

Note that a half-Nyquist zone is just the right-half of the corresponding Nyquist zone. Also note that L is chosen the number of length- $1/(2T)$ intervals that can fully cover the one-sided frequency band $f \in [0, B)$ of interest.

First, we define an LCL-TV operator that converts a scalar-valued signal to a vector-valued one.

Definition 5: Given a reference pair $(B, 1/T)$ and an input $Y(t)$, the *properizing FREQUENCY SHIFT (p-FRESH)* vectorizer is defined as a single-input multiple-output (SIMO) LCL-TV system, whose output $\mathbf{Y}(t)$ is given by

$$\mathbf{Y}(t) \triangleq \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_{2L}(t) \end{bmatrix} \quad (16a)$$

where the l th entry $Y_l(t)$, for $l = 1, 2, \dots, 2L$, is given by

$$Y_l(t) \triangleq \begin{cases} (Y(t)e^{-j2\pi f_1 t}) * g(t), & \text{for } l \leq L, \\ (Y(t)^* e^{-j2\pi f_{L+1-l} t}) * g(t), & \text{for } l > L, \end{cases} \quad (16b)$$

with $g(t)$ being the impulse response of the ideal brickwall filter having the half-Nyquist zone $\mathcal{F}^+$ as the support, i.e., the Fourier transform $G(f) \triangleq \mathcal{F}\{g(t)\}$ is given by

$$G(f) = \begin{cases} 1, & \forall f \in \mathcal{F}^+ \\ 0, & \text{elsewhere.} \end{cases} \quad (17)$$

Fig. 2-(a) shows how a p-FRESH vectorizer works in the time domain, where a scalar-valued signal $Y(t)$ is converted to a $2L$ -by-1 vector-valued signal $\mathbf{Y}(t)$. The p-FRESH vectorizer modulates $Y(t)$ and its complex conjugate $Y(t)^*$ each with L different carriers, of which frequencies are integer multiples of the reference rate $1/T$, and filters to form $\mathbf{Y}(t)$, of which entries are all strictly band-limited to the half-Nyquist zone $\mathcal{F}^+$.

Fig. 2-(b) shows how a p-FRESH vectorizer works in the frequency domain when a deterministic signal $s(t)$ with $L = 2$ is converted to a vector-valued signal $\mathbf{s}(f)$. Let $\mathbf{s}(f)$ be the element-wise Fourier transform of $\mathbf{s}(t)$. Then, to generate the l th entry $[s(f)]_l$ of $\mathbf{s}(f)$, the p-FRESH vectorizer left-shifts the Fourier transform $S(f)$ of $s(t)$ by f_l for $l \leq L$, while $S(-f)^*$ by f_{L+1-l} for $l > L$, and leaves only the signal component in $\mathcal{F}^+$.

Note that, given a signal, different reference pairs result in different p-FRESH vectorizations in general. It can be seen that, given an arbitrary signal with bandwidth B_0 , no information is lost during the vectorization as long as the reference bandwidth B satisfies

$$B_0 \leq B, \quad (18)$$

regardless of the reference rate $1/T$. Note also that the p-FRESH vectorizer processes both $Y(t)$ and $Y(t)^*$, like linear-conjugate linear [33] or, equivalently, widely linear filters [34] that are devised to process improper-complex random signals. The major difference is that, as it can be easily seen from Fig. 2-(b), the p-FRESH vectorizer filters out half the signal components in $Y(t)$ and $Y(t)^*$ to completely remove the redundancy during the vectorization.

³The abbreviation FRESH appears first in [30], where the cyclic Wiener filtering that is a generalization of the Wiener filtering for cyclostationary random processes is proposed.

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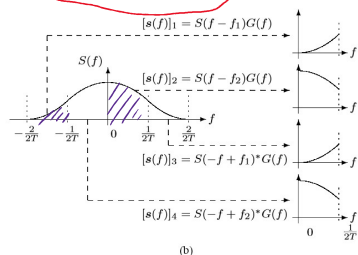
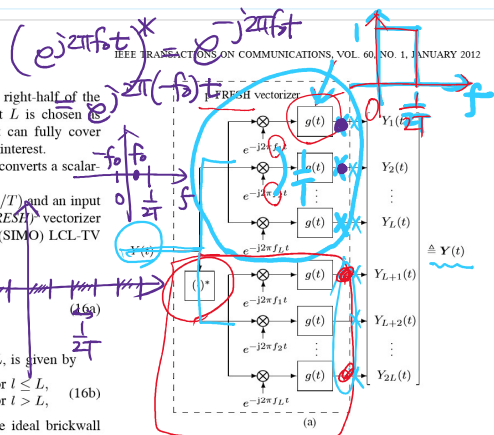


Fig. 2. (a) p-FRESH vectorization viewed in the time domain. (b) p-FRESH vectorization viewed in the frequency domain for a deterministic signal $s(t)$ with $L = 2$.

Second, we define an LCL-TV operator that converts a vector-valued signal to a scalar-valued one.

Definition 6: The p-FRESH scalarization of a $2L$ -by-1 vector-valued signal $\mathbf{X}(t)$ with reference pair $(B, 1/T)$ is defined as

$$X(t) \triangleq \sum_{l=1}^L \{X_l(t)e^{j2\pi f_l t} + (X_{L+1-l}(t)e^{j2\pi f_{L+1-l} t})^*\}. \quad (19)$$

Fig. 3 shows how a p-FRESH scalarizer works in the time domain, where a $2L$ -by-1 vector-valued random process $\mathbf{X}(t)$ is converted to a scalar-valued random process $X(t)$. In the frequency domain, the p-FRESH scalarizer right-shifts the l th input element $X_l(t)$ of $\mathbf{X}(t)$ by f_l , while the complex conjugate of the $(L+1-l)$ th element by $-f_l$, both for $l = 1, 2, \dots, L$, to construct the output $X(t)$.

If the input to the p-FRESH scalarizer is the p-FRESH vectorized version of an SOCS random process with a reference bandwidth $B \geq B_0$ and an arbitrary reference rate

$$\begin{cases} \theta = 0 \\ \theta = \frac{\pi}{2} \end{cases}$$

$$\square \quad \underline{z} = \underline{x} + j\underline{y}$$

$$E\{\underline{z}\} = \underline{\mu}_z = \underline{\mu}_x + j\underline{\mu}_y$$

$$\text{Cov}\{\underline{z}\} = E\{(\underline{z} - \underline{\mu}_z)(\underline{z} - \underline{\mu}_z)^H\} = \leftarrow \text{Cov}(\underline{x}), \text{Cov}(\underline{x}, \underline{y}), \text{Cov}(\underline{y})$$

$$\tilde{\text{Cov}}\{\underline{z}\} = E\{(\underline{z} - \underline{\mu}_z)(\underline{z} - \underline{\mu}_z)^T\} =$$

$$\square \text{ Def. proper} \Leftrightarrow \tilde{\text{Cov}}\{\underline{z}\} = \mathbf{0} \Leftrightarrow \text{Cov}(\underline{x}) = \text{Cov}(\underline{y}), \text{Cov}(\underline{x}, \underline{y}) = \mathbf{0}$$

$$\square \text{ Def. (i) circular} \Leftrightarrow \underline{z}e^{j\theta} \text{ has the same pdf as } \underline{z} \text{ for all } \theta \in [0, 2\pi).$$

$$\square \text{ Lemma. If } \underline{z} \text{ is circular, then } \underline{z} \text{ is proper.}$$

$$\square \text{ Def } \underline{z} = \underline{x} + j\underline{y} \text{ is complex Gaussian if } \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \text{ has Gaussian distribution.}$$

$$\square \text{ Def If } \underline{z} \in \mathbb{C}^N \text{ is proper then}$$

$$\begin{aligned} \text{for complex proper Gaussian } N=1 \rightarrow f_z(z) &= \frac{1}{\pi \sigma^2} \exp\left(-\frac{|z - \mu_z|^2}{\sigma^2}\right) \\ f_z(z) &= \frac{1}{\pi^N \sigma_1^2 \sigma_2^2 \dots \sigma_N^2} \exp\left(-(\underline{z} - \underline{\mu}_z)^H \underline{C}^{-1} (\underline{z} - \underline{\mu}_z)\right) \end{aligned}$$

$$\text{When } \underline{C} \text{ is invertible} \rightarrow \text{For real Gaussian, } f_x(x) = \frac{1}{\sqrt{(2\pi)^N \det(\underline{C})}} \exp\left(-\frac{(\underline{x} - \underline{\mu}_x)^T \underline{C}^{-1} (\underline{x} - \underline{\mu}_x)}{2}\right)$$

$$\square Q \text{ Pdf of improper Gaussian Random Vector?}$$

$$\underline{X} \sim \mathcal{N}(\underline{\mu}_x, \underline{C}_{xx})$$

$$\underline{Z} \sim \mathcal{CN}(\underline{\mu}_z, \underline{C}_z)$$

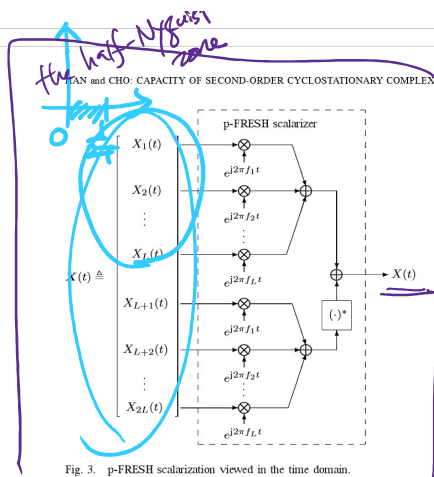


Fig. 3. p-FRESH scalarization viewed in the time domain.

$1/T$, then the p-FRESH scalarizer with the same reference pair as the vectorizer just performs the inverse operation of the p-FRESH vectorization. In this sense, a p-FRESH vectorization of the SOCS process may not be more than one of many possible representations that convert a scalar-valued signal or random process to an equivalent vector-valued one. However, the following lemma shows that an appropriate choice of the reference rate $1/T$ is crucial to capture the correlation structure in an SOCS process. In what follows, we define superscripts T and H as the transposition and the Hermitian transposition, respectively, and 0_{2L} and $0_{2L \times 2L}$ as the $2L$ -by-1 all-zero vector and the $2L$ -by- $2L$ all-zero matrix, respectively.

Lemma 1: If the reference rate $1/T$ of a p-FRESH vectorizer is chosen such that a zero-mean SOCS random process input $Y(t)$ with cycle period T_0 is also SOCS with cycle period T , then the output $Y(t)$ always becomes a proper-complex zero-mean vector-WSS process, i.e., if

$$T = KT_0 \quad (20)$$

for some positive integer K , then

$$\mu_Y(t) \triangleq \mathbb{E}\{Y(t)\} = 0_{2L}, \forall t, \quad (21a)$$

$$r_Y(t, s) \triangleq \mathbb{E}\{Y(t)Y(s)^H\} = r_Y(t-s), \forall t, \forall s, \text{ and} \quad (21b)$$

$$\tilde{r}_Y(t, s) \triangleq \mathbb{E}\{Y(t)Y(s)^T\} = 0_{2L \times 2L}, \forall t, \forall s, \quad (21c)$$

where $r_Y(\tau) \triangleq r_Y(\tau, 0)$ is a $2L$ -by- $2L$ matrix-valued function.

Proof: See Appendix A. ■

As far as the auto-correlation function $r_Y(t, s)$ is concerned, this lemma applying the p-FRESH vectorizer to an SOCS input with an arbitrary K is a mere extension of the result in [9], where the HSR is applied to a WSCS input with $K = 1$. However, it additionally shows that, regardless of the propriety of the SOCS input, the complementary auto-correlation function $\tilde{r}_Y(t, s)$ always vanishes, which is not

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the case with the HSR unless the input is proper-complex.⁴

Let $R_Y(f) \triangleq \mathcal{F}\{r_Y(\tau)\}$ be the $2L$ -by- $2L$ element-wise Fourier transform of the auto-correlation matrix, i.e., $R_Y(f)$ is the PSD matrix of the p-FRESH vectorizer output $Y(t)$. From Lemma 1, it can be seen that a p-FRESH vectorizer with $T = KT_0$ not only generates a sufficient statistic that allows a perfect reconstruction of the SOCS input $Y(t)$ but also tactically aligns its frequency components in such a way that the vector-valued output $Y(t)$ has no correlation among different frequency components and no complementary correlation at all on $f \in \mathcal{F}^+$. The next lemma points out one important property of the PSD matrix $R_Y(f)$ of $Y(t)$.

Lemma 2: Given a reference pair $(B, 1/T)$, the PSD matrix $R_Y(f)$ of the p-FRESH vectorizer output $Y(t)$ of an SOCS random process $Y(t)$ with cycle period T is a Hermitian-symmetric positive semi-definite matrix for all $f \in \mathcal{F}^+$.

Proof: See Appendix B. ■

This property is used in the frequency-domain reformulation of the problem, as shown in the next section. The following proposition shows how the p-FRESH vectorization of the channel output can be rewritten in terms of the p-FRESH vectorization of the channel input and that of the channel impulse response.

Proposition 1: The p-FRESH vectorization $Y(t)$ of the channel output $Y(t)$ can be rewritten as

$$Y(t) = \text{diag}\{h(t)\} * X(t) + N(t), \quad (22)$$

where $Y(t)$, $h(t)$, $X(t)$, and $N(t)$ are the p-FRESH vectorizations of $Y(t)$, $h(t)$, $X(t)$, and $N(t)$, respectively, and $\text{diag}\{h(t)\}$ denotes the diagonal matrix having the l th diagonal entry equal to the l th entry of $h(t)$.

Proof: By the definition (16b), the l th entry $[Y(t)]_l = Y_l(t)$ of $Y(t)$ can be rewritten, for $l \leq L$, as

$$Y_l(t) \triangleq (Y(t)e^{-j2\pi f_1 t}) * g(t) \quad (23a)$$

$$= ((h(t) * X(t) + N(t))e^{-j2\pi f_1 t}) * g(t) \quad (23b)$$

$$= \left(\int_{-\infty}^{\infty} h(\tau)X(t-\tau) d\tau \cdot e^{-j2\pi f_1 t} \right) * g(t) + N_l(t) \quad (23c)$$

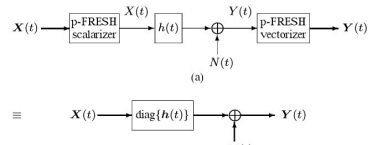
$$= \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_1 \tau} X_l(t-\tau) d\tau + N_l(t) \quad (23d)$$

$$= [h(t)]_l * [X(t)]_l + [N(t)]_l, \quad (23e)$$

where (23d) follows since the convolution with $g(t)$ acts only on signals parameterized by t , and (23e) follows since $g(t)$ is the impulse response of the ideal brickwall filter. Similarly, it can be shown that (23e) also holds for $l > L$. Therefore, we have (22). ■

This implies that, by placing a p-FRESH scalarizer and a p-FRESH vectorizer, respectively, at the input and the output of the SISO channel as shown in Fig. 4-(a), the SISO channel depicted in Fig. 1 can be converted to an equivalent MIMO channel shown in Fig. 4-(b). This equivalent MIMO channel is simpler than general MIMO channels, in that the channel matrix is always square and diagonal. However, the additive proper-complex Gaussian noise $N(t)$ is in general colored.

⁴As it will be discussed in the second paragraph of Section V, the selection of the center frequency is very important in down-converting a real bandpass random process to make it SOCS.



(c) The degree of freedom $N_p(f)$ at f is defined as the length of an effective Fourier transform of the p-FRESH vectorizer output.

When we design a band-limited signal $s(t)$ or a band-limited SOCS random process $X(t)$, such a removal of the entries in $s(f)$ or $R_X(f)$ is necessary because every entry is not always a free variable unless $L = 2BT$. From (25a) and (25b), it can be seen that $N_p(f)$ is given by

$$X \sim N(\mu_X, \Sigma_X)$$

$$\underline{Z} \sim \text{CN}(\mu_Z, C_{ZZ})$$

$$\underline{Z} \sim \text{CPN}(\mu_Z, C_{ZZ})$$

$$\underline{Z} \sim \text{CN}(\mu_Z, C_{ZZ}, \tilde{C}_{ZZ})$$

$$\underline{Z} \sim \text{CN}(\mu_Z, C_{ZZ}, 0)$$

$$(\underline{Z} - \mu_Z)e^{j\theta}$$

$$1-D \text{ circulant}$$

$$N-D \quad \underline{Z}e^{j\theta}, (\underline{Z} - \mu_Z)e^{j\theta}$$

$$N-D \quad U \underline{Z}$$

$$U(\underline{Z} - \mu_Z) \dots \text{w/ unitary matrix } U.$$

□ Theorem.

$$\underline{y} = H \underline{x} + \underline{n}$$

where $\underline{x}$ input vector $\in \mathbb{C}^M$
 $\underline{y}$ output " $\in \mathbb{C}^N$
 H ch. matrix $\in \mathbb{C}^{N \times M}$
 $\underline{n} \sim \text{CN}(0, C_{nn})$

□ SVD (Singular-Value Decomposition) ...

$$H = U \Sigma V^H$$

$$\begin{cases} U^H U = U U^H = I \\ V^H V = V V^H = I \\ [\Sigma]_{ij} = \sigma_i^2 \delta_{ij}, 1 \leq i \leq M, 1 \leq j \leq N \end{cases}$$

$$\left(\text{Recall } \Sigma \in \mathbb{C}^{M \times N} \right)$$

□ Covariance $L = \text{dim}(h) = (1/h)(\sigma_p)$

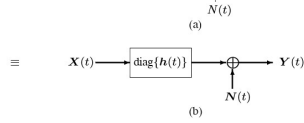


Fig. 4. (a) Conversion of a SISO channel to an equivalent MIMO channel by using a p-FRESH scalarizer and a p-FRESH vectorizer. (b) Equivalent MIMO channel with vector-valued colored noise.

Since the capacity of MIMO additive proper-complex white Gaussian noise channels is well known [25], we consider whitening the vector-WSS noise $N(t)$ of the equivalent MIMO channel as shown in the next section.

IV. CYCLIC WATER FILLING AND CAPACITY OF SOCS GAUSSIAN NOISE CHANNEL

In this section, the problem is re-formulated in the frequency domain and then the channel capacity is derived. Throughout the section, it is assumed that

$$B = B_0 \text{ and } T = KT_0 \quad (24)$$

for some positive integer K , unless otherwise specified. Thus, the p-FRESH vectorization of $Y(t)$ becomes a sufficient statistic and the SOCS noise becomes a vector-WSS process.

A. Noise Whitening and Problem Re-Formulation

To find a whitening filter, we need the following notions of effective Fourier transform of the p-FRESH vectorizer output, effective PSD matrix, degree of freedom, and effective element-wise inverse Fourier transform.

Definition 7: Suppose that a reference pair $(B, 1/T)$ is given for a p-FRESH vectorizer and that $L \triangleq \lceil 2BT \rceil$ is defined as before.

(a) The effective Fourier transform of the p-FRESH vectorizer output of a deterministic signal $s(t)$ is defined as a variable-length vector-valued function of f that is obtained by removing the L th and the $2L$ th entries of $s(f) \triangleq \mathcal{F}\{s(t)\}$ for

$$B - \frac{L-1}{2T} \leq f < \frac{1}{2T} \quad (25a)$$

when L is odd, and the 1st and the $(L+1)$ th entries for

$$0 \leq f < \frac{L}{2T} - B \quad (25b)$$

when L is even.

(b) The effective PSD matrix of the p-FRESH vectorizer output of an SOCS random process $X(t)$ with cycle period T is defined as a variable-size matrix-valued function of f that is obtained by removing the L th and the $2L$ th rows and columns of $R_X(f) \triangleq \mathcal{F}\{r_X(\tau)\}$ for f satisfying the condition (25a) when L is odd, while the 1st and the $(L+1)$ th rows and columns for f satisfying the condition (25b) when L is even.

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vectorizer output.

When we design a band-limited signal $s(t)$ or a band-limited SOCS random process $X(t)$, such a removal of the entries in $s(f)$ or $R_X(f)$ is necessary because every entry is not always a free variable unless $L = 2BT$. From (25a) and (25b), it can be seen that $N_p(f)$ is given by

$$N_p(f) \triangleq \begin{cases} 2L, & \text{for } f \in [0, B - (L-1)/(2T)], \\ 2L-2, & \text{otherwise} \end{cases} \quad (26a)$$

when L is odd, while

$$N_p(f) \triangleq \begin{cases} 2L-2, & \text{for } f \in [0, L/(2T) - B], \\ 2L, & \text{otherwise} \end{cases} \quad (26b)$$

when L is even. Thus, an effective Fourier transform of the p-FRESH vectorizer output becomes an $N_p(f)$ -by-1 vector and an effective PSD matrix becomes an $N_p(f)$ -by- $N_p(f)$ matrix at each $f \in \mathcal{F}^+$. For simplicity, we do not introduce new notations for the effective Fourier transforms and PSD matrices.

Definition 8: Given a reference pair $(B, 1/T)$, the element-wise inverse Fourier transform of an $N_p(f)$ -by- $N_p(f)$ matrix-valued function of $f \in \mathcal{F}^+$ is defined as a $2L$ -by- $2L$ matrix-valued function of t obtained as follows: First, construct a $2L$ -by- $2L$ matrix-valued function $W(f)$ of f as if the $N_p(f)$ -by- $N_p(f)$ matrix-valued function is the effective PSD matrix of $W(f)$. Second, construct a $2L$ -by- $2L$ matrix-valued function $W(t)$ of t by applying the inverse Fourier transform to every entry of $W(f)$.

