39}$$

$= \text{diag}\{\underline{h}(f)\}$

Proof: The first term in the integrand of (29) can be ignored because it is not dependent upon $\boldsymbol{w}(f)$. Hence, we just need to find $\boldsymbol{w}_{\text{lmmse}}(f)$ that minimizes

$$\boldsymbol{w}(f)^H \boldsymbol{R}(f) \boldsymbol{w}(f) - 2\Re\left(\boldsymbol{w}(f)^H (M(fT)\boldsymbol{p}(f))\right)$$

for all $f \in [-1/(2T), 1/(2T))$. This is an unconstrained quadratic optimization problem and the first-order necessary condition yields $\boldsymbol{R}(f)\boldsymbol{w}_{\text{lmmse}}(f) = M(fT)\boldsymbol{p}(f)$ for all $f \in [-1/(2T), 1/(2T))$. Note that we do not need linear/conjugate linear (LCL) filtering [16] to minimize the MSE because the noise is proper. In Lemma 3, it is shown that the Hermitian matrix $\boldsymbol{R}_N(f)$ is positive definite. Consequently, the matrices $\boldsymbol{R}(f)$ and $\boldsymbol{R}_N(f)$ are invertible for all f. Since $\boldsymbol{R}(f)$ is invertible, we obtain the unique solution (37a) and the MSE (38a). By using the matrix inversion lemma, we obtain

$$\boldsymbol{R}^{-1}(f)\boldsymbol{p}(f) = \frac{\boldsymbol{R}_N(f)^{-1}\boldsymbol{p}(f)}{1 + \frac{M(fT)}{T}\boldsymbol{p}(f)^H \boldsymbol{R}_N(f)^{-1}\boldsymbol{p}(f)} \tag{40}$$

which leads to (37b) and (38b). □

Note that this is the cyclic Wiener filter [17] that minimizes the MSE of the data symbol instead of the MSE of the desired signal.

B. Step 2: Weighted Water-Filling and Transmitter Optimization

Thanks to the receiver-only optimization performed in the previous subsection, the joint optimization problem now reduces to

$$\begin{aligned} &\underset{\boldsymbol{s}(f)}{\text{minimize}} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{TM(fT)}{1 + \frac{M(fT)}{T}\boldsymbol{s}(f)^H \boldsymbol{C}(f)\boldsymbol{s}(f)} df \\ &\text{subject to} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{M(fT)}{T} a(f)\, df = [illegible] \end{aligned} \tag{41}$$

(SRRC)
$|S(\xi)|$
$\frac{1}{2T} \quad \frac{1-\alpha}{2T} \quad \frac{1}{2T} \quad \frac{1+\alpha}{2T}$ ξ
$0 \le \alpha \le 1.$

□ $\min_x \; \max_y f(x,y)$

□ Thm. Under ....,

$\iint_{(x,y)\in\Omega} f(x,y)\, dx\, dy$

$= \int \left(\int f(x,y)\, dx \right) dy$

$((G))$ $\iint \cdots \int_{x \in \mathbb{R}^N} f(x)\, dx$

$\#$ $M=1$. $\underline{w}, \underline{p}, R$

$\varepsilon = 1 - 2\underline{w}^T \underline{p} + \underline{w}^T (\underline{p}\underline{p}^T + R_N) \underline{w}$

$\left.\frac{\partial \varepsilon}{\partial \underline{w}}\right|_{\underline{w}_{opt}} = 2(\underline{p}\underline{p}^T + R_N)\underline{w}_{opt} - 2\underline{p} = 0$

Sherman-Morrison Formula.

of this correspondence are effective ones, unless otherwise specified.

Lemma 3: If a second-order random process $N(t)$ is WSCS with period T, band-limited with bandwidth B, and positive definite, then its matrix-valued PSD is also positive definite for all $f \in [-1/(2T), 1/(2T))$, i.e, $\boldsymbol{R}_N(f) > 0, \forall f$.