Note that, by definition, the L th and the $2L$ th row and column of $W(f)$ have all zero entries for f satisfying the condition (25a), while the 1st and the $(L+1)$ th row and column have all zero entries for f satisfying the condition (25b).

Proposition 2: Let $W(t)$ be the element-wise inverse Fourier transform of the inverse square root of the effective PSD matrix of $N(t)$, i.e.,

$$W(t) = \mathcal{F}^{-1}\{R_{N_p}(f)^{-\frac{1}{2}}\}. \quad (27)$$

Then, the $2L$ -by- $2L$ MIMO LTI system with impulse response $W(t)$ whitens the noise $N(t)$.

Proof: Define the output of the MIMO LTI system as

$$\tilde{N}(t) \triangleq W(t) * N(t). \quad (28)$$

Then, its auto-correlation matrix $r_{\tilde{N}}(\tau)$ can be written as

$$r_{\tilde{N}}(\tau) \triangleq \mathbb{E}\{\tilde{N}(t)\tilde{N}^H(t-\tau)\} \quad (29a)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(t)r_N(\tau-t+s)W^H(s)ddts. \quad (29b)$$

Thus, the PSD matrix $R_{\tilde{N}}(f)$ is given by

$$R_{\tilde{N}}(f) \triangleq \mathcal{F}\{r_{\tilde{N}}(\tau)\} \quad (30a)$$

$$= R_N(f)^{-\frac{1}{2}} R_N(f) R_N(f)^{-\frac{1}{2}} = I_{N_p(f)}, \quad (30b)$$

where we used the fact that $R_N(f)^{-\frac{1}{2}}$ exists for all $f \in \mathcal{F}^+$ because the PSD matrix $R_{N_p}(f)$ of the interference $N_p(t)$ is positive semi-definite $\forall f \in \mathcal{F}^+$ by Lemma 2 and the PSD matrix $R_{N_A}(f)$ of the ambient noise $N_A(t)$ is a scaled

identity matrix $\forall f \in \mathcal{F}^+$. Therefore, the conclusion follows.

This MIMO whitening filter plays a crucial role in formulating and solving the optimization problem to find the channel capacity in the frequency domain. In what follows, all the functions of $f \in \mathcal{F}^+$ are effective ones, unless otherwise specified.

Theorem 1: The optimization problem to find the capacity of an SOCS Gaussian noise channel is given by

$$\begin{aligned} & \text{maximize} \int_{\mathcal{F}^+} \log_2 \det \left\{ I_{N_p(f)} + \tilde{H}(f) R_X(f) \tilde{H}^H(f) \right\} df \\ & \quad (R_X(f))_{f \in \mathcal{F}^+} \end{aligned} \quad (31a)$$

$$\text{subject to} \int_{\mathcal{F}^+} \text{tr}\{R_X(f)\} df = P, \quad (31b)$$

where $\text{tr}\{\cdot\}$ denote the trace, the PSD matrix $R_X(f)$ of $X(t)$ is a positive semi-definite $N(f)$ -by- $N(f)$ matrix for all $f \in \mathcal{F}^+$.

have independence among $N^{(k)}(t)$, for $k = 1, 2, \dots, K$. It is well known that the capacity of parallel Gaussian channels with independent noise components can be achieved by using independent channel codes [21]. Thus, the optimal input PSD matrix $R_X(f)$ becomes a block diagonal matrix. If we multiply complex exponential functions to $Y^{(k)}$, for $k = 1, 2, \dots, K$, such that $Y^{(k)}$ is shifted in the frequency domain to the interval $f \in [(k-1)/(2KT_0), k/(2KT_0))$, the mutual information to be maximized becomes

$$\sum_{k=1}^K \int_{\frac{k-1}{2KT_0}}^{\frac{k}{2KT_0}} \log_2 \det \left\{ I_{N_p(f)} + \tilde{H}(f) R_X(f) \tilde{H}^H(f) \right\} df, \quad (36)$$

which is identical to (31a) with $K = 1$, where $N_p(f)$, $\tilde{H}(f)$, and $R_X(f)$ are all defined for $K = 1$. Since the optimization problems have the same objective functions and constraints, the conclusion follows. ■

Recall $H \in \mathbb{C}^{M \times N}$

□ Covariance $H = \text{diag}(h) = \begin{bmatrix} |h_1|e^{j\phi_1} & & \\ & |h_2|e^{j\phi_2} & \\ & & \ddots \\ & & & |h_M|e^{j\phi_M} \end{bmatrix}$

arbitrary phase = $\begin{bmatrix} e^{j\phi} & & \\ & e^{j\phi_2} & \\ & & e^{j\phi_3+\theta_4} \end{bmatrix}$

□ Thm. If $n \sim \mathcal{CN}(0, \sigma^2 I)$ then $C \text{ (bits/use)} = \max_{P \succeq 0} \log \left(\det \left(I + \frac{P - \sigma^2 I}{\sigma^2} \right) \right)$ s.t. $\sum P_i = P = E\{\|x\|^2\}$.

□ Cor. If $n \sim \mathcal{CN}(0, C)$ then $C \succ 0$

$C = W D W^H$ $D = \text{diag}(d_1, d_2, \dots, d_M)$

$W = [w_1, w_2, \dots, w_M]$, $W W^H = W^H W = I$

$x \rightarrow [H] \rightarrow y \rightarrow [C]^{-\frac{1}{2}} \rightarrow z$

$z = C^{-\frac{1}{2}} H x = \tilde{C}^{-\frac{1}{2}} \tilde{x}$ (normalized eigen vectors of C)

where $C^{-\frac{1}{2}} = W D^{-\frac{1}{2}} W^H = W \begin{bmatrix} \frac{1}{\sqrt{d_1}} & & \\ & \frac{1}{\sqrt{d_2}} & \\ & & \ddots \\ & & & \frac{1}{\sqrt{d_M}} \end{bmatrix} W^H$

$\tilde{z} = \begin{bmatrix} \sim \\ H \end{bmatrix} x + \tilde{m}$

(31a) $\int_{\mathcal{F}^+} \text{tr}\{\mathbf{R}_{\mathbf{X}}(f)\} df = P$,
 (31b) subject to $\int_{\mathcal{F}^+} \text{tr}\{\mathbf{R}_{\mathbf{X}}(f)\} df = P$,
 where $\text{tr}\{\cdot\}$ denote the trace, the PSD matrix $\mathbf{R}_{\mathbf{X}}(f)$ of $\mathbf{X}(t)$ is a positive semi-definite $N_p(f)$ -by- $N_p(f)$ matrix for all $f \in \mathcal{F}^+$, and the noise equivalent channel $\tilde{\mathbf{H}}(f)$ at $f \in \mathcal{F}^+$ is defined as

$$\tilde{\mathbf{H}}(f) \triangleq \mathbf{R}_{\mathbf{N}}(f)^{-\frac{1}{2}} \mathbf{H}(f), \quad (32)$$

where

$$\mathbf{H}(f) \triangleq \text{diag}\{h(f)\}. \quad (33)$$

Proof: If we place the whitening filter $\mathbf{W}(t)$ defined by (27) at the output of the p-FRESH vectorizer in Fig. 4 (a), then the observation model (22) can be converted equivalently to

$$\tilde{\mathbf{Y}}(t) = \tilde{\mathbf{H}}(t) * \mathbf{X}(t) + \tilde{\mathbf{N}}(t), \quad (34)$$

where $\tilde{\mathbf{N}}(t)$ is defined in (28), and $\tilde{\mathbf{Y}}(t)$ and $\tilde{\mathbf{H}}(t)$ are defined as $\tilde{\mathbf{Y}}(t) \triangleq \mathbf{W}(t) * \mathbf{Y}(t)$ and $\tilde{\mathbf{H}}(t) \triangleq \mathbf{W}(t) * \text{diag}\{h(t)\}$, respectively. Thus, the MIMO channel shown in Fig. 4 (a) is equivalent to that in Fig. 5 (a). Since the additive Gaussian noise $\tilde{\mathbf{N}}(t)$ is white as shown in Proposition 2, the optimization problem to find the capacity of the SOCS Gaussian noise channel can be formulated in exactly the same way as that of a MIMO channel with a proper-complex white Gaussian noise [26], where the proper-complex Gaussianity of the channel input is necessary to achieve the capacity. Therefore, the conclusion follows. ■

Before we move to the next subsection where the solution to (31) is derived, we examine whether a different choice of K may result in a different value of the channel capacity or not.

Proposition 3: For any positive integer K , the problem described in (31) is equivalent to that with $K=1$.

Proof: Suppose that $\mathbf{Y}(t)$ is the output of the p-FRESH vectorizer when the input is $Y(t)$ and $T = KT_0$ for some $K > 1$. By pre-multiplying a permutation matrix, the elements of $Y(t)$ can be rearranged as

$$\tilde{\mathbf{Y}}(t) \triangleq \begin{bmatrix} \mathbf{Y}^{(1)}(t) \\ \mathbf{Y}^{(2)}(t) \\ \vdots \\ \mathbf{Y}^{(K)}(t) \end{bmatrix}, \quad (35)$$

where the l th entry of $\mathbf{Y}^{(k)}(t)$, for $k=1, 2, \dots, K$, is defined as $[\mathbf{Y}^{(k)}(t)]_l \triangleq Y_{(l-1)K+k}(t)$. If the permuted vector-valued noise $\tilde{\mathbf{N}}(t)$ and its elements $\mathbf{N}^{(k)}(t)$, for $k=1, 2, \dots, K$, are similarly defined, then, by Lemma 1 with $K=1$, we

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$$(36) \quad \sum_{k=1}^K \int_{\mathcal{F}^+} \text{tr}\{\mathbf{R}_{\mathbf{X}}(f)\} df = P$$

which is identical to (31a) with $K=1$, where $N_p(f)$, $\tilde{\mathbf{H}}(f)$, and $\mathbf{R}_{\mathbf{X}}(f)$ are all defined for $K=1$. Since the optimization problems have the same objective functions and constraints, the conclusion follows. ■

B. Optimal Solution and Its Property

In this subsection, the optimal solution to (31) is derived and its property is investigated. For this, consider the singular value decomposition

$$\tilde{\mathbf{H}}(f) = \mathbf{U}(f) \mathbf{\Gamma}(f) \mathbf{V}(f)^H \quad (37)$$

of the $N_p(f)$ -by- $N_p(f)$ noise equivalent channel matrix $\tilde{\mathbf{H}}(f)$ for each $f \in \mathcal{F}^+$, where $\mathbf{U}(f)$ and $\mathbf{V}(f)$ are unitary matrices, and $\mathbf{\Gamma}(f)$ is a diagonal matrix having the singular values $\gamma_n(f)$, for $n=1, 2, \dots, N_p(f)$, as the diagonal entries.

Theorem 2: Let $\mathbf{R}_{\mathbf{X}, \text{opt}}(f)$ be an $N_p(f)$ -by- $N_p(f)$ diagonal matrix whose n th diagonal entry is given by

$$[\mathbf{R}_{\mathbf{X}, \text{opt}}(f)]_{n,n} \triangleq \left[\nu_{\text{opt}} - \frac{1}{\gamma_n(f)^2} \right]^+, \quad (38a)$$

where ν_{opt} is the unique solution to

$$\int_{\mathcal{F}^+} \sum_{n=1}^{N_p(f)} \left[\nu_{\text{opt}} - \frac{1}{\gamma_n(f)^2} \right]^+ df = P \quad (38b)$$

with $[x]^+ \triangleq (x + |x|)/2$ denoting the positive part of x .⁵ Then, the optimal solution to (31) is given by

$$\mathbf{R}_{\mathbf{X}, \text{opt}}(f) = \mathbf{V}(f) \mathbf{R}_{\mathbf{X}, \text{opt}}(f) \mathbf{V}(f)^H, \quad (39)$$

and, consequently, the capacity of the SOCS Gaussian noise channel is given by

$$C_{\text{SOCS}} = \int_{\mathcal{F}^+} \sum_{n=1}^{N_p(f)} \log_2 (1 + [\gamma_n(f)^2 \nu_{\text{opt}} - 1]^+) df. \quad (40)$$

Proof: By linearly pre-coding the signal $\mathbf{X}(t)$ as $\tilde{\mathbf{X}}(t) \triangleq \mathcal{F}^{-1}\{\mathbf{V}(f)^H\} * \mathbf{X}(t)$ and transforming the whitened observation $\tilde{\mathbf{Y}}(t)$ as $\tilde{\mathbf{Y}}(t) \triangleq \mathcal{F}^{-1}\{\mathbf{U}(f)^H\} * \tilde{\mathbf{Y}}(t)$, we obtain $\tilde{\mathbf{Y}}(t) = \mathcal{F}^{-1}\{\mathbf{\Gamma}(f)\} * \tilde{\mathbf{X}}(t) + \tilde{\mathbf{N}}(t)$, where $\tilde{\mathbf{N}}(t) \triangleq \mathcal{F}^{-1}\{\mathbf{U}(f)^H\} * \tilde{\mathbf{N}}(t)$. Thus, the MIMO channel shown in Fig. 5 (a) is equivalent to that in Fig. 5 (b). Because the proper-complex noise process $\tilde{\mathbf{N}}(t)$ is white and $\mathbf{U}(f)$ is unitary, the noise process $\tilde{\mathbf{N}}(t)$ is also proper-complex and white. Thus, the integrand of the objective function in (31a)

⁵If $\gamma_{n_0}(f) = 0$ for some n_0 , then $[\mathbf{R}_{\mathbf{X}, \text{opt}}(f)]_{n_0, n_0} = 0$ and n_0 is excluded from the summation in (38b).

$$\tilde{\mathbf{X}} = \tilde{\mathbf{H}} \mathbf{x} + \tilde{\mathbf{n}}$$

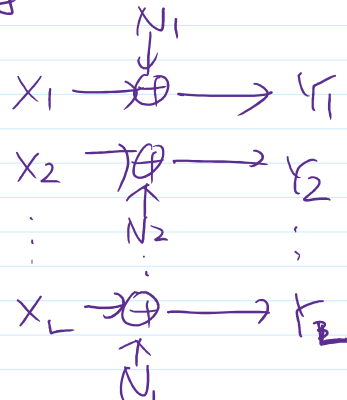
$$\begin{bmatrix} -\frac{1}{2} \\ \text{diag}\{h\} \end{bmatrix} \neq \text{diag}$$

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} a \mathbf{e}_1^T \\ b \mathbf{e}_2^T \end{bmatrix} = a c_1 \mathbf{e}_1^T + b c_2 \mathbf{e}_2^T$$

□

□ Preliminaries

(i) Waterfilling Procedure for parallel Gaussian Channels

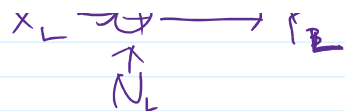


$$\sum_{i=1}^L \mathbb{E}\{|X_i|^2\} \leq P$$

N_i 's are independent & $N_i \sim \mathcal{CN}(0, \sigma_i^2)$
 $\sigma_i^2 > 0, \forall i$

$$\max_{f_{\mathbf{X}}(\mathbf{x})} I(\mathbf{X}; \mathbf{Y})$$

(→ frequency-selective CT channel)

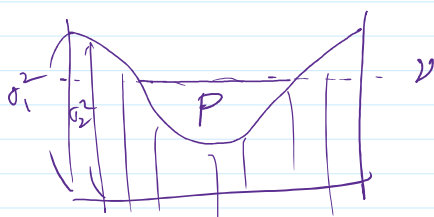
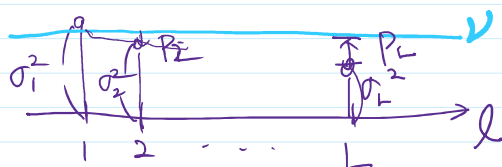


$$\min_{f_X(x)} \mathcal{I}(X, Y)$$

$$f_X(x)$$

$$\text{s.t.} \quad \sum_{i=1}^L \mathbb{E}[X_i^2] \leq P$$

$$\sum_{i=1}^L [\underline{\nu} - \sigma_i^2]^+ = P$$

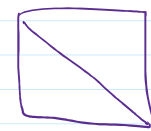


$$C [\text{bits/ch. use}] = \sum_{k=1}^L \log_2 \left(1 + \frac{P_i}{\sigma_i^2} \right)$$

(ii) SVD

$$H = U \Sigma V^H$$

$$\begin{cases} U \in \mathbb{C}^{M \times M} \\ \Sigma \in \mathbb{R}^{M \times N} \\ V \in \mathbb{C}^{N \times N} \end{cases} \quad \begin{aligned} &U U^H = U^H U = I_M \\ &\Sigma \in \mathbb{R}_+^{M \times N}, \quad [\Sigma]_{ij} = \sigma_{ij}^2 \delta_{ij} \\ &V V^H = V^H V = I_N \end{aligned}$$



$$H = \begin{bmatrix} 1 & e^{j\theta_1} & e^{j\theta_2} \\ 2 & e^{j\theta_1} & e^{j\theta_2} \\ 3 & e^{j\theta_1} & e^{j\theta_2} \end{bmatrix}$$

$$\underline{y} = H \underline{x} + \underline{n} = U \Sigma V^H \underline{x} + \underline{n} \quad \underline{\tilde{n}} = U^H \underline{n}$$

$$\underline{y} = \underline{1}^T \underline{x} + \underline{n} = \underline{U} \underline{\Sigma} \underline{V}^H \underline{x} + \underline{n} \quad \underline{\tilde{n}} = \underline{U}^H \underline{n}$$

$$\underline{\tilde{z}} = \underline{U}^H \underline{y} = \underline{\Sigma} \underline{V}^H \underline{x} + \underline{\tilde{n}} = \underline{\Sigma} (\underline{\tilde{x}} + \underline{\tilde{n}})$$

$$\tilde{x}_1 \rightarrow \tilde{z}_{N-1}$$

$$\tilde{x}_2 \rightarrow \tilde{z}_{N-2}$$

$$\vdots \rightarrow \tilde{z}_{N-1}$$

$$\tilde{P}_N = 0 \quad \text{second-order}$$

□ $N(t)$ is improper-complex cyclostationary Gaussian RP w/ cycle period $T > 0$.

□ (G) $X(t)$ is a second-order improper-complex WSCS RP w/ " " $T > 0$

Q. How can we stationarize $X(t)$?
A? ~~FRESH~~ vectorizer FREQUENCY shift.
~~VFT~~ modified?

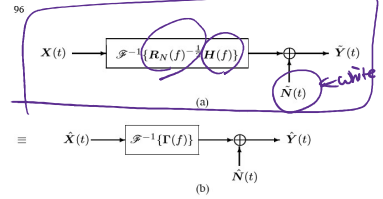
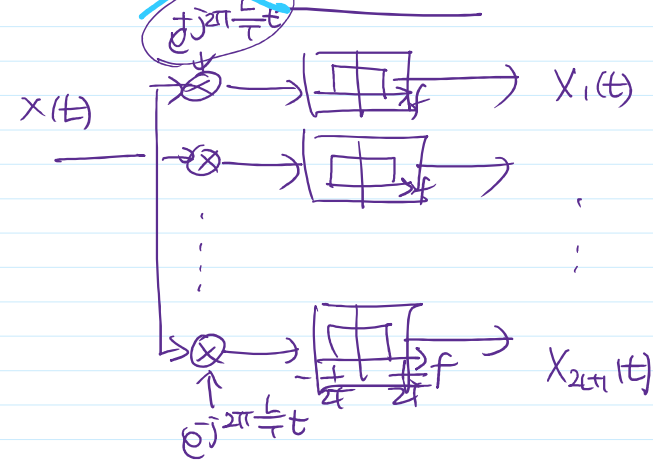


Fig. 5. (a) Conversion of the MIMO channel to an equivalent MIMO channel with vector-valued white noise. (b) Equivalent parallel MIMO channel with vector-valued white noise.

can be rewritten as $\log_2 \det\{\mathbf{I}_{N_r(f)} + \mathbf{\Gamma}(f) \mathbf{R}_X(f) \mathbf{\Gamma}(f)^H\}$, where

$$\mathbf{R}_X(f) \triangleq \mathbf{V}(f)^H \mathbf{R}_X(f) \mathbf{V}(f), \quad (41)$$

and the constraint (31b) can be rewritten as $\int_{\mathcal{F}} \text{tr}\{\mathbf{R}_X(f)\} df = P$ with a positive semi-definite matrix-valued function $\mathbf{R}_X(f)$ as the decision parameter. This implies that we now have a MIMO proper-complex Gaussian noise channel with input $\tilde{\mathbf{X}}(t)$, channel $\mathcal{F}^{-1}\{\mathbf{\Gamma}(f)\}$, output $\tilde{\mathbf{Y}}(t)$, and average power constraint P . Therefore, from the water filling for parallel Gaussian channels, we have (38) and (40). Also by (41), we have (39).

We call this water filling procedure described in Theorem 2 the cyclic water filling (CWF) because the resultant optimal input signal to the SISO channel is SOCS in general, which is proved in the following theorem.

Theorem 3: The optimal input to the SOCS Gaussian noise channel is an SOCS Gaussian random process with the same cycle period as the noise.

Proof: Let $\tilde{X}_{l,\text{opt}}(t)$ be the l th entry of the optimal input $\tilde{\mathbf{X}}_{\text{opt}}(t)$ to the equivalent MIMO channel. Then, as (19), the optimal input $\tilde{X}_{l,\text{opt}}(t)$ to the SISO channel can be written as

$$\tilde{X}_{l,\text{opt}}(t) = \sum_{i=1}^L \left\{ \tilde{X}_{l,\text{opt}}(t) e^{j2\pi f_i t} + (\tilde{X}_{l+L,\text{opt}}(t) e^{j2\pi f_i t})^* \right\}, \quad (42)$$

Now, the auto-correlation function $r_{\tilde{X}_{l,\text{opt}}}(t, s) \triangleq \mathbb{E}\{\tilde{X}_{l,\text{opt}}(t) \tilde{X}_{l,\text{opt}}(s)^*\}$ of $\tilde{X}_{l,\text{opt}}(t)$ can be written as

$$r_{\tilde{X}_{l,\text{opt}}}(t, s) = \sum_{i=1}^L \sum_{i'=1}^L \left\{ [r_{\tilde{\mathbf{X}},\text{opt}}(t-s)]_{l,l'} e^{j2\pi(f_i t - f_{i'} s)} + ([r_{\tilde{\mathbf{X}},\text{opt}}(t-s)]_{l+L,l'+L})^* e^{-j2\pi(f_i t - f_{i'} s)} \right\}, \quad (43a)$$

$$= r_{\tilde{\mathbf{X}},\text{opt}}(t+T, s+T), \quad \forall t, \forall s, \quad (43b)$$

where we used the fact that, by (15b), $f_i T$ and $f_{i'} T$ are always integers for all l and l' , and that the optimal input $\tilde{\mathbf{X}}_{\text{opt}}(t)$ has a vanishing complementary auto-correlation function. Similarly, the complementary auto-correlation function $\bar{r}_{\tilde{\mathbf{X}},\text{opt}}(t, s) \triangleq \mathbb{E}\{\tilde{\mathbf{X}}_{\text{opt}}(t) \tilde{\mathbf{X}}_{\text{opt}}(s)^*\}$ of $\tilde{\mathbf{X}}_{\text{opt}}(t)$ can be written as

$$\bar{r}_{\tilde{\mathbf{X}},\text{opt}}(t, s) = \sum_{i=1}^L \sum_{i'=1}^L \left\{ [r_{\tilde{\mathbf{X}},\text{opt}}(t-s)]_{l,l'} e^{j2\pi(f_i t - f_{i'} s)} + ([r_{\tilde{\mathbf{X}},\text{opt}}(t-s)]_{l+L,l'+L})^* e^{-j2\pi(f_i t - f_{i'} s)} \right\}, \quad (44a)$$

$$= \bar{r}_{\tilde{\mathbf{X}},\text{opt}}(t+T, s+T), \quad \forall t, \forall s, \quad (44b)$$

Thus, $\tilde{X}_{\text{opt}}(t)$ is SOCS with cycle period T . In addition, $\tilde{X}_{\text{opt}}(t)$ is Gaussian as shown in Theorem 2. Moreover, by Proposition 3, any optimal solution to (31) with $K > 1$ is also an optimal solution to the problem with $K = 1$, which means the cycle period of an optimal solution $\tilde{X}_{\text{opt}}(t)$ is T_0 . Therefore, the conclusion follows. ■

Note that this theorem does not claim that the optimal input signal $\tilde{X}_{\text{opt}}(t)$ cannot be second-order stationary. It can be second-order stationary, because a second-order stationary process is also an SOCS process for any cycle period $T > 0$. Note also that the optimal input is in general improper-complex. However, as it is shown in [24] and can be seen easily, the optimal input is always proper-complex if the SOCS complex Gaussian noise is proper.

V. DISCUSSIONS AND NUMERICAL RESULTS

In this section, we provide discussions and numerical results. First, we outline the proof that the worst-case interference is Gaussian given the second-order properties of the SOCS interference. By taking the Nyquist-rate samples from the output of the equivalent MIMO channel in Fig. 5 with a vector-valued white noise, the problem can be converted to a mutual information game over a discrete-time memoryless channel [22], [23]. If B_0 is an integer multiple of $1/(2T_0)$, then the result in [22, Lemma 1] is directly applicable to the sampled output of the equivalent channel in Fig. 5-(a) to show the Gaussianity of the worst-case interference. Otherwise, the result in [23] combined with the independence bound and the water-filling argument [21] can be applied to each output branch of the equivalent channel in Fig. 5-(b).

Second, we discuss how to handle the cases with an improper-complex SOCS noise having a non-zero offset between the center frequency of symmetry and the center frequency of the communication channel. So far, it is assumed that they are the same. Unlike a proper-complex SOCS noise, an improper-complex SOCS noise exhibits non-zero spectral correlation not only between the components that are integer multiples of the symbol rate $1/T$ apart but also between two components that are equally distant from the center frequency. Thus, a real bandpass noise that is down-convertible to an improper-complex SOCS random process no longer results in an SOCS complex envelope if a different center frequency is chosen for the down-conversion. Suppose that there is a non-zero offset between these two center frequencies. Then, to make the output of the p-FRESH vectorizer a proper-complex vector WSS random process, we need to choose the center frequency of symmetry as the origin in complex baseband. If we choose B to include the entire communication channel as before, then the interval $f \in [-B, B]$ may include a frequency band that is not a part of the communication channel. So, we need to set the channel response $H(f)$ and the interference component in the 2-D PSDs $R_N(f, f')$ and $R_N(f, f')$ to zero outside the communication channel, before performing the CWF.

Next, we discuss how to handle the cases with unequal frequency bands for signal transmission and reception. Suppose that the frequency bands for signal transmission and reception are overlapping but not identical. Then, we need

(2L+1) branches

$$\begin{aligned} & \sum_{l=1}^L \sum_{s=1}^S \left\{ \left(\mathbf{r}^H \mathbf{X}_{\text{opt}}(t-s) \right)_{L+l, t}^* e^{-j2\pi(f_l t - f_l s)} \right\} \quad (44a) \\ & = \tilde{\mathbf{r}}_{X, \text{opt}}(t+T, s+T), \forall t, \forall s. \quad (44b) \end{aligned}$$

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to re-define the frequency band for transmission as described in [20, Section III-B] and set the channel response $H(f)$ to zero outside this transmit band, before performing the CWF or possibly the modified CWF discussed above.

Next, we provide numerical results that demonstrate the effectiveness of the CWF in exploiting the cyclostationarity of the noise. For simplicity, it is assumed that the interference $N_I(t)$ is modeled as (9), where the data sequences $(c_1[n])_n$ and $(c_2[n])_n$ are independent, each data sequence consists of independent and identically distributed zero-mean real Gaussian random variables with variance σ_c^2 for $i = 1, 2$, and the transmit pulses satisfy the relation $p_1(t) = p(t)$ and $p_2(t) = p(t - \Delta)$. In what follows, the variance σ_c^2 and the offset Δ are chosen as $\sigma_c^2 = 0$ for DSB, SSB, and VSB PAMs, $\sigma_c^2 = \sigma_s^2$ and $\Delta = 0$ for QAM, and $\sigma_c^2 = \sigma_s^2$ but $\Delta = T_0/2$ for SQPSK. Let $p(f)$ and $\hat{p}(f)$ be the responses of the p-FRESH vectorizer to the inputs $p(t)$ and $\hat{p}(t - \Delta)$. Then, it can be shown that the PSD matrix $\mathbf{R}_{N_I}(f)$ of $N_I(t)$ is given by

$$\mathbf{R}_{N_I}(f) = \frac{\sigma_c^2}{T_0} \mathbf{p}(f) \mathbf{p}(f)^H + \frac{\sigma_s^2}{T_0} \hat{\mathbf{p}}(f) \hat{\mathbf{p}}(f)^H, \quad (45)$$

under this assumption. Note that $\Delta = 0$ does not imply $\hat{p}(f) = \hat{p}(f)$ in general, because the p-FRESH vectorizer is not a linear but a linear-conjugate linear operator. It can be also shown that $N_I(t)$ is proper-complex if and only if $\mathbf{R}_{N_I}(f)$ is a block diagonal matrix in which each block is $(N_p(f)/2)$ -by- $(N_p(f)/2)$ for all $f \in \mathcal{F}^*$.

A root raised-cosine (RRC) pulse with roll-off factor $\beta \in [0, 1]$ is used as $p(t)$ except the SSB and the VSB modulations, where its filtered version is used instead. In addition to the bandwidth assumption (12a), $H(f)$ is assumed to be unity on the frequency band $[-B_0, B_0]$, where the bandwidth B_0 and the roll-off factor β are related to the cycle period T_0 as

$$B_0 T_0 = \frac{1+\beta}{2}, \quad (46)$$

i.e., $N_I(t)$ as well as $N_A(t)$ fully occupies the entire frequency band of interest. Under this flat channel assumption, it turns out that a result obtained by using the RRC pulse with roll-off factor β is the same as that obtained by using any square root Nyquist pulse with excess bandwidth β . This is because, though the PSD matrix $\mathbf{R}_{N_I}(f)$ in (45) changes, its eigenvalue distribution does not change. For convenience, the passband bandwidth $W \triangleq 2B_0$ is used throughout this section.

The first results are to compare the performance of the CWF with that of the ordinary water filling (OWF), which ignores the spectral correlation in the cyclostationary noise $N(t)$. The interference is generated by using QAM, thus proper-complex, and leads to the power P_I . Then, it can be shown that the capacity of this SOCS proper-complex Gaussian noise channel is given by

$$\begin{aligned} C &= \frac{W\beta}{1+\beta} \log_2 \left(1 + \frac{\min(P, \beta P_I)}{N_0 \frac{W\beta}{1+\beta}} \right) \\ &+ \frac{W}{1+\beta} \log_2 \left(1 + \frac{\frac{[P - \beta P_I]^+}{1+\beta}}{N_0 \frac{W}{1+\beta} + P_I} \right) \text{ [bps]}, \quad (47) \end{aligned}$$

for $0 \leq \beta \leq 1$. This is because, though $N_I(t)$ physically occupies the entire frequency band, it occupies in effect only

Next, we discuss how to handle the cases with unequal frequency bands for signal transmission and reception. Suppose that the frequency bands for signal transmission and reception are overlapping but not identical. Then, we need

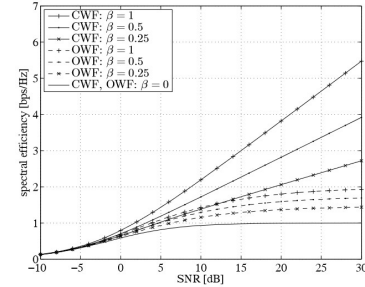


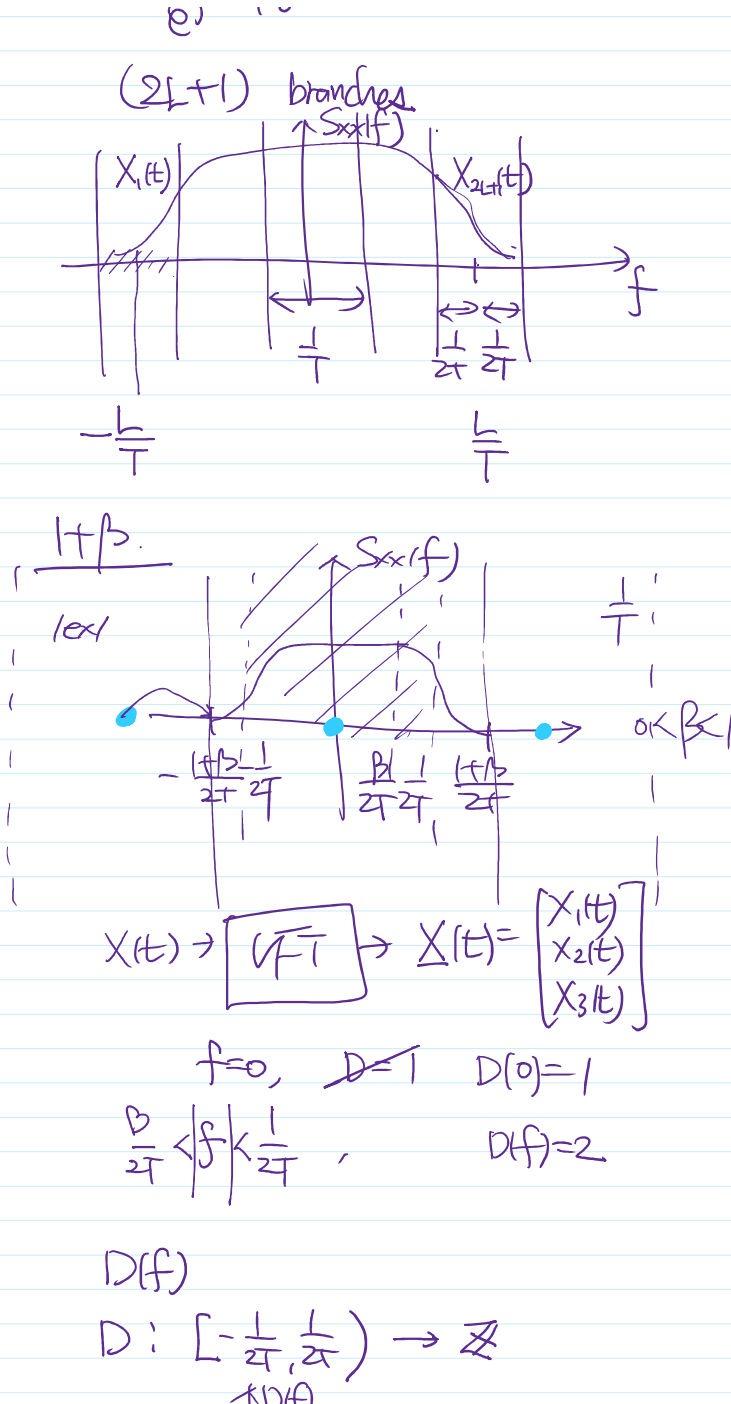
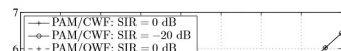
Fig. 6. Comparison of the CWF and the OWF with $\text{SIR} = 0$ dB when a proper-complex Gaussian interference is generated by using QAM. The same performance is observed when an improper-complex Gaussian interference is generated by using SQPSK.

a fraction $1/(W T_0) = 1/(1+\beta)$ of the frequency band and leaves $\beta/(1+\beta)$ unoccupied, as evidenced by the pre-log coefficients in (47). Fig. 6 shows the spectral efficiency C/W as functions of the signal-to-noise ratio (SNR) $P/(N_0 W)$ for various roll-off factors. The signal-to-interference ratio (SIR) P/P_I is fixed at 0 dB. Note that the CWF that exploits the spectral correlation in the cyclostationary interference significantly outperforms the OWF except the case with $\beta = 0$, where $N_I(t)$ degenerates to a proper-complex white Gaussian noise without any spectral correlation. It can be also shown that the improper-complex interference generated by using SQPSK results in the same capacity as (47). This is because, though the PSD matrix changes, its eigenvalue distribution does not change.

The second results are to compare the performance of the CWF with that of the OWF, when the interference is generated by using DSB PAM, thus improper-complex. Since $N_I(t)$ exists only in the real part, the OWF separately processes the real and the imaginary parts. Then, it can be shown that the capacity of this SOCS improper-complex Gaussian noise channel is given by

$$\begin{aligned} C &= \frac{W(1+2\beta)}{2+2\beta} \\ &\times \log_2 \left(1 + \frac{\min(P, (1+2\beta)P_I)}{N_0 \frac{W(1+2\beta)}{2+2\beta}} \right) \\ &+ \frac{W}{2+2\beta} \log_2 \left(1 + \frac{\frac{[P - (1+2\beta)P_I]^+}{2+2\beta}}{N_0 \frac{W}{2+2\beta} + P_I} \right) \text{ [bps]}, \quad (48) \end{aligned}$$

for $0 \leq \beta \leq 1$. This is because $N_I(t)$ occupies in effect only a fraction $1/(2+2\beta)$ of the frequency band and leaves $(1+2\beta)/(2+2\beta)$ unoccupied, as evidenced by the pre-log coefficients in (48). Similar to Fig. 6, Fig. 7 shows the spectral efficiency versus SNR with the SIR fixed at 0 dB. Note again that the CWF significantly outperforms the OWF except the



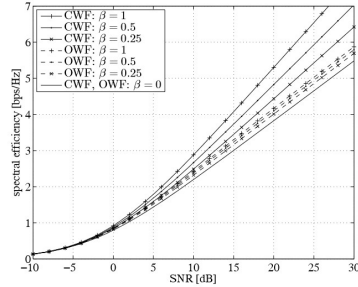


Fig. 7. Comparison of the CWF and the OWF with SIR = 0 dB when an improper-complex Gaussian interference is generated by using DSB or SSB PAM.

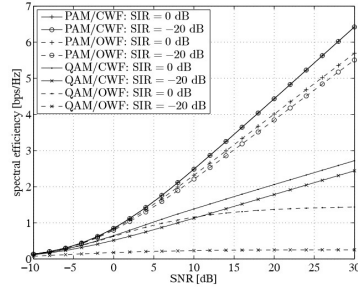


Fig. 8. Comparison of the CWF and the OWF when $\beta = 0.25$ and SIR = 0 and -20 dB.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we have derived the capacity of an SOCS complex Gaussian noise channel. By using the p-FRESH vectorization of the input, the impulse response, and the output of the channel, the SISO channel is converted into an equivalent MIMO channel and its capacity is derived in the frequency domain. A geometric procedure called the CWF is proposed to construct an optimal input distribution that achieves the capacity. It is shown that the CWF significantly outperforms the OWF that ignores the spectral correlation of the SOCS noise. This result can be applied to various interference types of PAM, QAM, and their variants including the WCDMA short-code and IEEE 802.11 DS/SS signals, which have massive spectral redundancies.

Although the CWF achieves the capacity, the knowledge on the center frequency of the SOCS noise is required when the complex noise is improper. Thus, extensions to the cases with an uncertainty in the center frequency or with a channel impairment in the form of phase jittering are interesting future research directions. Deriving the capacity and the capacity-achieving schemes for a channel with a non-Gaussian noise or with multiple spectrally overlapping improper-complex SOCS interferences having different center frequencies is another future work that warrants further investigation. Particularly, in the latter case, the use of the spectral correlation density function [4, Ch. 1] or its variant will be necessary to characterize the interferences in the cycle frequency and the spectral frequency domains. It is also an interesting future research to derive practical signal processing structures that provide the same properizing and stationarizing operation as the p-FRESH vectorizer.

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APPENDIX

A. Proof of Lemma 1

Proof: It is straightforward to show (21a) by using $\mu_Y(t) = 0, \forall t$. To show (21b) and (21c), similar to (13a) and (13b), let $\{R_Y^{(k)}(f)\}_{k \in \mathbb{Z}}$ and $\{\tilde{R}_Y^{(k)}(f)\}_{k \in \mathbb{Z}}$ be the functions such that $R_Y(f, f') = \sum_{k=-\infty}^{\infty} R_Y^{(k)}(f - k/T) \delta(f - f' - k/T)$ and $\tilde{R}_Y(f, f') = \sum_{k=-\infty}^{\infty} \tilde{R}_Y^{(k)}(f - k/T) \delta(f - f' - k/T)$. Then, the cross-correlation between $Y_i(t)$ and $Y_{i'}(s)$ for $i, i' \leq L$ is given by

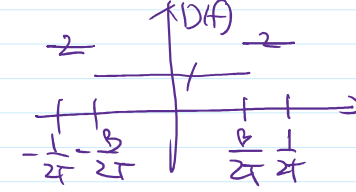
LTI filter output sampled at $t = 0$ is given by

$$\mathbb{E} \left\{ \left\| A(-t)^H * Y(t) \right\|_{t=0}^2 \right\} = \sum_{l=1}^{2L} \sum_{l'=1}^{2L} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_l(\tau)^* [R_Y(\tau, \tau')]_{l,l'} a_{l'}(\tau') d\tau d\tau' \quad (52a)$$

$$= \sum_{l=1}^{2L} \sum_{l'=1}^{2L} \int_{\mathcal{F}^+} A_l(f)^* [R_Y(f)]_{l,l'} A_{l'}(f) df \quad (52b)$$

$$= \int_{\mathcal{F}^+} a(f)^H R_Y(f) a(f) df \geq 0, \quad (52c)$$

$$D: \left[-\frac{1}{2\beta}, \frac{1}{2\beta}\right) \rightarrow \mathbb{R}$$



$$\frac{1}{T} \int_{-\frac{1}{2\beta}}^{\frac{1}{2\beta}} D(f) df = 1 + \beta$$

□ p-FRESH vectorizer

↑
properizing

Q. Show

$\Delta(t)$ is proper!

A. It suffices to show

$$\mathbb{E} \{ \Delta(t) \Delta^H(s) \} = 0 \quad \forall (t, s)$$

□ Why?