Proof: By using Proposition 4, Definition 2 can be rewritten as

$$\int_{-\frac{1}{2T}}^{\frac{1}{2T}} \boldsymbol{w}(f)^H \boldsymbol{R}_N(f)\, \boldsymbol{w}(f) df > 0$$

for all nonzero $w(t) \in \mathbb{L}^2(B)$. Hence, the conclusion follows. □

Theorem 2: The vectorized Fourier transform $\boldsymbol{w}_{\mathrm{lmmse}}(f)$ of the optimum receiver waveform is obtained as

$$\boxed{\boldsymbol{w}_{\mathrm{lmmse}}(f) = M(fT)\left(\boldsymbol{R}(f)^{-1}\boldsymbol{p}(f)\right)} \tag{37a}$$

$$= \frac{M(fT)\boldsymbol{R}_N(f)^{-1}\boldsymbol{H}(f)\boldsymbol{s}(f)}{1 + \frac{M(fT)}{T}\boldsymbol{s}(f)^H \boldsymbol{C}(f)\boldsymbol{s}(f)} \tag{37b}$$

$$\begin{aligned} &\underset{\boldsymbol{s}(f)}{\text{minimize}} && \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{T M(fT)}{1 + \frac{M(fT)}{T}\boldsymbol{s}(f)^H \boldsymbol{C}(f)\boldsymbol{s}(f)} df \\ &\text{subject to} && \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{M(fT)}{T} a(f) df = A \end{aligned} \tag{41}$$

$\|\underline{s}(f)\|^2$

which is a constrained optimization problem to find the jointly optimum transmitter waveform $s_{\mathrm{opt}}(t)$, or equivalently, $\boldsymbol{s}_{\mathrm{opt}}(f)$. Note that, given $\boldsymbol{s}_{\mathrm{opt}}(f)$, the jointly optimum receiver waveform is determined by (37b).

Using the definition of $a(f)$, we can rewrite the objective functional as

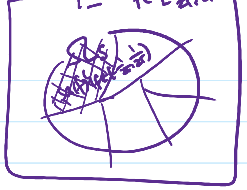
$$\int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{T M(fT)}{1 + \frac{M(fT)}{T} a(f) \left[\frac{\boldsymbol{s}(f)^H \boldsymbol{C}(f)\boldsymbol{s}(f)}{\boldsymbol{s}(f)^H \boldsymbol{s}(f)}\right]} df. \tag{42}$$

The ~~Euler~~'s quotient satisfies

$$0 \le \lambda_{\min}(f) \le \frac{\boldsymbol{s}(f)^H \boldsymbol{C}(f)\boldsymbol{s}(f)}{\boldsymbol{s}(f)^H \boldsymbol{s}(f)} \le \lambda_{\max}(f) \tag{43}$$

Rayleigh's quotient

$\{\underline{s}(f)\}$ $f \in [-\frac{1}{2T}, \frac{1}{2T}]$



$\frac{1}{2T}$ $\Big|\frac{1-\alpha}{2T}\ \frac{1}{2T}\ \frac{1+\alpha}{2T}$

$0 \le \alpha \le 1.$

$\underline{s}(f)$

$\underline{s}(0) = 1$

$\underline{s}(\frac{1}{2T}) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$\underline{s}(\frac{1}{2T}+\delta) = \begin{bmatrix} \frac{1}{2}+\beta(\alpha) \\ \frac{1}{2}-\beta(\alpha) \end{bmatrix}$

$$\min_{a(f)} \int \boxed{\min_{\underline{s}(f)} \text{———}}\, df$$

$$\text{s.t.} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{M(fT)}{T} a(f) df = A$$

$\Big|_{\underline{w}_{\mathrm{opt}}}$

$\Rightarrow \quad \underline{w}_{\mathrm{opt}} = (\underline{p}\,\underline{p}^T + R_N)^{-1}\underline{p}.$

$= R^{-1}\underline{p}$

$\underline{u} = \underline{v} = \underline{p}$

$A = R$

Sherman–Morrison formula.

$$(A + \underline{u}\,\underline{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\underline{u}\,\underline{v}^T A^{-1}}{1 + \underline{v}^T A^{-1}\underline{v}}$$

$$= \left(R_N^{-1} - \frac{R_N^{-1}\underline{p}\,\underline{p}^T R_N^{-1}}{1 + \underline{p}^T R_N^{-1}\underline{p}}\right)\underline{p}$$

$$= \frac{(1 + \underline{p}^T R_N^{-1}\underline{p}) R_N^{-1}\underline{p} - R_N^{-1}\underline{p}(\underline{p}^T R_N^{-1}\underline{p})}{1 + \underline{p}^T R_N^{-1}\underline{p}}$$

$$= \frac{R_N^{-1}\underline{p} + \cancel{R_N^{-1}\underline{p}(\underline{p}^T R_N^{-1}\underline{p}) - R_N^{-1}\underline{p}(\underline{p}^T R_N^{-1}\underline{p})}}{1 + \underline{p}^T R_N^{-1}\underline{p}}$$