$x(t)$ a real-valued signal of finite energy
 $X(f)$ CFT of $x(t)$

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{j2\pi f t} dt \\ &= \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi f t} dt \\ &= \left(\int_{-\infty}^{\infty} x(t) e^{j2\pi f t} dt \right)^* \\ &= \{ X(-f) \}^* \\ &= X^*(-f), \quad \forall f \end{aligned}$$

□ Improper-Complex RP

(13b), let $\{R_Y^{(s)}(f)\}_{k \in \mathbb{Z}}$ and $\{R_Y^{(s)}(f)\}_{k \in \mathbb{Z}}$ be the functions such that $R_Y(f, f') = \sum_{k=-\infty}^{\infty} R_Y^{(k)}(f - k/T) \delta(f - f' - k/T)$ and $\tilde{R}_Y(f, f') = \sum_{k=-\infty}^{\infty} \tilde{R}_Y^{(k)}(f - k/T) \delta(f - f' - k/T)$. Then, the cross-correlation between $Y_l(t)$ and $Y_{l'}(s)$ for $l, l' \leq L$ is given by

$$\mathbb{E}\{Y_l(t)Y_{l'}(s)^*\} = \int_{-B}^B \int_{-B}^B G(f - f_l) e^{j2\pi(f - f_l)t} R_Y(f, f') G(f' - f_{l'})^* e^{-j2\pi(f' - f_{l'})s} df df' \quad (49a)$$

$$= \int_{\mathcal{F}^+} \int_{\mathcal{F}^+} R_Y(f, f') e^{j2\pi(f - f_{l'})t} e^{-j2\pi(f_{l'} - f_{l'})s} df df' \quad (49b)$$

$$= \int_{\mathcal{F}^+} R_Y^{(l-l')}(f + f_{l'}) e^{j2\pi f(t-s)} df, \quad (49c)$$

where Parseval's relation is used twice in (49a) and the identity $f_l - f_{l'} = (l - l')/T$ is used in (49c). In the same way, we have

$$\mathbb{E}\{Y_l(t)Y_{l'}(s)^*\} = \begin{cases} \int_{\mathcal{F}^+} \tilde{R}_Y^{(l-l'+L)}(f + f_{l'-L}) e^{j2\pi f(t-s)} df, & \text{for } l \leq L < l', \\ \int_{\mathcal{F}^+} \tilde{R}_Y^{(l'-l+L)}(-f - f_{l'}) e^{j2\pi f(t-s)} df, & \text{for } l' \leq L < l, \\ \int_{\mathcal{F}^+} \tilde{R}_Y^{(l'-l)}(-f - f_{l'-L}) e^{j2\pi f(t-s)} df, & \text{for } l, l' > L. \end{cases} \quad (50)$$

Thus, the cross-correlation $\mathbb{E}\{Y_l(t)Y_{l'}(s)^*\}$ becomes a function only of $t - s, \forall l, l'$.

On the other hand, the complementary cross-correlation between $Y_l(t)$ and $Y_{l'}(s)$ for $l, l' \leq L$ is given by

$$\mathbb{E}\{Y_l(t)Y_{l'}(s)\} = \int_{-B}^B \int_{-B}^B G(f - f_l) e^{j2\pi(f - f_l)t} \tilde{R}_Y(f, -f') G(f' - f_{l'}) e^{j2\pi(f' - f_{l'})s} df df' \quad (51a)$$

$$= \int_{\mathcal{F}^+} \int_{\mathcal{F}^+} \tilde{R}_Y(f, -f') e^{j2\pi(f - f_l)t} e^{j2\pi(f' - f_{l'})s} df df' \quad (51b)$$

$$= 0. \quad (51c)$$

This is because the impulse fences of $\tilde{R}_Y(f, -f')$ on $f = -f' - m/T$ do not cross the integration area $(f, f') \in \mathcal{F}^+ \times \mathcal{F}^+$. In the same way, we have $\mathbb{E}\{Y_l(t)Y_{l'}(s)\} = 0$ for other cases of l, l' . Thus, the complementary cross-correlation $\mathbb{E}\{Y_l(t)Y_{l'}(s)\}$ becomes zero, $\forall l, \forall l'$. Therefore, the conclusion follows. ■

B. Proof of Lemma 2

Proof: The Hermitian symmetry follows straightforwardly from the element-wise Fourier transform of $\mathbf{r}_Y(\tau) = \mathbb{E}\{\mathbf{Y}(t)\mathbf{Y}(t - \tau)^H\} = \mathbb{E}\{\mathbf{Y}(t - \tau)\mathbf{Y}(t)^H\}^H = \mathbf{r}_Y(-\tau)^H$. Now, consider filtering the p-FRESH vectorizer output $\mathbf{Y}(t)$ by using the LTI filter with impulse response $\mathbf{A}(-t)^H \triangleq \text{diag}\{a_1(-t)^*, a_2(-t)^*, \dots, a_{2L}(-t)^*\}$. Then, the power of the

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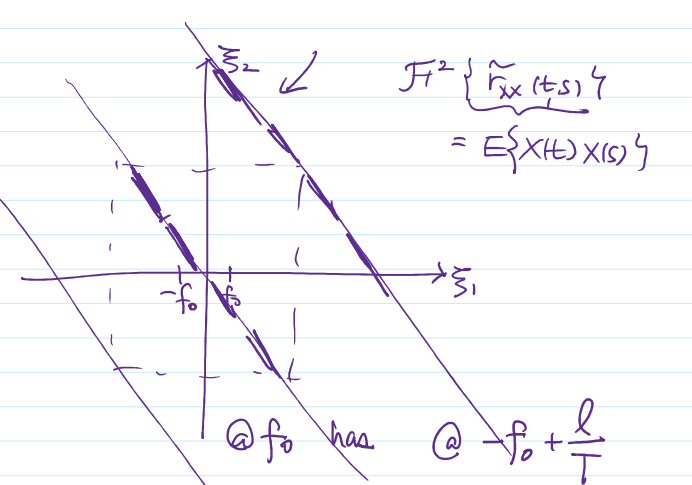
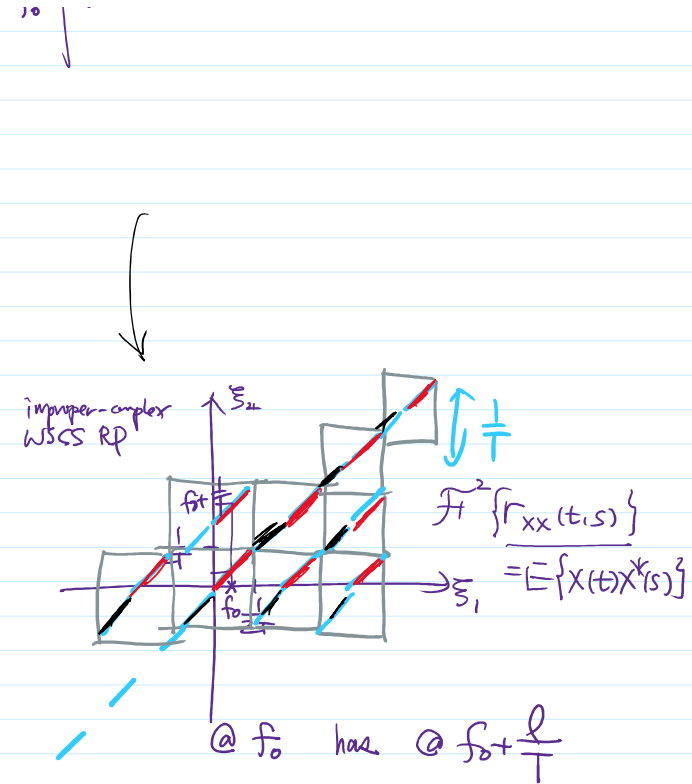
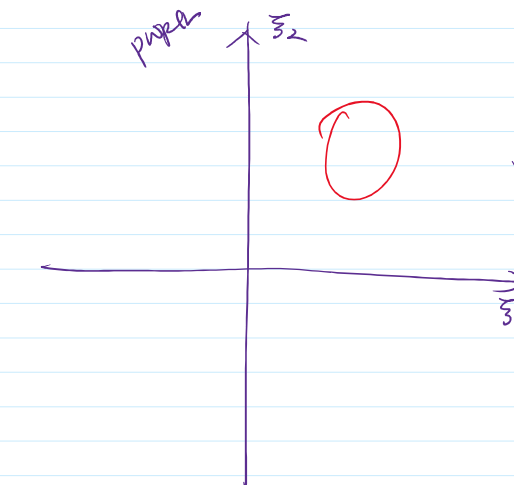
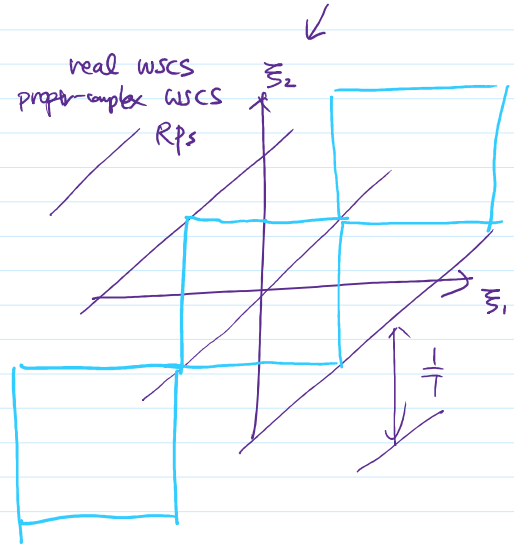
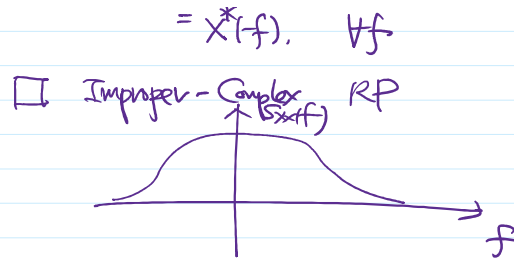
$$\sum_{l=1}^{2L} \sum_{l'=1}^{2L} \int_{\mathcal{F}^+} A_l(f)^* [R_Y(f)]_{l,l'} A_{l'}(f) df \quad (52b)$$

$$= \int_{\mathcal{F}^+} \mathbf{a}(f)^H R_Y(f) \mathbf{a}(f) df \geq 0, \quad (52c)$$

where $\mathbf{a}(f)$ is the $2L$ -by-1 vector of the element-wise Fourier transform of $[a_1(t), a_2(t), \dots, a_{2L}(t)]^T$. Since $a_l(t)$ is arbitrary for all l , (52c) implies $\mathbf{a}(f)^H R_Y(f) \mathbf{a}(f) \geq 0, \forall \mathbf{a}(f), \forall f \in \mathcal{F}^+$. Therefore, the conclusion follows. ■

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Asymptotic Frequency-Shift Properizer for Block Processing of Improper-Complex Second-Order Cyclostationary Random Processes

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Asymptotic Frequency Shift Properizer for Block Processing of Improper-Complex Second-Order Cyclostationary Random Processes

Jeongho Yeo and Joon Ho Cho, Member, IEEE

Abstract—In this paper, the block processing of a discrete-time (DT) improper-complex second-order cyclostationary (SOCS) random process is considered. In particular, it is of interest to find a preprocessing operation that enables the adoption of conventional signal processing techniques and algorithms developed for the filtering of proper-complex signals and that leads to computationally efficient near-optimal postprocessing. An invertible linear-conjugate linear (LCL) operator named the DT frequency shift (FRESH) properizer is first proposed. It is shown that the DT FRESH properizer converts a DT improper-complex SOCS random process input to an equivalent DT proper-complex SOCS random process output by utilizing the information only about the cycle period of the input. An invertible LCL block processing operator named the asymptotic FRESH properizer is then proposed that mimics the operation of the DT FRESH properizer but processes a finite number of consecutive samples of a DT improper-complex SOCS random process. It is shown that the output of the asymptotic FRESH properizer is not proper but asymptotically proper and that its frequency-domain covariance matrix converges to a highly structured block matrix with diagonal blocks as the block size tends to infinity. Two representative estimation and detection problems are presented to demonstrate that asymptotically optimal low-complexity postprocessors can be easily designed by exploiting these asymptotic second-order properties of the output of the asymptotic FRESH properizer.

Index Terms—Asymptotic analysis, complexity reduction, improper-complex random process, properization, second-order cyclostationarity.

I. INTRODUCTION

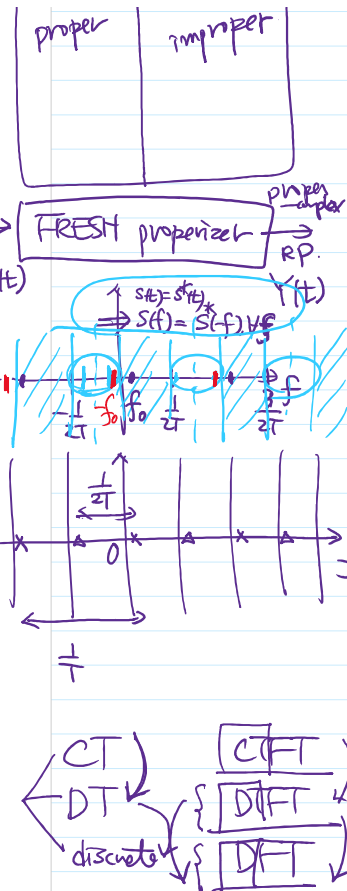
IT IS well known that the complex envelope of a real-valued bandpass wide-sense stationary (WSS) random process is proper, i.e., its complementary auto-covariance (a.k.a. the pseudo-covariance) function vanishes [1]. However, there are a lot of important improper-complex random processes that do not have vanishing complementary auto-covariance functions [2]–[4]. For example, the complex envelope of a real-valued bandpass nonstationary signal is not necessarily proper and even the complex envelope of a real-valued bandpass WSS

signal becomes improper in the presence of the interference between its in-phase and quadrature components [5]. In order to fully capture the statistical properties of a complex-valued signal, including all the second-order statistics, the filtering of augmented signals has been proposed. This so-called widely linear (WL) filtering processes either the signal augmented by its complex conjugate or the real part of the signal augmented by the imaginary part [5], [6], where the former is referred to as the linear-conjugate linear (LCL) filtering [7].

On the other hand, many digitally modulated signals are well modeled by wide-sense cyclostationary (WSCS) random processes [8], [9], whose complex envelopes possess periodicity in all variables of the mean and the auto-covariance functions. In [10], it is shown that a WSCS random process can be represented by an equivalent WSS vector random process. In [11], to efficiently extract the correlation structure of a WSCS random process in the time and the frequency domains, the translation series representation (TSR) and the harmonic series representation (HSR) are proposed, which are based on linear periodically time-varying (PTV) processings. Such second-order structure of a proper-complex WSCS random process has long been exploited in the design of many communications and signal processing systems including presence detectors [12], [13], estimators [14]–[16], and optimal transceivers [17]–[19] under various criteria.

The complex envelopes of the majority of digitally modulated signals are proper and WSCS. However, as well documented in [5], there still remain many other digitally modulated signals such as pulse amplitude modulation (PAM), offset quaternary phase-shift keying (OQPSK), and Gaussian minimum shift keying (GMSK), that are not only improper-complex WSCS but also second-order cyclostationary (SOCS) [20], i.e., the complementary auto-covariance function is also periodic in all its variables with the same period as the mean and the auto-covariance functions. To exploit both cyclostationarity and impropriety, the LCL FREquency Shift (FRESH) filter has been proposed in [21] that combines signal augmentation and linear PTV processing.

Recently, another LCL PTV operator called the properizing FRESH (p-FRESH) vectorizer is proposed in [20] and [22] by non-trivially extending the HSR. The p-FRESH vectorizer converts an *improper-complex* SOCS scalar random process to an equivalent *proper-complex* WSS vector random process by exploiting the frequency-domain correlation and complementary correlation structures that are rigorously examined in [3]. By successfully deriving the capacity of an SOCS Gaussian



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noise channel, it is shown that the optimal channel input is improper-complex SOCS in general when the interfering signal is improper-complex SOCS. It is also well demonstrated that such properization provides the advantage of enabling the adoption of the conventional signal processing techniques and algorithms that utilize only the correlation but not the complementary correlation structure. These results warrant further research in communications and signal processing on the efficient construction and processing of improper-complex SOCS random processes.

In this paper, we consider the block processing of a complex-valued random vector that is obtained by taking a finite, but a very large, number of consecutive samples from a discrete-time (DT) improper-complex SOCS random process. To proceed, the p-FRESH vectorization developed for continuous-time (CT) SOCS random processes is first extended to DT improper-complex SOCS random processes. Instead of straightforwardly modifying the CT p-FRESH vectorizer that properizes as well as vectorizes the input CT improper-complex SOCS random process, an invertible LCL operator is proposed in this paper that does not vectorize but only properizes the input DT improper-complex SOCS random process. Thus, unlike the p-FRESH vectorizer, this LCL operator named the DT FRESH properizer generates a scalar-valued random process. Specifically, the DT FRESH properizer transforms a DT improper-complex SOCS scalar random process into an equivalent proper-complex SOCS scalar random process with the cycle period that is twice the cycle period of the input process. Similar to the p-FRESH vectorizer, the DT FRESH properizer also enables the adoption of the conventional signal processing techniques and algorithms that are already developed for the filtering of proper-complex random processes.

To make this idea of pre-processing by properization better suited for digital signal processing, an invertible LCL block processing operator is then proposed that mimics the operation of the DT FRESH properizer. Although the augmentation of the improper-complex random vector by its complex conjugate can generate a sufficient statistic [5], it is not only a redundant information of twice the length of the original observation vector but also improper. Thus, the augmenting pre-processor does not allow the direct application of the conventional techniques and algorithms dedicated to the block processing of proper-complex random vectors. Motivated by how the DT FRESH properizer works in the frequency domain, the pre-processor proposed in this paper utilizes the centered discrete Fourier transform (DFT) and the information only about the cycle period of the DT improper-complex SOCS random process in order to convert a finite number of consecutive samples of the random process to an equivalent random vector. Unlike the simple augmentation of the improper-complex random vector by its complex conjugate, the output of this LCL block processing operator is not redundant in that it has the same length as its input. Unlike the output of the DT FRESH properizer, the output of this LCL block processing operator is not proper but approximately proper for sufficiently large block size. Thus, the pre-processor is named the asymptotic FRESH properizer. Specifically, the asymptotic FRESH properizer

makes the sequence of the complementary covariance matrices of the output asymptotically equivalent [23] to the sequence of all-zero matrices.

In [24], it is shown that the complex-valued random vector consisting of a finite number of consecutive samples of a DT *proper-complex* SOCS random process has its frequency-domain covariance matrix that approaches a block matrix with diagonal blocks as the number of samples increases. This is because the periodicity in the second-order statistics of the DT SOCS random process naturally leads to a sequence of block Toeplitz covariance matrices as the number of samples increases, the sequence of block Toeplitz matrices is asymptotically equivalent to a sequence of block circulant matrices, and a block circulant matrix becomes a block matrix with diagonal blocks when pre- and post-multiplied by DFT and inverse DFT matrices, respectively. Since the asymptotic FRESH properizer mimics the DT FRESH properizer and the DT FRESH properizer outputs a DT *proper-complex* SOCS random process, it naturally becomes of interest to examine the asymptotic property of the frequency-domain covariance matrix of the asymptotic FRESH properizer output. It turns out that the output of the asymptotic FRESH properizer has the same property discovered in [24].

Such properties of the covariance and the complementary covariance matrices may be used in designing, under various optimality criteria, low-complexity post-processors that follow the asymptotic FRESH properizer. In particular, a post-processor can be developed that approximates the output of the asymptotic FRESH properizer by a proper-complex random vector having the block matrix with diagonal blocks that is asymptotically equivalent to the exact frequency-domain covariance matrix as its frequency-domain covariance matrix. Of course, this technique makes the post-processor suboptimal that performs most of the main operations in the frequency domain. As the two asymptotic properties strongly suggest, however, if the block size is large enough, then the performance degradation may be negligible. It is already shown in [24] that this is the case for the asymptotic property only of the covariance matrix, where a suboptimal frequency-domain equalizer is proposed that approximates the frequency-domain covariance matrix of a proper-complex SOCS interference by an asymptotically equivalent block matrix with diagonal blocks. It turns out that this equalizer not only achieves significantly lower computational complexity than the exact linear minimum mean-square error (LMMSE) equalizer by exploiting the block structure of the frequency-domain covariance matrix, but also is asymptotically optimal in the sense that its average mean-squared error (MSE) approaches that of the LMMSE equalizer as the number of samples tends to infinity.

To demonstrate that asymptotic optimality and low complexity can be simultaneously achievable by employing the asymptotic FRESH properizer, we consider two representative estimation and detection problems. A conventional technique for the estimation and the detection of CT real-valued random processes with finite-time observation is to use the Karhunen-Loève (KL) expansion of the observation [25]. In [26], the estimation and the detection of CT improper-complex random

processes with finite-time observation are considered, and the KL expansion of the augmented observation is proposed. Similarly, the estimation and the detection of DT improper-complex random processes with finite-time observation can be approached by representing the augmented observation vector with a linear combination of the eigenvectors of its correlation matrix. However, as shown in [24] for the equalization in a proper-complex SOCS random process, such a straightforward approach without exploiting the cyclostationarity results in a high computational complexity in general for a large sample size.

First, for a DT improper-complex SOCS random signal in additive proper-complex white noise, a low-complexity signal estimator is proposed that is a linear function of the output of the asymptotic FRESH properizer. It is shown that the average MSE performance of the estimator approaches that of the widely linear minimum mean-squared error (WLMSE) estimator as the number of samples tends to infinity. Second, for a DT improper-complex SOCS Gaussian random signal in additive proper-complex white Gaussian noise, a low-complexity signal presence detector is proposed whose test statistic is a quadratic function of the output of the asymptotic FRESH properizer. It is shown that the test statistic converges to the exact likelihood ratio test (LRT) statistic that is a quadratic function of the augmented observation vector with probability one (w.p. 1) as the number of samples tends to infinity. Note that in both cases the asymptotic FRESH properizer as the pre-processor utilizes only the information about the cycle period. Thus, the adoption of adaptive estimation and detection algorithms may be possible that are developed for the processing of proper-complex random vectors. This advantage of employing the asymptotic FRESH properizer may be taken in other communications and signal processing problems involving the block processing of DT improper-complex SOCS random processes.

As mentioned earlier, the LCL-type augmentation of the signal mainly used in this paper is not the only way to WL filtering. We provide a brief discussion on how the proposed WL block processing can be realized by an alternative WL block processing that utilizes the real part of the signal augmented by the imaginary part. In particular, the same estimation and detection problems are examined now with this real-valued signal representation of the complex signal. The optimal estimator and detector and their computational complexities are derived. It is shown that the all real-valued augmented signal after being properly permuted has a similar asymptotic property in the frequency domain to the LCL-type augmented signal. It is also shown that the redundancy in the augmented signal can be removed as it is done in the asymptotic FRESH properization. By exploiting these properties, we propose asymptotically optimal estimator and detector whose computational complexities are the same as those adopting the asymptotic FRESH properizer as the pre-processor. However, this alternative approach to the block processing of DT improper-complex SOCS random processes may not directly adopt the conventional signal processing techniques and algorithms that are already developed for the filtering of proper-complex random processes. Moreover,

the WL processing using complex-valued representation of the signal provides more mathematical elegance and implementational advantage, as pointed out in [4], [5, Ch. 1], and [6].

The rest of this paper is organized as follows. In Section II, definitions and lemmas related to SOCS random processes are provided and the DT FRESH properizer is proposed. In Section III, the asymptotic FRESH properizer is proposed and the second-order properties of its output are analyzed. In Sections IV and V, the application of the asymptotic FRESH properizer is considered to exemplary estimation and detection problems, respectively. In Section VI, an alternative WL block processing is discussed that utilizes the real part of the signal augmented by the imaginary part. Finally, concluding remarks are offered in Section VII.

Throughout this paper, the operator $E\{\cdot\}$ denotes the expectation, the operator $\mathcal{F}\{x[n]\} \triangleq \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$ denotes the discrete-time Fourier transform (DTFT) of an absolutely summable sequence $(x[n])_n$, and the function $\delta(\cdot)$ denotes the Dirac delta function. The sets $\mathbb{Z}$ and $\mathbb{N}$ are the sets of all integers and of all positive integers, respectively. The operator $\times$ denotes the Cartesian product between two sets. The superscripts $*$, T , and H denote the complex conjugation, the transpose, and the Hermitian transpose, respectively. The operator $*$ denotes the convolution.¹ The operators $\odot$ and $\otimes$ denote the Hadamard product and the Kronecker product, respectively. The matrices $\mathbf{1}_N$, $\mathbf{I}_N$, $\mathbf{O}_N$, and $\mathbf{O}_{M,N}$ denote the N -by- N all-one matrix, the N -by- N identity matrix, the N -by- N all-zero matrix, and the M -by- N all-zero matrix, respectively. The matrix $\mathbf{P}_N$ denotes the N -by- N backward identity matrix whose (m,n) th entry is given by 1 for $m+n = N+1$, and 0 otherwise. The operator $[A]_{m,n}$ and $\text{tr}(A)$ denote the (m,n) th entry and the trace of a matrix A , respectively. The operator $[x]_n$ denotes the n th entry of a vector x . To describe the computational complexity, we will use the big-O notation $\mathcal{O}(g(N))$ defined as $f(N) = \mathcal{O}(g(N))$ if and only if there exist a positive constant M and a real number N_0 such that $|f(N)| \leq M|g(N)|$, $\forall N > N_0$.

II. DT FRESH PROPERIZER

In this section, the notion of DT second-order cyclostationarity is introduced and an LCL PTV operator is proposed that converts a DT improper-complex SOCS random process into an equivalent DT proper-complex SOCS random process. In [20], the notion of CT second-order cyclostationarity is introduced and an LCL PTV operator named p-FRESH vectorizer is proposed that converts an input CT SOCS random process to an equivalent CT proper-complex WSS vector random process. In [27], another LCL PTV operator named CT FRESH properizer is proposed that converts a CT improper-complex SOCS random process into an equivalent CT proper-complex SOCS random process. In this section, the notion of second-order cyclostationarity is extended to DT processes and the DT counterpart of the CT FRESH properizer is also proposed. Note that, unlike the p-FRESH vectorizer, the CT and the DT FRESH properizers do not vectorize but only

¹There should be no confusion from the superscript $*$ that denotes the complex conjugation.

properize the input improper-complex SOCS random process. Similar to the CT FRESH properizer, the DT FRESH properizer as a pre-processor enables the adoption of the conventional signal processing techniques and algorithms that utilize only the correlation but not the complementary correlation structure of the signal.

A. DT SOCS Random Processes

In this subsection, definitions and lemmas related to the DT SOCS random processes are provided.

Definition 1: Given a DT complex-valued random process $X[n]$ with a finite power, i.e., $E\{|X[n]|^2\} < \infty, \forall n$, the mean, the auto-correlation, and the complementary auto-correlation functions of $X[n]$ are defined, respectively, as

$$\mu_X[n] \triangleq E\{X[n]\}, \quad (1a)$$

$$r_X[n, m] \triangleq E\{X[n]X[m]^*\}, \quad (1b)$$

$$\tilde{r}_X[n, m] \triangleq E\{X[n]X[m]\}. \quad (1c)$$

The auto-covariance and the complementary auto-covariance functions can be defined straightforwardly. Throughout this paper, all DT complex-valued random processes are assumed to be of finite power, i.e., $|r_X[n, m]| \leq E\{|X[n]|^2\} < \infty, \forall n, \forall m$.

Definition 2: The spectral correlation function $R_X(f, f')$ and the complementary spectral correlation function $\tilde{R}_X(f, f')$ of a DT complex-valued random process $X[n]$ are defined as

$$R_X(f, f') \triangleq \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} r_X[n, m] e^{j2\pi(fn - f'm)} \quad (2a)$$

and

$$\tilde{R}_X(f, f') \triangleq \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{r}_X[n, m] e^{j2\pi(fn + f'm)}, \quad (2b)$$

respectively, if they exist.

The spectral correlation function has also been referred to as bifrequency spectral correlation function [28], cointensity spectrum [29], generalized spectrum [30], and Loève bifrequency spectrum [31]. Similarly, the complementary spectral correlation function has also been referred to as complementary Loève spectrum [32].

Since the spectral correlation function and the complementary spectral correlation function are the DT double Fourier transforms of $r_X[n, m]$ and $\tilde{r}_X[n, m]$, respectively, they are always periodic in both variables f and f' with the common period 1. The set of all DT complex-valued random processes can be partitioned into two subsets by using the following definition.

Definition 3: [1, Definition 2] A DT complex-valued random process $X[n]$ is proper if its complementary auto-correlation function vanishes, i.e., $\tilde{r}_X[n, m] = 0, \forall n, \forall m$, and is improper otherwise.

Two types of stationarity can be defined as follows by using the second-order moments of a DT complex-valued random process.

Definition 4: [33, Sec. II-B] A DT complex-valued random process $X[n]$ is second-order stationary (SOS) if, $\forall m, \forall n$,

$$\mu_X[n] = \mu_X[0], \quad (3a)$$

$$r_X[n, m] = r_X[n - m, 0], \quad (3b)$$

$$\tilde{r}_X[n, m] = \tilde{r}_X[n - m, 0]. \quad (3c)$$

Definition 5: A DT complex-valued random process $X[n]$ is SOCS with cycle period $M \in \mathbb{N}$ if, $\forall n, \forall m$,

$$\mu_X[n] = \mu_X[n + M], \quad (4a)$$

$$r_X[n, m] = r_X[n + M, m + M], \quad (4b)$$

$$\tilde{r}_X[n, m] = \tilde{r}_X[n + M, m + M]. \quad (4c)$$

We are mainly interested in the DT SOCS random processes in this paper. Note that a DT SOS random process can be viewed as a special DT SOCS random process with cycle period 1. Note also that the above definition of a DT SOCS random process is a straightforward extension of the definition of a CT SOCS random process in [20]. For the ease of comparison with the results in [20], we use the time indexes m and n in the orders appearing in Definitions 1–5.

In the following lemmas, the implications of the second-order cyclostationarity are provided in the time and the frequency domains, respectively.

Lemma 1: For a DT SOCS random process $X[n]$ with cycle period $M \in \mathbb{N}$, there exist $(r_X^{(k)})_{k=0}^{M-1}$ and $(\tilde{r}_X^{(k)})_{k=0}^{M-1}$ such that

$$r_X[n, m] = \sum_{k=0}^{M-1} r_X^{(k)}[n - m] e^{j2\pi kn/M}, \quad (5a)$$

and

$$\tilde{r}_X[n, m] = \sum_{k=0}^{M-1} \tilde{r}_X^{(k)}[n - m] e^{j2\pi kn/M}. \quad (5b)$$

Proof: Since $r_X^{(k)}[n, l] \triangleq r_X[n, n - l]$ is periodic in n with period M and is finite, there exist the DT Fourier series coefficients $(r_X^{(k)})_{k=0}^{M-1}$ for each l such that $r_X^{(k)}[n, l] = \sum_{m=0}^{M-1} r_X^{(k)}[l] e^{j2\pi km/M}$. By replacing l with $n - m$, we obtain (5a). Similarly, we obtain (5b). Therefore, the conclusion follows. $\square$

Lemma 2: For a DT SOCS random process $X[n]$ with cycle period $M \in \mathbb{N}$, the spectral correlation function and the complementary spectral correlation function are given, respectively, by

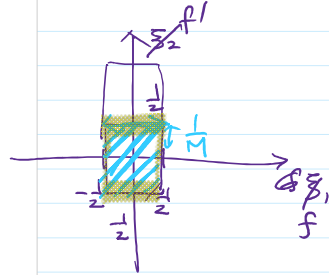
$$R_X(f, f') = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{M-1} R_X^{(k)}\left(f - \frac{k}{M}\right) \delta\left(f - f' - \frac{k}{M} - l\right) \quad (6a)$$

and

$$\tilde{R}_X(f, f') = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{M-1} \tilde{R}_X^{(k)}\left(f - \frac{k}{M}\right) \delta\left(f + f' - \frac{k}{M} - l\right), \quad (6b)$$

where $R_X^{(k)}(f) \triangleq \mathcal{F}\{r_X^{(k)}[n]\}$ and $\tilde{R}_X^{(k)}(f) \triangleq \mathcal{F}\{\tilde{r}_X^{(k)}[n]\}$.

Proof: It is straightforward by applying the definitions (2a) and (2b) in Definition 2, respectively, to (5a) and (5b) in Lemma 1. $\square$



$$x[n] \\ X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn} \\ e^{j2\pi(f+f_0)n} = e^{j2\pi f n} \text{ if } f_0 \in \mathbb{Z}.$$

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YEO AND CHO: ASYMPTOTIC FREQUENCY-SHIFT PROPERIZER

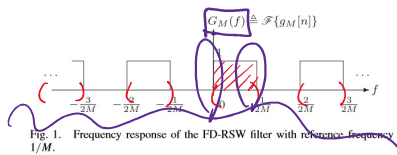
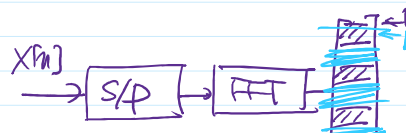
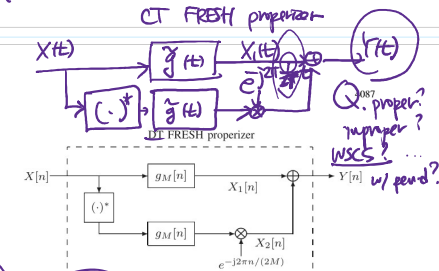


Fig. 1. Frequency response of the FD-RSW filter with reference frequency $1/M$.



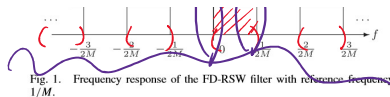


Fig. 1. Frequency response of the FD-RSW filter with reference frequency $1/M$.

B. DT FRESH Properizer

In this subsection, an LCL PTV operator is proposed that always outputs an equivalent DT proper-complex random process regardless of the propriety of the input DT SOCS random process, under the assumption that the parameter called the reference frequency of the operator satisfies a certain condition. In what follows, all SOCS random processes are DT processes unless otherwise specified.

To proceed, the following frequency-selective filter is defined that has half of the entire frequency band as its stopband.

Definition 6: Given a reference frequency $1/M$, a linear time-invariant system with impulse response $g_M[n]$ is called the frequency-domain raised square wave (FD-RSW) filter if its frequency response $G_M(f) = \mathcal{F}\{g_M[n]\}$ is given by

$$G_M(f) \triangleq \begin{cases} 0, & \text{for } -\frac{1}{2M} \leq f < 0, \\ 1, & \text{for } 0 \leq f < \frac{1}{2M}, \end{cases} \quad (7)$$

and satisfying $G_M(f) \triangleq G_M(f + 1/M)$.

Fig. 1 shows the frequency response of the FD-RSW filter with reference frequency $1/M$, which alternately passes the frequency components of the input signal in every other interval of bandwidth $1/(2M)$. Hereafter, G_M denotes the support of this FD-RSW filter. By using the impulse response $g_M[n]$ of the FD-RSW filter with reference frequency $1/M$, we can define an LCL PTV filter as follows.

Definition 7: Given a reference frequency $1/M$ and an input $X[n]$, the DT FRESH properizer is defined as a single-input single-output LCL PTV system, whose output is given by

$$Y[n] \triangleq X[n] * g_M[n] + (X[n]^* * g_M[n]) e^{-j2\pi n/(2M)}. \quad (8)$$

Fig. 2 shows how the DT FRESH properizer works in the time domain, where $X[n]$ is the input, $Y[n]$ is the output, and $X_1[n]$ and $X_2[n]$ are, respectively, the first and the second terms on the right side of (8). Note that, unlike the upper branch where $X[n]$ is processed by the FD-RSW filter to generate $X_1[n]$, $X[n]^*$ is processed in the lower branch by the FD-RSW filter and multiplied by a complex-exponential function $e^{-j2\pi n/(2M)}$ to generate $X_2[n]$. Even though $-1/(2M)$ is chosen as the frequency of the complex-exponential function, any $(2k+1)/(2M)$ for $k \in \mathbb{Z}$ can be chosen. The reason why

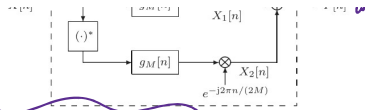


Fig. 2. DT FRESH properizer with reference frequency $1/M$ viewed in the time domain.

this is so becomes clear once the DT FRESH properizer is viewed in the frequency domain.

The following lemma makes this invertibility argument more precise.

Lemma 3: The DT FRESH properizer with reference frequency $1/M$ is identical to its inverse mapping, i.e.,

$$X[n] = Y[n] * g_M[n] + (Y[n]^* * g_M[n]) e^{-j2\pi n/(2M)}, \quad (9)$$

where $Y[n]$ is the output of the DT FRESH properizer with reference frequency $1/M$ when its input is $X[n]$.

Proof: Straightforward by substituting (8) into the right side of (9). $\square$

Note that, at this point, the DT FRESH properizer may not be more than one of many possible invertible operators. The reason why this operator is named the properizer will become clear once the second-order property of the output is analyzed as follows when its input is a zero-mean SOCS random process. In [27], similar property of band-limited CT SOCS random processes and its proof can be found. Thus, we omit the detailed proof of the following theorem, which can be straightforwardly obtained by substituting CT parameters in [27] with DT parameters.

Theorem 1: If the input $X[n]$ to the DT FRESH properizer with reference frequency $1/M$ is a zero-mean SOCS random process with cycle period M , then the output $Y[n]$ becomes a zero-mean proper-complex SOCS random process with cycle period $2M$, i.e., the mean, the auto-correlation, and the complementary auto-correlation functions of $Y[n]$ satisfy

$$\mu_Y[n] \triangleq \mathbb{E}\{Y[n]\} = 0, \quad (10a)$$

$$r_Y[n, m] \triangleq \mathbb{E}\{Y[n]Y[m]^*\} = r_Y[n+2M, m+2M], \quad (10b)$$

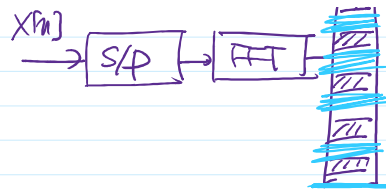
$$\tilde{r}_Y[n, m] \triangleq \mathbb{E}\{Y[n]Y[m]\} = 0, \quad (10c)$$

$\forall m, \forall n$, respectively.

Proof: See the proof of [27, Proposition 1]. $\square$

It is already shown that the amount of the frequency shift in the second term of (8) can be any $(2k+1)/(2M)$, for $k \in \mathbb{Z}$, to satisfy the invertibility of the DT FRESH properizer. It can be also shown that Theorem 1 holds for any frequency shift $(2k+1)/(2M)$, for $k \in \mathbb{Z}$. Moreover, since any integer multiple of M is also a cycle period of an SOCS random process with cycle period M , the random process can be FRESH properized by using any reference frequency $1/(kM)$, $\forall k \in \mathbb{N}$. Thus, it is not unique to FRESH properize an SOCS random process.

This theorem shows that the DT FRESH properizer in general doubles the cycle period at the cost of the propriety



ex

$$X(t) = \sum_{n=-\infty}^{\infty} b_n p(t-nT) e^{j\omega n}$$

$b_n \in \{+1, -1\}$
BPSK signal is proper
complex RP w/
cycle period T . $\downarrow$ proper
cycle period $2T$

Q $X(t) \rightarrow \text{FRESH-p} \rightarrow Y(t)$
cycle period T cycle period $2T$

of the output. However, it does not double the cycle period if the input is already proper.

Corollary 1: If the input $X[n]$ to the DT FRESH properizer with reference frequency $1/M$ is a zero-mean proper-complex SOCS random process with cycle period M , then the output $Y[n]$ is a zero-mean proper-complex SOCS random process with cycle period M .

Proof: See the proof of [27, Corollary 1]. Due to the same reason as Theorem 1, we again omit the detailed proof. $\square$

Theorem 1 also has the following consequence.

Corollary 2: An optimal linear processing of the output of the DT FRESH properizer always leads to the same result as the optimal widely linear processing of the input of the DT FRESH properizer does.

Proof: Let $X[n]$ and $Y[n]$ be the input and the output of the DT FRESH properizer with relation (7) and (3). Because the DT FRESH properization is invertible as shown in Lemma 3, the performance of an optimal processing of $X[n]$ always equal to that of the processing of $Y[n]$ under the same optimality criterion [25, pp. 289–290]. Thus, an optimal widely linear processing of $X[n]$ is equivalent to the widely linear processing of $Y[n]$ under the same criterion. Therefore, the conclusion follows by the propriety of $Y[n]$ shown in Theorem 1. $\square$

It is noteworthy that Theorem 1 does not imply that any FRESH properizer can properize an improper SOCS random process with cycle period M .

Corollary 3: If $k_1 M = k_2 M'$ for integers k_1 and k_2 where k_1 is not divisible by k_2 , then the output of the DT FRESH properizer with reference frequency $1/M'$ becomes an improper-complex SOCS random process with cycle period $2k_1 M$.

Proof: It can be straightforwardly shown by using the CT version proof of [27, Th. 1], where we just need to replace the reference frequency of the FRESH properizer by $k_2/(k_1 M)$. $\square$

In theory, the DT FRESH properizer is enough to deal with improper-complex SOCS random processes by using conventional linear processing techniques and algorithms. However, the FD-RSW filter is not realizable in practice due to its shape in the frequency domain.

III. ASYMPTOTIC FRESH PROPERIZER

In this section, the block processing of an SOCS random process is considered. Motivated by how the DT FRESH properizer works in the frequency domain, an LCL block operator is proposed that converts a finite number of consecutive samples of an SOCS random process to an equivalent random vector. Unlike the DT FRESH properizer proposed in the previous section, this invertible operator does not directly make the complementary covariance matrix of the output vector vanish. Instead, it is shown that the LCL operator makes the complementary covariance matrix of the output vector approach all-zero matrix as the number of samples tends to infinity. This is why it is named the asymptotic FRESH properizer.

A. Asymptotic FRESH Properizer and Its Inverse Operator

Let $\mathbf{x}$ be the length- MN vector obtained by taking the MN consecutive samples of a DT signal, where and in what follows it is assumed that N is a positive even number. Motivated by the DT FRESH properizer, we introduce in this subsection an LCL block operator and its inverse.

To proceed, some definitions are provided.

Definition 8: The centered DFT matrix $\mathbf{W}_{MN}$ is defined as an (MN) -by- (MN) matrix whose (m, n) th entry, for $m, n \in \{1, \dots, MN\}$, is given by

$$[\mathbf{W}_{MN}]_{m,n} \triangleq \frac{1}{\sqrt{MN}} e^{-j2\pi(m-c_{MN})(n-c_{MN})/(MN)}, \quad (11)$$

where $c_{MN} \triangleq (MN+1)/2$.

It is well known that the matrix-vector multiplication with a centered DFT matrix can be implemented with low computational complexity [34], as the multiplication with an ordinary DFT matrix is efficiently implemented by using the fast Fourier transform algorithm.

Definition 9: Given M and an even number N , the (MN) -by- (MN) matrix $\mathbf{G}_{M,N}$ is defined as

$$\mathbf{G}_{M,N} \triangleq \mathbf{I}_M \otimes \begin{bmatrix} \mathbf{O}_{N/2} & \mathbf{O}_{N/2} \\ \mathbf{O}_{N/2} & \mathbf{I}_{N/2} \end{bmatrix}. \quad (12)$$

Similar to the FD-RSW pulse, the matrix $\mathbf{G}_{M,N}$ will be called the raised square wave (RSW) matrix because, when pre-multiplied to a column vector or a matrix, it turns the $((m-1)N+n)$ th row, for $m = 1, \dots, M$ and $n = 1, \dots, N/2$, into all zeros, i.e., it alternately nulls every other band of $N/2$ consecutive rows.

Definition 10: Given M and an even number N , the (MN) -by- (MN) matrix $\mathbf{S}_{M,N}$ is defined as

$$\mathbf{S}_{M,N} \triangleq \begin{bmatrix} \mathbf{O}_{MN-N/2, N/2} & \mathbf{I}_{MN-N/2} \\ \mathbf{I}_{N/2} & \mathbf{O}_{N/2, MN-N/2} \end{bmatrix}. \quad (13)$$

Note that the matrix $\mathbf{S}_{M,N}$, when pre-multiplied to a column vector or a matrix, circularly shifts the rows by $N/2$, which corresponds to multiplying $e^{-j2\pi n/(2M)}$ in the second term of the right side of (8). Now, we are ready to introduce an LCL operator that is the block-processing counterpart to the DT FRESH properizer.

Definition 11: Given M and an even number N , the LCL operator $\mathbf{f}$ with input $\mathbf{x}$ and output $\mathbf{y} = \mathbf{f}(\mathbf{x})$, both of length MN , is called the asymptotic FRESH properizer if

$$\mathbf{f}(\mathbf{x}) \triangleq \mathbf{W}_{MN}^H (\mathbf{G}_{M,N} \mathbf{W}_{MN} \mathbf{x} + \mathbf{S}_{M,N} \mathbf{G}_{M,N} \mathbf{W}_{MN} \mathbf{x}^*). \quad (14)$$

Note that the input $\mathbf{x}$ is pre-multiplied by the centered DFT matrix $\mathbf{W}_{MN}$ and the RSW matrix $\mathbf{G}_{M,N}$, while the complex conjugate $\mathbf{x}^*$ is multiplied additionally by the circular shift matrix $\mathbf{S}_{M,N}$ to generate the frequency-domain output $\mathbf{W}_{MN} \mathbf{y}$.

Fig. 3 shows $\mathbf{W}_{MN} \mathbf{x}$, $\mathbf{G}_{M,N} \mathbf{W}_{MN} \mathbf{x}$, and $\mathbf{S}_{M,N} \mathbf{G}_{M,N} \mathbf{W}_{MN} \mathbf{x}^*$, when the l th entry of the $\mathbf{W}_{MN} \mathbf{x}$ is denoted by x_l^f . Note that, similar to the DT FRESH properization, the locations of all possible non-zero rows of $\mathbf{G}_{M,N} \mathbf{W}_{MN} \mathbf{x}$ and $\mathbf{S}_{M,N} \mathbf{G}_{M,N} \mathbf{W}_{MN} \mathbf{x}^*$ do not overlap, which makes $\mathbf{W}_{MN} \mathbf{y}$ contain all the entries of $\mathbf{W}_{MN} \mathbf{x}$ without any distortion. Note also that the amount of the circular shift of $\mathbf{S}_{M,N} \mathbf{G}_{M,N} \mathbf{W}_{MN} \mathbf{x}^*$ by any kN for $k \in \mathbb{Z}$ generates the signal

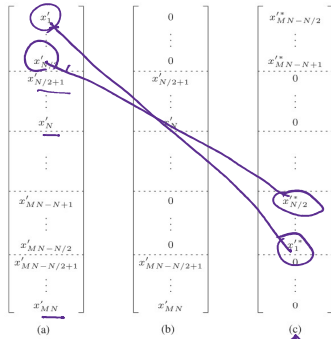


Fig. 3. Illustration that shows how the asymptotic FRESH properizer works in the frequency domain of W_{MN}^H . (b) $G_{M,N}W_{MN}^H$. (c) $S_{M,N}G_{M,N}W_{MN}^H$.

that contains the same information as $S_{M,N}G_{M,N}W_{MN}^H x^*$ does. The following lemma makes this invertibility argument more precise.

Lemma 4: From the output y of the asymptotic FRESH properizer with parameters M and N , the input x of the asymptotic FRESH properizer can be recovered as

$$x = W_{MN}^H G_{M,N} W_{MN} y \quad (15a)$$

$$\triangleq f^{-1}(y) \quad (15b)$$

Proof: By substituting (14) into the right side of (15a), we have $W_{MN}^H G_{M,N}^2 W_{MN} x + W_{MN}^H G_{M,N} S_{M,N} G_{M,N} W_{MN} x^* + W_{MN}^H P_{MN} G_{M,N} S_{M,N}^T G_{M,N} W_{MN} x^* + W_{MN}^H P_{MN} G_{M,N} S_{M,N}^T S_{M,N} G_{M,N} W_{MN} x$. The second and the third terms vanish since $G_{M,N} S_{M,N} G_{M,N} = O_{MN}$ and $G_{M,N} S_{M,N}^T G_{M,N} = O_{MN}$, respectively. Thus, the right side of (15a) becomes $W_{MN}^H (G_{M,N}^2 + P_{MN} G_{M,N}^2 P_{MN}) W_{MN} x$. Moreover, it can be easily shown that $G_{M,N}^2 = G_{M,N}$ and $G_{M,N} + P_{MN} G_{M,N} P_{MN} = I_{MN}$, because $P_{MN} = P_M \otimes P_N$ by the properties of the Kronecker product. Therefore, the conclusion follows. $\square$

Note that, at this point, the asymptotic FRESH properizer may not be more than one of many possible invertible operators. The reason why this operator is named the asymptotic properizer will become clear once the second-order property of the output is analyzed in the next subsection when its input is a finite number of consecutive samples of a zero-mean SOCS random process.

B. Second-Order Properties of Output of Asymptotic FRESH Properizer

Let $\tilde{x}$ be the length- $2MN$ augmented vector defined as

$$\tilde{x} \triangleq \begin{bmatrix} x \\ x^* \end{bmatrix} \quad (16)$$

where and in what follows it is assumed that x consists of a finite number of consecutive samples of a zero-mean SOCS random process $X[n]$ with cycle period $M \in \mathbb{N}$. Then, the output $y = f(x)$ in (14) of the asymptotic FRESH properizer can be rewritten as

$$y = W_{MN}^H \tilde{G}_{M,N} \tilde{W}_{MN} \tilde{x} \quad (17a)$$

$$\triangleq W_{MN}^H \tilde{y}, \quad (17b)$$

where the (MN) -by- $(2MN)$ matrix $\tilde{G}_{M,N}$ and the $(2MN)$ -by- $(2MN)$ matrix $\tilde{W}_{MN}$ are given by

$$\tilde{G}_{M,N} \triangleq [G_{M,N} \ S_{M,N} G_{M,N}] \quad (18a)$$

and

$$\tilde{W}_{MN} \triangleq \begin{bmatrix} W_{MN} & O_{MN} \\ O_{MN} & W_{MN} \end{bmatrix}, \quad (18b)$$

respectively, and $\tilde{y}$ is the centered DFT of y . Thus, the covariance matrix $R_y \triangleq E\{yy^H\}$ and the complementary covariance matrix $\tilde{R}_y \triangleq E\{\tilde{y}\tilde{y}^H\}$ of y are given by

$$R_y = W_{MN}^H \tilde{G}_{M,N} \tilde{W}_{MN} R_{\tilde{x}} \tilde{W}_{MN}^H \tilde{G}_{M,N}^H W_{MN} \quad (19a)$$

and

$$\tilde{R}_y = W_{MN}^H \tilde{G}_{M,N} \tilde{W}_{MN} \tilde{R}_{\tilde{x}} \tilde{W}_{MN}^H \tilde{G}_{M,N}^H W_{MN}^*, \quad (19b)$$

respectively, where $R_{\tilde{x}} \triangleq E\{\tilde{x}\tilde{x}^H\}$ and $\tilde{R}_{\tilde{x}} \triangleq E\{\tilde{x}\tilde{x}^T\}$ denote the covariance and the complementary covariance matrices of the augmented vector $\tilde{x}$, respectively.

Throughout this paper, we call

$$R_{\tilde{y}} \triangleq E\{\tilde{y}\tilde{y}^H\} = W_{MN} R_y W_{MN}^H \quad (20a)$$

and

$$\tilde{R}_{\tilde{y}} \triangleq E\{\tilde{y}\tilde{y}^T\} = W_{MN} \tilde{R}_y W_{MN}^T \quad (20b)$$

the frequency-domain covariance and the frequency-domain complementary covariance matrices, respectively.

To analyze the asymptotic second-order properties of y , we briefly review definitions and related lemmas for asymptotic equivalence between two sequences of matrices.

Definition 12 [23]: Let $(A_k)_k$ and $(B_k)_k$ be two sequences of N_k -by- N_k matrices with $N_k \rightarrow \infty$ as $k \rightarrow \infty$. Then, $(A_k)_k$ and $(B_k)_k$ are asymptotically equivalent and denoted by $A_k \sim B_k$ if 1) the strong norms of A_k and B_k are uniformly bounded, i.e., there exists a constant c such that $\|A_k\|, \|B_k\| \leq c < \infty, \forall k$, and 2) the weak norm of $A_k - B_k$ vanishes asymptotically, i.e., $\lim_{k \rightarrow \infty} |A_k - B_k| = 0$, where the strong norm $\|A\|$ and the weak norm $|A|$ of an N_k -by- N_k matrix A are defined as $\|A\| \triangleq \max_{x \neq 0} \sqrt{x^H A^H A x} / x^H x$ and $|A| \triangleq \sqrt{\text{tr}(A A^H)} / N_k$, respectively.

Lemma 5: If two sequences $(A_k)_k$ and $(B_k)_k$ of N_k -by- N_k matrices are asymptotically equivalent and if the strong norms of N_k -by- N_k matrices C_k and D_k are uniformly bounded, then $C_k A_k D_k \sim C_k B_k D_k$.

Proof: Since $C_k \sim C_k$ and $D_k \sim D_k$, it is straightforward to show the conclusion by applying [23, Th. 1-(3)] twice. $\square$

Lemma 6: If $A_k \sim B_k$ and $C_k \sim D_k$, then $(A_k + C_k) \sim (B_k + D_k)$.

Proof: By applying the triangle inequality, it is straightforward to show that the strong norms of $A_k + C_k$ and

$B_k + D_k$ are uniformly bounded and that the weak norm of $A_k + C_k - B_k - D_k$ vanishes asymptotically. Therefore, the conclusion follows. $\square$

For brevity, in what follows, two square matrices A_k and B_k of the same size will be said to be asymptotically equivalent if $A_k \sim B_k$. Now, we introduce our definition of asymptotic propriety.

Definition 13: A sequence $(y_k)_k$ of length- N_k complex-valued random vectors is asymptotically proper if the complementary covariance matrix of y_k is asymptotically equivalent to the N_k -by- N_k all-zero matrix.

For brevity, a random vector y_k will be said to be asymptotically proper if the sequence $(y_k)_k$ is asymptotically proper. Before showing that the output y of the asymptotic FRESH properizer is asymptotically proper, the second-order properties of the augmented vector $\tilde{x}$ are examined as follows.

Proposition 1: If the random vector x is obtained by taking the MN consecutive samples from a zero-mean SOCS random process with cycle period M , then the covariance matrix R_x and the complementary covariance matrix $\tilde{R}_x$ of the augmented vector $\tilde{x}$ defined in (16) satisfy

$$\tilde{W}_{MN} R_x \tilde{W}_{MN}^H \sim \Omega \quad (21a)$$

and

$$\tilde{W}_{MN} \tilde{R}_x \tilde{W}_{MN}^T \sim \tilde{\Omega}, \quad (21b)$$

respectively, where the $(2MN)$ -by- $(2MN)$ matrices Ω and $\tilde{\Omega}$ are given by

$$\Omega \triangleq (\tilde{W}_{MN} R_x \tilde{W}_{MN}^H) \odot (I_{2M} \otimes I_N) \quad (22a)$$

and

$$\tilde{\Omega} \triangleq (\tilde{W}_{MN} \tilde{R}_x \tilde{W}_{MN}^T) \odot (I_{2M} \otimes P_N), \quad (22b)$$

respectively.

Proof: By definition, the left sides of (21a) and (21b) can be rewritten, respectively, as

$$\tilde{W}_{MN} R_x \tilde{W}_{MN}^H = \left[\begin{array}{c|c} W_{MN} R_x W_{MN}^H & W_{MN} R_x W_{MN}^H \\ \hline W_{MN} R_x^* W_{MN}^H & W_{MN} R_x^* W_{MN}^H \end{array} \right] \quad (23a)$$

and

$$\tilde{W}_{MN} \tilde{R}_x \tilde{W}_{MN}^T = \left[\begin{array}{c|c} W_{MN} \tilde{R}_x W_{MN}^T & W_{MN} R_x W_{MN}^T \\ \hline W_{MN} R_x^* W_{MN}^T & W_{MN} \tilde{R}_x^* W_{MN}^T \end{array} \right]. \quad (23b)$$

Since both R_x and $\tilde{R}_x$ are (MN) -by- (MN) block Toeplitz matrices with block size M -by- M , each submatrix on the right side of (23a) is asymptotically equivalent to the Hadamard product of $(I_M \otimes I_N)$ and the submatrix itself as shown in [24, Proposition 1]. Thus, by using the fact that $A \sim B$ if each submatrix of A is asymptotically equivalent to the corresponding submatrix of B [24, Lemma 5], we obtain (21a). Similar to (21a), it can be shown by using the property $W_{MN}^T = W_{MN}^H P_{MN}$ of the centered DFT matrix that each submatrix on the right side of (23b) is asymptotically equivalent to Hadamard product of $(I_M \otimes P_N)$ and the submatrix itself. Thus, we obtain (21b). Therefore, the conclusion follows. $\square$

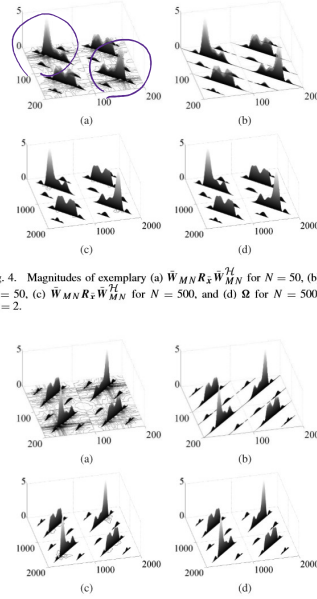


Fig. 4. Magnitudes of exemplary $\tilde{W}_{MN} R_x \tilde{W}_{MN}^H$ for $N = 50$, (b) for $N = 50$, (c) $\tilde{W}_{MN} R_x \tilde{W}_{MN}^H$ for $N = 500$, and (d) Ω for $N = 500$, when $M = 2$.

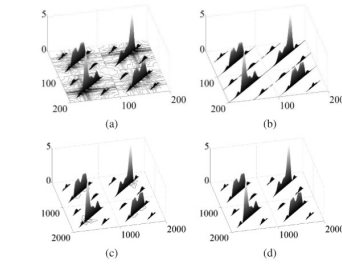
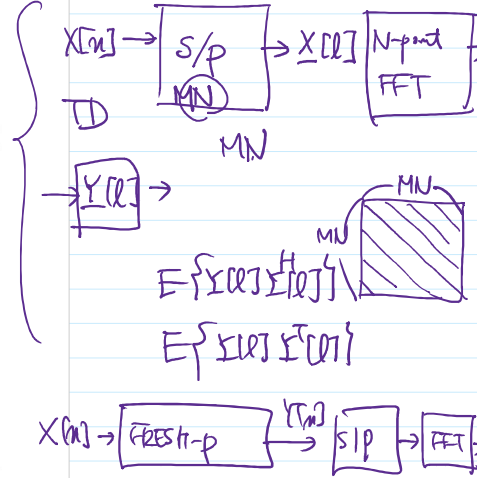


Fig. 5. Magnitudes of exemplary $\tilde{W}_{MN} \tilde{R}_x \tilde{W}_{MN}^T$ for $N = 50$, (b) for $N = 50$, (c) $\tilde{W}_{MN} \tilde{R}_x \tilde{W}_{MN}^T$ for $N = 500$, and (d) $\tilde{\Omega}$ for $N = 500$, when $M = 2$.

Figs. 4 and 5 show the entry-by-entry magnitudes of the exemplary frequency-domain covariance and complementary covariance matrices. To generate a zero-mean improper-complex SOCS random process, equally-likely BPSK symbols are linearly modulated with the square-root raised cosine (SRRC) pulse having roll-off factor 0.5 and transmitted at the symbol rate of $1/T_s$ [symbols/sec]. After the BPSK signal passes through the frequency selective channel with impulse response $h(t) = 0.22e^{j\pi/3}\delta(t - T_s) + 0.44e^{j\pi/2}\delta(t - 3T_s) + 0.87e^{j\pi/6}\delta(t - 5T_s)$, the SOCS random process $X[n]$ is obtained by 2-times over-sampling the received signal, i.e., the sampling rate is $2/T_s$ [samples/sec]. Thus, the zero-mean SOCS random process $X[n]$ has the cycle period 2. As shown in (21a), it can be seen in Fig. 4 that $\tilde{W}_{MN} R_x \tilde{W}_{MN}^H$ approaches Ω as N increases. Similarly, as shown in (21b), it can be seen in Fig. 5 that $\tilde{W}_{MN} \tilde{R}_x \tilde{W}_{MN}^T$ approaches $\tilde{\Omega}$ as N increases.

Now, the asymptotic second-order properties of the output y of the asymptotic FRESH properizer are examined as follows.



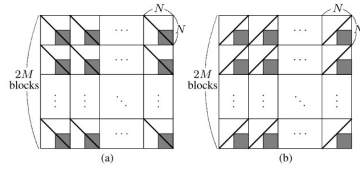


Fig. 6. Illustrations that show how to construct (a) $\mathbf{R}_y$ and (b) $\tilde{\mathbf{R}}_y$ from $\tilde{\mathbf{W}}_{MN}\mathbf{R}_x\tilde{\mathbf{W}}_{MN}^H \sim \mathbf{\Omega}$ and $\tilde{\mathbf{W}}_{MN}\tilde{\mathbf{R}}_x\tilde{\mathbf{W}}_{MN}^H \sim \mathbf{\tilde{\Omega}}$, respectively. Thick diagonal and anti-diagonal lines represent all possible non-zero entries of $\mathbf{\Omega}$ and $\mathbf{\tilde{\Omega}}$, respectively.

Theorem 2: If the input $\mathbf{x}$ to the asymptotic FRESH properizer with parameters M and N is obtained by taking the MN consecutive samples from an SOCS random process with cycle period M , then the covariance and the complementary covariance matrices of the output $\mathbf{y}$ of the asymptotic FRESH properizer satisfy

$$\mathbf{R}_y \sim \mathbf{W}_{MN}^H \mathbf{\Sigma} \mathbf{W}_{MN} \quad (24a)$$

and

$$\tilde{\mathbf{R}}_y \sim \mathbf{O}_{MN}, \quad (24b)$$

where an (MN) -by- (MN) matrix $\mathbf{\Sigma}$ is given by

$$\mathbf{\Sigma} \triangleq \tilde{\mathbf{G}}_{M,N} \mathbf{\Omega} \tilde{\mathbf{G}}_{M,N}^H \quad (25a)$$

$$= \mathbf{R}_y \odot (\mathbf{I}_{2M} \otimes \mathbf{I}_{N/2}). \quad (25b)$$

Proof: By applying Lemmas 5 and 6 to (19a) and (21a), we obtain $\mathbf{R}_y \sim \tilde{\mathbf{G}}_{M,N} \mathbf{\Omega} \tilde{\mathbf{G}}_{M,N}^H$. Since $\tilde{\mathbf{G}}_{M,N} \mathbf{\Omega} \tilde{\mathbf{G}}_{M,N}^H$ can be rewritten as $(\tilde{\mathbf{G}}_{M,N} \tilde{\mathbf{W}}_{MN} \mathbf{R}_x \tilde{\mathbf{W}}_{MN}^H \tilde{\mathbf{G}}_{M,N}^H) \odot (\tilde{\mathbf{G}}_{M,N} (\mathbf{I}_{2M} \otimes \mathbf{I}_N) \tilde{\mathbf{G}}_{M,N}^H)$ by the definition (22a), we obtain $\mathbf{R}_y \sim \mathbf{\Sigma}$ from $\tilde{\mathbf{G}}_{M,N} (\mathbf{I}_{2M} \otimes \mathbf{I}_N) \tilde{\mathbf{G}}_{M,N}^H = \mathbf{I}_{2M} \otimes \mathbf{I}_{N/2}$. Similarly, $\tilde{\mathbf{R}}_y \sim (\tilde{\mathbf{G}}_{M,N} \tilde{\mathbf{W}}_{MN} \tilde{\mathbf{R}}_x \tilde{\mathbf{W}}_{MN}^H \tilde{\mathbf{G}}_{M,N}^H) \odot (\tilde{\mathbf{G}}_{M,N} (\mathbf{I}_{2M} \otimes \mathbf{P}_N) \tilde{\mathbf{G}}_{M,N}^H)$ by (19b), (21b), and (22b). Thus, we obtain $\tilde{\mathbf{R}}_y \sim \mathbf{O}_{MN}$ from $\tilde{\mathbf{G}}_{M,N} (\mathbf{I}_{2M} \otimes \mathbf{P}_N) \tilde{\mathbf{G}}_{M,N}^H = \mathbf{O}_{MN}$. Therefore, the conclusion follows by Lemma 5. $\square$

By Definition 13 and the above theorem, the output $\mathbf{y}$ of the asymptotic FRESH properizer is indeed asymptotically proper. Fig. 6-(a) illustrates how the frequency-domain covariance matrix $\mathbf{R}_y$ of the output $\mathbf{y}$ of the asymptotic FRESH properizer is constructed from $\tilde{\mathbf{W}}_{MN} \mathbf{R}_x \tilde{\mathbf{W}}_{MN}^H$. The thick diagonal lines in each block of size N -by- N represent all possible non-zero entries of $\mathbf{\Omega}$ that is asymptotically equivalent to $\tilde{\mathbf{W}}_{MN} \mathbf{R}_x \tilde{\mathbf{W}}_{MN}^H$. Then, $\mathbf{R}_y$ that is asymptotically equivalent to $\mathbf{\Sigma}$ is obtained by collecting the shaded sub-blocks of size $(N/2)$ -by- $(N/2)$. Fig. 6-(b) illustrates how the frequency-domain complementary covariance matrix $\tilde{\mathbf{R}}_y$ of the output $\mathbf{y}$ of the asymptotic FRESH properizer is constructed from $\tilde{\mathbf{W}}_{MN} \tilde{\mathbf{R}}_x \tilde{\mathbf{W}}_{MN}^H$. The thick anti-diagonal lines in each block of size N -by- N represent all possible non-zero entries of $\mathbf{\tilde{\Omega}}$ that is asymptotically equivalent to $\tilde{\mathbf{W}}_{MN} \tilde{\mathbf{R}}_x \tilde{\mathbf{W}}_{MN}^H$. Then, $\tilde{\mathbf{R}}_y$ that is asymptotically equivalent to $\mathbf{O}_{MN}$ is obtained by collecting the shaded sub-blocks of size $(N/2)$ -by- $(N/2)$.

It is already shown that the amount of the circular shift of $\mathbf{G}_{M,N} \mathbf{W}_{MN} \mathbf{x}^*$ in (12) can be any $N(2k+1)/2$, for $k \in \mathbb{Z}$, to satisfy the invertibility of the asymptotic FRESH properizer. It can be also shown that Theorem 2 holds for every circular shift by $N(2k+1)/2$, for $k \in \mathbb{Z}$. Moreover, since $\mathbf{R}_x$ and $\tilde{\mathbf{R}}_x$ that are (MN) -by- (MN) block Toeplitz matrices with block size M -by- M can be viewed as block Toeplitz matrices with block size (lM) -by- (lM) , $\forall l \in \mathbb{N}$, the sampled vector $\mathbf{x}$ can be asymptotically properized by using any parameters lM and N/l , for all $l \in \mathbb{N}$ such that N/l is even. Thus, similar to the DT FRESH properizer, it is not unique to asymptotically properize a finite number of consecutive samples of a zero-mean SOCS random process with cycle period M .

In the following two sections, two representative estimation and detection problems are presented to demonstrate that low-complexity asymptotically optimal post-processors can be easily designed by exploiting the asymptotic second-order properties of the output of the asymptotic FRESH properizer.

IV. APPLICATION OF ASYMPTOTIC FRESH PROPERIZER TO SIGNAL ESTIMATION PROBLEM

In this section, given a finite number of consecutive samples of a zero-mean improper-complex SOCS random process in additive proper-complex white noise, the minimum mean squared error estimation of the random process is considered when only the second-order statistics of the observation vector are provided. It is well known [5] that an optimal estimator is the WLMSE estimator that linearly processes the observation vector augmented by its complex conjugate. The optimality of the use of the augmented vector still holds even if the output of the asymptotic FRESH properizer is processed, because the output as an equivalent observation vector is still improper. Instead, motivated by Theorem 2, we propose to use the LMMSE estimator that processes the output of the asymptotic FRESH properizer as if the signal component is a proper-complex random vector with its frequency-domain covariance matrix being equal to the masked version of the exact frequency-domain covariance matrix. It turns out that this suboptimal linear estimator is asymptotically optimal in the sense that the difference of its average MSE performance from that of the WLMSE estimator having higher computational complexity converges to zero as the number of samples tends to infinity.

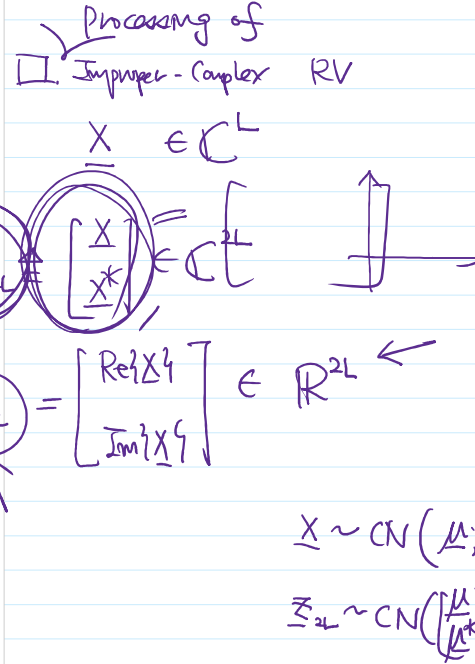
A. Asymptotically Optimal Low-Complexity Estimator

Let $\mathbf{r}$ be the length- MN observation vector modeled by

$$\mathbf{r} = \mathbf{x} + \mathbf{v}, \quad (26)$$

where $\mathbf{x}$ is the desired signal to be estimated that consists of the MN consecutive samples of a zero-mean improper-complex SOCS random process with cycle period $M \in \mathbb{N}$, and $\mathbf{v}$ is the additive proper-complex white noise vector.

Given the observation model (26) with the first- and second-order moments $\mathbf{E}\{\mathbf{x}\} = \mathbf{0}_{MN}$, $\mathbf{E}\{\mathbf{x}\mathbf{x}^H\} = \mathbf{R}_x$, $\mathbf{E}\{\mathbf{x}\mathbf{x}^T\} = \tilde{\mathbf{R}}_x$, $\mathbf{E}\{\mathbf{v}\} = \mathbf{0}_{MN}$, $\mathbf{E}\{\mathbf{v}\mathbf{v}^H\} = \sigma^2 \mathbf{I}_{MN}$, and $\mathbf{E}\{\mathbf{v}\mathbf{v}^T\} = \mathbf{O}_{MN}$



of $\mathbf{x}$ and $\mathbf{v}$, the WLMSE estimation problem can be formulated as

$$\begin{aligned} & \underset{\mathbf{x}_{\text{wl}}}{\text{minimize}} \quad e^{(N)} \\ & \text{subject to} \quad \mathbf{x}_{\text{wl}} = \mathbf{F}_1 \mathbf{r} + \mathbf{F}_2 \mathbf{r}^*, \end{aligned} \quad (27)$$

where the average MSE $e^{(N)}$ is defined as

$$e^{(N)} \triangleq \frac{1}{MN} \mathbf{E}\{\|\mathbf{x} - \mathbf{x}_{\text{wl}}\|^2\}. \quad (28)$$

In the following lemma, the optimal solution to the above estimation problem is provided along with its average MSE performance. It is well known [5] that the optimal estimator that minimizes the average MSE without the wide linearity constraint becomes this WLMSE estimator if the desired signal vector and the additive noise vector are both Gaussian.

Lemma 7: The WLMSE estimator $\mathbf{x}_{\text{opt}}$ of $\mathbf{x}$ as the solution to the optimization problem (27) is given by

$$\mathbf{x}_{\text{opt}} = \mathbf{F}_{1,\text{opt}} \mathbf{r} + \mathbf{F}_{2,\text{opt}} \mathbf{r}^* \quad (29)$$

where

$$\mathbf{F}_{1,\text{opt}} \triangleq (\mathbf{R}_x - \tilde{\mathbf{R}}_x \mathbf{R}_r^{-1} \tilde{\mathbf{R}}_x^*) (\mathbf{R}_r - \tilde{\mathbf{R}}_x \mathbf{R}_r^{-1} \tilde{\mathbf{R}}_x^*)^{-1} \quad (30a)$$

and

$$\mathbf{F}_{2,\text{opt}} \triangleq (\tilde{\mathbf{R}}_x - \mathbf{R}_x \mathbf{R}_r^{-1} \tilde{\mathbf{R}}_x^*) (\mathbf{R}_r^* - \tilde{\mathbf{R}}_x^* \mathbf{R}_r^{-1} \tilde{\mathbf{R}}_x^*)^{-1}. \quad (30b)$$

with $\mathbf{R}_r \triangleq \mathbf{E}\{\mathbf{r}\mathbf{r}^H\} = \sigma^2 \mathbf{I}_{MN} + \mathbf{R}_x$. Moreover, the average MSE $e_{\text{opt}}^{(N)} \triangleq \mathbf{E}\{\|\mathbf{x} - \mathbf{x}_{\text{opt}}\|^2\}/(MN)$ is given by

$$e_{\text{opt}}^{(N)} = \frac{1}{MN} \text{tr}(\mathbf{R}_x - \mathbf{F}_{1,\text{opt}} \mathbf{R}_x - \mathbf{F}_{2,\text{opt}} \tilde{\mathbf{R}}_x^*). \quad (31)$$

Proof: See [5, Sec. 5.4] and the references therein. $\square$

Apparently, the WLMSE estimator in (29) requires the computation of two matrices $\mathbf{F}_{1,\text{opt}}$ and $\mathbf{F}_{2,\text{opt}}$ to be pre-multiplied to $\mathbf{r}$ and $\mathbf{r}^*$, respectively. In this brute-force approach, as it can be seen from (30a) and (30b), the major burden in computing these matrices comes from the multiplications and the inversions of (MN) -by- (MN) matrices, which require $\mathcal{O}(M^3 N^3)$ complex-valued scalar multiplications. Instead, we may take an alternative approach to reduce the computational complexity. Similar to $\tilde{\mathbf{x}}$ in (16), let $\tilde{\mathbf{r}}$ be the augmented observation vector of $\mathbf{r}$. Then, it can be shown [5, Result 5.3] that $\mathbf{x}_{\text{opt}}$ given by (29) is identical to the upper MN elements of the LMMSE estimate

$$\mathbf{R}_{\tilde{\mathbf{x}}} \left(\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\tilde{\mathbf{x}}} \right)^{-1} \tilde{\mathbf{r}} \quad (32)$$

of $\tilde{\mathbf{x}}$, where $\mathbf{R}_{\tilde{\mathbf{x}}}$ denotes the covariance matrix of $\tilde{\mathbf{x}}$. Now, the following lemma enables us to reduce the complexity in computing $\mathbf{x}_{\text{opt}}$ to $\mathcal{O}(M^3 N^2)$.

Lemma 8: The covariance matrix $\mathbf{R}_{\tilde{\mathbf{x}}}$ of $\tilde{\mathbf{x}}$ can be converted into a $(2MN)$ -by- $(2MN)$ block Toeplitz matrix with block size $(2M)$ -by- $(2M)$ by performing row and column permutations.

Proof: Let $\mathbf{E}_{2,MN}$ be the $(2MN)$ -by- $(2MN)$ matrix that permutes the rows of the post-multiplied matrix in such a way that the $(MN(m-1)+n)$ th row of the post-multiplied matrix becomes the $(2(n-1)+m)$ th row, for $m=1,2$ and $n=1,2,\dots,MN$. Since $\mathbf{x}$ consists of MN consecutive samples of a zero-mean SOCS random process $X[n]$ with

cycle period M , if this invertible operator is used to rearrange the order of the augmented vector $\tilde{\mathbf{x}}$ defined in (16), then we have

$$\mathbf{E}_{2,MN} \tilde{\mathbf{x}} = \begin{bmatrix} [\mathbf{x}]_1 & [\mathbf{x}]_1^* & [\mathbf{x}]_2 & [\mathbf{x}]_2^* & \cdots & [\mathbf{x}]_{MN} & [\mathbf{x}]_{MN}^* \end{bmatrix}^T. \quad (33)$$

Note that this is nothing but the vector consisting of $2MN$ consecutive samples of the random process $\tilde{X}[n]$ defined by using $X[n]$ as $\tilde{X}[2n] \triangleq X[n]$ and $\tilde{X}[2n+1] \triangleq X[n]^*$ for all n . It can be straightforwardly shown that $\tilde{X}[n]$ is a zero-mean SOCS random process with cycle period $2M$, which implies that $r_{\tilde{X}}[n, m] \triangleq \mathbf{E}\{\tilde{X}[n]\tilde{X}[m]^*\} = r_{\tilde{X}}[n+2M, m+2M]$, $\forall m, \forall n$. Therefore, the conclusion follows. $\square$

The above lemma combined with the computationally efficient inversion technique in [35] for a block Toeplitz matrix enables us to perform the inversion of $(\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\tilde{\mathbf{x}}})$ in (32) now only with $\mathcal{O}(M^3 N^2)$ complex-valued scalar multiplications. Also, the matrix-vector multiplications $(\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\tilde{\mathbf{x}}})^{-1} \tilde{\mathbf{r}}$ and $\mathbf{R}_{\tilde{\mathbf{x}}} \{(\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\tilde{\mathbf{x}}})^{-1} \tilde{\mathbf{r}}\}$ in (32) require only $\mathcal{O}(M^2 N^2)$ complex-valued scalar multiplications. Therefore, the overall computational complexity of the WLMSE estimator is $\mathcal{O}(M^3 N^2)$.

The WLMSE estimator can be realized also in the frequency domain. Note, however, that the computational complexity does not decrease. Of course, the frequency-domain conversion of the observation vector requires only $\mathcal{O}(MN \log(MN))$ of computational complexity when the fast Fourier transform (FFT) is used. It can be shown that the brute-force approach based on (29) to the frequency-domain estimation requires the overall computational complexity of $\mathcal{O}(M^3 N^3)$. Instead, we may take an alternative approach based on (32), where the LMMSE estimate of $\tilde{\mathbf{W}}_{MN} \tilde{\mathbf{x}}$ is given by

$$\tilde{\mathbf{W}}_{MN} \mathbf{R}_{\tilde{\mathbf{x}}} \tilde{\mathbf{W}}_{MN}^H \left(\sigma^2 \mathbf{I}_{2MN} + \tilde{\mathbf{W}}_{MN} \mathbf{R}_{\tilde{\mathbf{x}}} \tilde{\mathbf{W}}_{MN}^H \right)^{-1} \tilde{\mathbf{W}}_{MN} \tilde{\mathbf{r}}. \quad (34)$$

However, it involves the inversion of a matrix that is not highly structured, which leads to the overall computational complexity of the frequency-domain approach using $\tilde{\mathbf{W}}_{MN} \tilde{\mathbf{r}}$ no better than the time-domain approach using $\tilde{\mathbf{r}}$.

An alternative observation model that is equivalent to the original one in (26) can be obtained by applying the asymptotic FRESH properization to $\mathbf{r}$ as

$$\mathbf{s} = \mathbf{y} + \mathbf{w}, \quad (35)$$

where $\mathbf{s} = \mathbf{f}(\mathbf{r})$, $\mathbf{y} = \mathbf{f}(\mathbf{x})$, and $\mathbf{w} = \mathbf{f}(\mathbf{v})$. Since the equivalent observation vector $\mathbf{s}$ is not proper in general, the linear processing of $\mathbf{s}$ and $\mathbf{s}^*$ is still needed for the second-order optimal estimation of $\mathbf{x}$. Instead, we propose to use a suboptimal linear estimator as follows, which is a function only of $\mathbf{s}$.

Definition 14: The proposed estimate $\mathbf{x}_p$ of $\mathbf{x}$ is defined by

$$\mathbf{x}_p = \mathbf{f}^{-1}(\mathbf{F}_p \mathbf{s}), \quad (36)$$

where $\mathbf{F}_p$ is given by

$$\mathbf{F}_p \triangleq \mathbf{W}_{MN}^H \Sigma (\sigma^2 \mathbf{I}_{MN} + \Sigma)^{-1} \mathbf{W}_{MN}. \quad (37)$$

Note from (36) and (37) that $\Sigma(\sigma^2 \mathbf{I}_{MN} + \Sigma)^{-1} \mathbf{W}_{MN} \mathbf{s}$ is the estimate of $\mathbf{W}_{MN} \mathbf{y} = \mathbf{W}_{MN} \mathbf{f}(\mathbf{x})$ and, consequently, that $\mathbf{F}_p \mathbf{s}$ is the estimate of $\mathbf{y} = \mathbf{f}(\mathbf{x})$. Let $\mathbf{y}_p \triangleq \mathbf{F}_p \mathbf{s}$. Then, the estimate $\mathbf{x}_p$ of $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{y})$ is obtained as (36) by applying the inverse operation $\mathbf{f}^{-1}$ to $\mathbf{y}_p$.

It can be immediately seen that $\mathbf{F}_p$ is chosen to make $\mathbf{y}_p$ the LMMSE estimate of $\mathbf{y}$ when the covariance and the complementary covariance matrices of $\mathbf{y}$ are set equal to $\mathbf{W}_{MN}^H \Sigma \mathbf{W}_{MN}$ and $\mathbf{O}_{MN}$ appearing in (24a) and (24b), respectively. This idea of using the asymptotically equivalent matrices is motivated by Theorem 2, which naturally leads to the asymptotic optimality of the proposed estimator as shown in what follows.

First, the invariance of the Euclidean norm under the asymptotic FRESH properizer is shown.

Lemma 9: The input $\mathbf{x}$ and the output $\mathbf{y} = \mathbf{f}(\mathbf{x})$ of the asymptotic FRESH properizer have the same Euclidean norm, i.e., $\|\mathbf{x}\| = \|\mathbf{y}\|$.

Proof: Since the centered DFT matrix $\mathbf{W}_{MN}$ is unitary, we have $\|\mathbf{x}\| = \|\mathbf{W}_{MN} \mathbf{x}\|$. Let y_l and y'_l denote the l th entries of $\mathbf{y}$ and $\mathbf{W}_{MN} \mathbf{x}$, respectively. Then, we have $\|\mathbf{W}_{MN} \mathbf{x}\|^2 = \sum_{l=1}^{MN} |y'_l|^2 = \sum_{l=1}^{MN} |y_l|^2 = \|\mathbf{y}\|^2$ by Definition 11 of $\mathbf{f}(\mathbf{x})$ in (14), which is illustrated in Fig. 3. Therefore, the conclusion follows. $\square$

Second, the average MSE of the proposed estimator is derived.

Lemma 10: The average MSE $\varepsilon_p^{(N)} \triangleq \mathbf{E}[\|\mathbf{x} - \mathbf{x}_p\|^2]/(MN)$ of the proposed estimator is given by

$$\varepsilon_p^{(N)} = \frac{1}{MN} \text{tr}\{\Sigma - \Sigma(\sigma^2 \mathbf{I}_{MN} + \Sigma)^{-1} \Sigma\}. \quad (38)$$

Proof: The linearity of the asymptotic FRESH properizer leads to $\mathbf{y} - \mathbf{y}_p = \mathbf{f}(\mathbf{x} - \mathbf{x}_p)$. Then, by Lemma 9, the average MSE performance can be written as $\varepsilon_p^{(N)} = \mathbf{E}[\|\mathbf{y} - \mathbf{y}_p\|^2]/(MN)$. Since $\mathbf{W}_{MN}$ is unitary and $\mathbf{y}_p = \mathbf{F}_p \mathbf{s}$, it can be shown that we have

$$\varepsilon_p^{(N)} = \frac{1}{MN} \text{tr}\{\mathbf{R}_{\hat{\mathbf{y}}} - 2\Sigma(\sigma^2 \mathbf{I}_{MN} + \Sigma)^{-1} \mathbf{R}_{\hat{\mathbf{y}}} + \Sigma(\sigma^2 \mathbf{I}_{MN} + \Sigma)^{-1} \times (\sigma^2 \mathbf{I}_{MN} + \mathbf{R}_{\hat{\mathbf{y}}})(\sigma^2 \mathbf{I}_{MN} + \Sigma)^{-1} \Sigma\}. \quad (39)$$

It can be also shown that if $\tilde{\mathbf{A}} \triangleq \mathbf{A} \odot (\mathbf{I}_{2M} \otimes \mathbf{I}_{N/2})$ and $\tilde{\mathbf{B}} \triangleq \mathbf{B} \odot (\mathbf{I}_{2M} \otimes \mathbf{I}_{N/2})$ for (MN) -by- (MN) matrices $\mathbf{A}$ and $\mathbf{B}$ then $\text{tr}(\mathbf{A}) = \text{tr}(\tilde{\mathbf{A}})$ and $\text{tr}(\tilde{\mathbf{A}}\tilde{\mathbf{B}}) = \text{tr}(\tilde{\mathbf{A}}\tilde{\mathbf{B}})$. Thus, the right side of (39) is invariant under replacing $\mathbf{R}_{\hat{\mathbf{y}}}$ with $\Sigma = \mathbf{R}_{\hat{\mathbf{y}}} \odot (\mathbf{I}_{2M} \otimes \mathbf{I}_{N/2})$ in (25b). Therefore, (39) can be simplified to (38). $\square$

Now, the asymptotic optimality of the proposed estimator is provided.

Theorem 3: The average MSE of the proposed estimator approaches that of the WLMSE estimator as the number of samples tends to infinity in the sense that

$$\lim_{N \rightarrow \infty} (\varepsilon_p^{(N)} - \varepsilon_{\text{opt}}^{(N)}) = 0. \quad (40)$$

Proof: It is shown in [5, Sec. 5.4] that the average MSE $\varepsilon_{\text{opt}}^{(N)}$ in (31) of the WLMSE estimator can be simplified as

$$\varepsilon_{\text{opt}}^{(N)} = \frac{1}{2MN} \text{tr}\{\mathbf{R}_{\hat{\mathbf{x}}} - \mathbf{R}_{\hat{\mathbf{x}}}(\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\hat{\mathbf{x}}})^{-1} \mathbf{R}_{\hat{\mathbf{x}}}\}. \quad (41)$$

Now, we define $\hat{\mathbf{\Omega}} \triangleq \tilde{\mathbf{W}}_{MN}^H \mathbf{\Omega} \tilde{\mathbf{W}}_{MN}$ and introduce

$$\hat{\varepsilon}_{\text{opt}}^{(N)} \triangleq \frac{1}{2MN} \text{tr}\{\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}(\sigma^2 \mathbf{I}_{2MN} + \hat{\mathbf{\Omega}})^{-1} \hat{\mathbf{\Omega}}\} \quad (42a)$$

$$= \frac{1}{2MN} \text{tr}\{\mathbf{\Omega} - \mathbf{\Omega}(\sigma^2 \mathbf{I}_{2MN} + \mathbf{\Omega})^{-1} \mathbf{\Omega}\}, \quad (42b)$$

which is obtained by replacing $\mathbf{R}_{\hat{\mathbf{x}}}$ in (41) with $\hat{\mathbf{\Omega}}$ and by using the fact that $\tilde{\mathbf{W}}_{MN}$ is unitary. Note that $\tilde{\mathbf{W}}_{MN} \mathbf{R}_{\hat{\mathbf{x}}} \tilde{\mathbf{W}}_{MN}^H \sim \mathbf{\Omega}$ as shown in (21a). This introduction of $\hat{\varepsilon}_{\text{opt}}^{(N)}$ is motivated by the result in [24, Th. 1], where a suboptimal estimator is proposed for the estimation of a proper-complex cyclostationary random signal and its average MSE is shown to approach that of the LMMSE estimator as the number of samples tends to infinity. Since $\mathbf{R}_{\hat{\mathbf{x}}}$ is a covariance matrix, it is positive semidefinite, which implies that all the eigenvalues of $(\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\hat{\mathbf{x}}})^{-1}$ in (41) are upper bounded by $1/\sigma^2$. Note that $\mathbf{\Omega}$ and $\hat{\mathbf{\Omega}}$ are also positive semidefinite because $(\mathbf{I}_{2M} \otimes \mathbf{I}_{N/2})$ in (22a) is positive semidefinite. Similarly, all the eigenvalues of $(\sigma^2 \mathbf{I}_{2MN} + \hat{\mathbf{\Omega}})^{-1}$ in (42a) are also upper bounded by $1/\sigma^2$. Thus, the strong norms $\|(\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\hat{\mathbf{x}}})^{-1}\|$ and $\|(\sigma^2 \mathbf{I}_{2MN} + \hat{\mathbf{\Omega}})^{-1}\|$ are uniformly upper bounded by $1/\sigma^2$ for any matrix size. Recall that if $\mathbf{A}_k \sim \mathbf{B}_k$ and $\|\mathbf{A}_k^{-1}\|, \|\mathbf{B}_k^{-1}\| \leq c < \infty$, $\forall k$, for a positive constant c , then $\mathbf{A}_k^{-1} \sim \mathbf{B}_k^{-1}$ [23, Th. 1-(4)]. Thus, we have $(\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\hat{\mathbf{x}}})^{-1} \sim (\sigma^2 \mathbf{I}_{2MN} + \hat{\mathbf{\Omega}})^{-1}$. Recall also that if two sequences $(\mathbf{A}_k)_k$ and $(\mathbf{B}_k)_k$ of N_k -by- N_k matrices are asymptotically equivalent then $\lim_{k \rightarrow \infty} \text{tr}(\mathbf{A}_k - \mathbf{B}_k)/N_k = 0$ [23, Corollary 1]. Thus, combined with Lemmas 5 and 6, we have $\lim_{N \rightarrow \infty} (\varepsilon_{\text{opt}}^{(N)} - \hat{\varepsilon}_{\text{opt}}^{(N)}) = 0$. Therefore, in order to show (40), it now suffices to show $\varepsilon_{\text{opt}}^{(N)} = \hat{\varepsilon}_{\text{opt}}^{(N)}$.

Let $\mathbf{E}_{2M,N}$ be the $(2MN)$ -by- $(2MN)$ matrix that permutes the rows of the post-multiplied matrix in such a way that the $(N(m-1) + n)$ th row of the post-multiplied matrix becomes the $(2M(n-1) + m)$ th row for $m = 1, 2, \dots, 2M$ and $n = 1, 2, \dots, N$, i.e., the rows having indexes $(N(m-1) + n)$, for $m = 1, 2, \dots, 2M$, are grouped for each n . Then, $\mathbf{E}_{2M,N} \mathbf{\Omega} \mathbf{E}_{2M,N}^T$ is a $(2MN)$ -by- $(2MN)$ block diagonal matrix with block size $(2M)$ -by- $(2M)$, because $\mathbf{\Omega}$ is a $(2MN)$ -by- $(2MN)$ block matrix with diagonal blocks of block size N -by- N . Similarly, the (MN) -by- (MN) matrix $\mathbf{E}_{2M,N/2}$ is defined. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the (MN) -by- (MN) matrix that are defined as $\mathbf{E}_1 \triangleq \mathbf{P}_{MN}(\mathbf{I}_{N/2} \otimes \hat{\mathbf{E}}_{2M})\mathbf{E}_{2M,N/2}$ and $\mathbf{E}_2 \triangleq (\mathbf{I}_{N/2} \otimes \hat{\mathbf{E}}_{2M})\mathbf{E}_{2M,N/2}$, respectively, where the $(2M)$ -by- $(2M)$ matrix $\hat{\mathbf{E}}_{2M}$ permutes the rows of the post-multiplied matrix in such a way that the $(2m)$ th row becomes the m th row and that the $(2m-1)$ th row becomes the $(M+m)$ th row for $m = 1, 2, \dots, M$. Then, the definition $\Sigma \triangleq \mathbf{G}_{M,N} \mathbf{\Omega} \mathbf{G}_{M,N}^H$ in (25a) leads to

$$\mathbf{E}_{2M,N} \mathbf{\Omega} \mathbf{E}_{2M,N}^T = \begin{bmatrix} \mathbf{E}_1 \Sigma^* \mathbf{E}_1^T & \mathbf{O}_{MN} \\ \mathbf{O}_{MN} & \mathbf{E}_2 \Sigma \mathbf{E}_2^T \end{bmatrix}. \quad (43)$$

Since $\mathbf{E}_{2M,N}$, $\mathbf{E}_1$, and $\mathbf{E}_2$ are all permutation matrices, $\hat{\varepsilon}_{\text{opt}}^{(N)}$ in (42a) can be rewritten as $\hat{\varepsilon}_{\text{opt}}^{(N)} = \text{tr}\{\Sigma^* - \Sigma^*(\sigma^2 \mathbf{I}_{2MN} + \Sigma^*)^{-1} \Sigma^* + \Sigma - \Sigma(\sigma^2 \mathbf{I}_{2MN} + \Sigma)^{-1} \Sigma\}/(2MN)$, which leads to $\hat{\varepsilon}_{\text{opt}}^{(N)} = \varepsilon_{\text{opt}}^{(N)}$ because Σ is Hermitian symmetric. Therefore, the conclusion follows. $\square$

The proposed estimator requires the pre-processing of $\mathbf{r}$ to obtain $\mathbf{s}$, the computation and the multiplication of

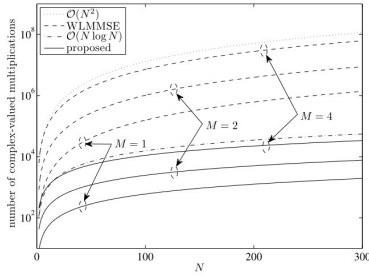


Fig. 7. Computational complexities of the WLMSE and the proposed estimators.

F_p , and the inverse operation on y_p to obtain x_p . As it can be seen from (14) and (15), the major burden in the asymptotic FRESH properization and its inverse operation comes from the multiplications of the (MN) -by- (MN) centered DFT matrix. This can be efficiently implemented with only $O(MN \log(MN))$ complex-valued scalar multiplications by using, e.g., the fast algorithm in [34]. As it can be seen from (37), the major burden in computing F_p comes from the inversion of $(\sigma^2 I_{2MN} + \Sigma)$. The inversion of an (MN) -by- (MN) block matrix with diagonal blocks of size N -by- N requires $O(M^3 N)$ complex-valued scalar multiplications [24]. Thus, this inversion of $(\sigma^2 I_{2MN} + \Sigma)$ has the same order $O(M^3 N)$ of computational complexity because Σ is an (MN) -by- (MN) block matrix with diagonal blocks of size $(N/2)$ -by- $(N/2)$.

Since we consider the block processing where M is a fixed small number and N is much larger than M , the overall complexities of the WLMSE and the proposed estimators as functions of N can be rewritten now as $O(N^2)$ and $O(N \log N)$, respectively. Thus, the computational complexity of the proposed estimator is much lower than that of the WLMSE estimator. Though not major, an additional complexity reduction comes from the fact that the proposed estimator is linear that requires the computation and multiplication of one matrix F_p instead of two matrices $F_{1,\text{opt}}$ and $F_{2,\text{opt}}$, all with $MN (\gg 1)$ columns.

B. Numerical Results

In this subsection, numerical results are provided that show the computational efficiency and the asymptotic optimality of the proposed estimator.

The first result is to compare the complexity of the proposed estimator with that of the WLMSE estimator. Fig. 7 shows that the number of complex-valued multiplications needed in computing the WLMSE and the proposed estimators for cycle periods $M = 1, 2$, and 4. Recall that the computational complexities of the WLMSE and the proposed estimators

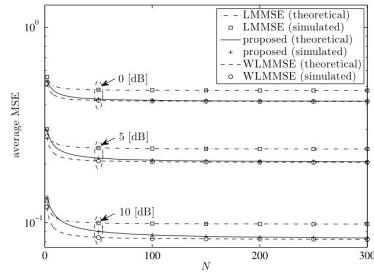


Fig. 8. Average MSEs of the LMMSE, the WLMSE and the proposed estimators.

are $O(N^2)$ and $O(N \log N)$ for a fixed integer M , respectively. It can be seen from Fig. 7 that the computational complexity of the proposed estimator is much lower than that of the WLMSE estimator. As predicted, the approximation motivated by Theorem 2 leads to this significant complexity reduction.

The second result is to show the asymptotic optimality of the proposed estimator. We consider the case where an improper-complex SOCS random process is obtained by uniformly sampling a CT zero-mean improper-complex SOCS random process. The CT random process is, e.g., a Gaussian jamming signal, generated by using two independent real-valued symbol sequences with the SRRC pulse having roll-off factor 0.22 and symbol rate $1/T_s$. Throughout the simulations, the data symbols are uncorrelated zero-mean Gaussian random variables with in-phase symbol variance being four-times the quadrature symbol variance. The frequency-selective channel is modeled by a tapped-delay line with impulse response $h(t) = \sum_{k=1}^5 a_k \delta(t - \tau_k/2)$, where the tap delay τ_k 's are i.i.d. uniform random variables on $[0, T_s, 2T_s, \dots, 15T_s]$ and the tap coefficient a_k 's are independent proper-complex zero-mean Gaussian random variables with variance $E\{a_k^2\} = e^{-\tau_k/(3T_s)}$. The overall channel is then normalized to meet the signal to noise ratio used in the numerical result. The channel output is sampled at 2-times the symbol rate of the OQPSK symbols, which results in the DT zero-mean improper-complex SOCS random process with cycle period $M = 2$. Thus, the entries of $\mathbf{x}$ in (26) are the MN consecutive samples of the random process. Fig. 8 shows that the average MSEs of the WLMSE and the proposed estimators versus N for symbol energy per noise density $E_s/N_0 = 0, 5$, and 10 dB, where the theoretical results are evaluated by (31) and (38) while the simulated results are obtained from 10^5 Monte-Carlo runs. It can be seen that, as shown in Theorem 3, the average MSE of the proposed estimator approaches that of the WLMSE estimator as N increases. It can be also seen that the LMMSE estimator has higher average MSE than the proposed and the WLMSE estimators.

V. APPLICATION OF ASYMPTOTIC FRESH PROPERIZER TO SIGNAL PRESENCE DETECTION PROBLEM

In this section, again given a finite number of consecutive samples, now the signal presence detection of a zero-mean improper-complex SOCS Gaussian random process is considered in additive proper-complex white Gaussian noise. It is well known [36] that the likelihood ratio is a sufficient statistic for all binary hypothesis tests under any optimality criterion. In [5], it is shown that the exact LRT statistic can be written as a quadratic function of the augmented observation vector when the improper-complex random process is *Gaussian*. Similar to the estimation problem in the previous section, this optimality of the use of the augmented vector still holds even if the output of the asymptotic FRESH properizer as an equivalent observation vector is processed. Again motivated by Theorem 2, we propose to use this equivalent observation vector of half the length of the augmented vector as if the signal component is a proper-complex random vector with its frequency-domain covariance matrix being equal to the masked version of the exact frequency-domain covariance matrix. It turns out that this suboptimal test statistic is asymptotically optimal in the sense that its difference from the exact LRT statistic having higher computational complexity converges to zero w.p. 1 as the number of samples tends to infinity.

A. Asymptotically Optimal Low-Complexity Detector

Let $\mathbf{r}$ be the length- MN observation vector modeled by

$$\begin{aligned} \mathcal{H}_0 : \mathbf{r} &= \mathbf{v} \\ \text{versus} \\ \mathcal{H}_1 : \mathbf{r} &= \mathbf{x} + \mathbf{v}, \end{aligned} \quad (44)$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are the null and the alternative hypotheses, respectively. The desired signal $\mathbf{x}$ to be detected consists of the MN consecutive samples of a zero-mean improper-complex SOCS Gaussian random process with cycle period $M \in \mathbb{N}$, and $\mathbf{v}$ is the additive proper-complex white Gaussian noise vector.

Throughout this section, the definitions and the notations of the second-order statistics of $\mathbf{x}$, $\mathbf{v}$, and $\mathbf{r}$ follow those in the previous section except that under the null hypothesis the second-order statistic of $\mathbf{r}$ does not contain the desired signal component. Similar to (16), the augmented observation vector is denoted by $\bar{\mathbf{r}}$. Then, the LRT statistic of the observation vector $\mathbf{r}$ is provided as follows.

Lemma 11: Given the observation model (44), the LRT statistic is given by

$$T_{\text{LRT}}^{(N)} = \frac{1}{2MN} \bar{\mathbf{r}}^H \bar{\mathbf{R}}_{\bar{\mathbf{r}}} \bar{\mathbf{r}}, \quad (45)$$

where $\bar{\mathbf{R}}_{\bar{\mathbf{r}}} \triangleq \sigma^{-2} \mathbf{I}_{2MN} - (\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\bar{\mathbf{r}}})^{-1}$.

Proof: It is straightforward by using the probability density function (PDF) of the augmented improper-complex Gaussian random vector [37]. For details, see [5, Sec. 7.4]. $\square$

Note that, since the length- MN observation vector $\mathbf{r}$ is improper and Gaussian, the LRT statistic $T_{\text{LRT}}^{(N)}$ in (45) is a quadratic form of the length- $2MN$ augmented observation

vector $\bar{\mathbf{r}}$. Under any optimality criterion such as the Neyman-Pearson, the Bayesian, and the minimax criteria, the optimal detector computes $T_{\text{LRT}}^{(N)}$, compares it with an optimal threshold η , and then declares $\mathcal{H}_1$ if $T_{\text{LRT}}^{(N)} > \eta$ and $\mathcal{H}_0$ otherwise.

The computation of $T_{\text{LRT}}^{(N)}$ requires the matrix inversion of $(\sigma^2 \mathbf{I}_{2MN} + \mathbf{R}_{\bar{\mathbf{r}}})$, which is the $(2MN)$ -by- $(2MN)$ covariance matrix of the augmented observation vector $\bar{\mathbf{r}}$ under $\mathcal{H}_1$, and the matrix-vector multiplication. The major burden in computing $T_{\text{LRT}}^{(N)}$ comes from the matrix inversion and it requires $\mathcal{O}(M^3 N^2)$ complex-valued scalar multiplications when Lemma 8 is used. The matrix-vector multiplication requires only $\mathcal{O}(M^2 N^2)$ complex-valued scalar multiplications. Thus, the overall computational complexity of the LRT statistic is $\mathcal{O}(M^3 N^2)$. Due to the same reason as the frequency-domain WLMSE estimation in Section IV-A, the frequency-domain approach using $\bar{\mathbf{W}}_{MN} \bar{\mathbf{r}}$ is no better in computational complexity than the time-domain approach using $\bar{\mathbf{r}}$.

An alternative observation model that is equivalent to the original one in (44) can be obtained by applying the asymptotic FRESH properization to $\mathbf{r}$ as

$$\begin{aligned} \mathcal{H}_0 : s &= \mathbf{w} \\ \text{versus} \\ \mathcal{H}_1 : s &= \mathbf{y} + \mathbf{w}, \end{aligned} \quad (46)$$

where $s = f(\mathbf{r})$, $\mathbf{y} = f(\mathbf{x})$, and $\mathbf{w} = f(\mathbf{v})$. Since the equivalent observation vector s is not proper in general, the augmentation of s and s^* is still needed to compute the exact LRT statistic of s . Instead, we propose to use a suboptimal LRT statistic as follows, which is a quadratic function only of s .

Definition 15: Given the observation model (46), the proposed test statistic of s is defined by

$$T_p^{(N)} \triangleq \frac{1}{MN} s^H \mathbf{W}_{MN}^H \bar{\Sigma} \mathbf{W}_{MN} s, \quad (47)$$

where $\bar{\Sigma} \triangleq \sigma^{-2} \mathbf{I}_{MN} - (\sigma^2 \mathbf{I}_{MN} + \Sigma)^{-1}$.

It can be immediately seen that $T_p^{(N)}$ can be viewed as the exact LRT statistic of s when the covariance and the complementary covariance matrices of $\mathbf{y}$ are set equal to $\mathbf{W}_{MN}^H \Sigma \mathbf{W}_{MN}$ and $\mathbf{O}_{MN}$ appearing in (24a) and (24b), respectively. Similar to the estimation problem in the previous section, this idea of using the asymptotically equivalent matrices is again motivated by Theorem 2, which naturally leads to the asymptotic equivalence of the LRT statistic $T_{\text{LRT}}^{(N)}$ in (45) and the proposed test statistic $T_p^{(N)}$ in (47).

Now, the asymptotic optimality of the proposed test statistic is provided.

Theorem 4: The proposed test statistic approaches the exact LRT statistic as the number of samples tends to infinity in the sense that

$$\lim_{N \rightarrow \infty} (T_p^{(N)} - T_{\text{LRT}}^{(N)}) = 0, \text{ w.p. } 1, \quad (48)$$

under $\mathcal{H}_0$ and $\mathcal{H}_1$.

Proof: We define $\bar{\Omega} \triangleq \sigma^{-2} \mathbf{I}_{2MN} - (\sigma^2 \mathbf{I}_{2MN} + \Omega)^{-1}$ and introduce a suboptimal test statistic

$$\hat{T}_{\text{LRT}}^{(N)} \triangleq \frac{1}{2MN} \bar{\mathbf{r}}^H \bar{\mathbf{W}}_{MN}^H \bar{\Omega} \bar{\mathbf{W}}_{MN} \bar{\mathbf{r}}, \quad (49)$$

which is obtained by replacing $\hat{\mathbf{R}}_{\bar{x}}$ in (45) with $\bar{\mathbf{W}}_{MN}^H \bar{\Omega} \bar{\mathbf{W}}_{MN}$ and by using the fact that $\bar{\mathbf{W}}_{MN}$ is unitary. This introduction of $\hat{T}_{\text{LRT}}^{(N)}$ is motivated by $\bar{\mathbf{W}}_{MN} \hat{\mathbf{R}}_{\bar{x}} \bar{\mathbf{W}}_{MN}^H \sim \bar{\Omega}$ in (21a). Similar to the proof of Theorem 3, it can be shown that $\hat{\mathbf{R}}_{\bar{x}} \sim \bar{\mathbf{W}}_{MN}^H \bar{\Omega} \bar{\mathbf{W}}_{MN}$. In [38], the LRT problems are considered where the covariance matrix of a proper-complex Gaussian signal vector is either Toeplitz or block Toeplitz. It is shown under both $\mathcal{H}_0$ and $\mathcal{H}_1$ that a suboptimal test statistic, which is obtained by replacing the covariance matrix in the quadratic form of the LRT statistic with its asymptotically equivalent one, converges to the exact LRT statistic w.p. 1 as the number of samples tends to infinity [38, Propositions 1 and 3]. By applying this result, we have $\lim_{N \rightarrow \infty} (\hat{T}_{\text{LRT}}^{(N)} - \hat{T}_{\text{LRT}}^{(N)}) = 0$ w.p. 1 because $\hat{\mathbf{R}}_{\bar{x}} \sim \bar{\mathbf{W}}_{MN}^H \bar{\Omega} \bar{\mathbf{W}}_{MN}$. Therefore, in order to show (48), it now suffices to show $\hat{T}_{\text{LRT}}^{(N)} = T_p^{(N)}$.

Let $\hat{s}$ be the frequency-domain equivalent observation vector that is defined as $\hat{s} \triangleq \bar{\mathbf{W}}_{MN} \mathbf{s}$. Then, by using the permutation matrices $\mathbf{E}_{2M,N}$, $\mathbf{E}_1$, and $\mathbf{E}_2$ that are defined in the proof of Theorem 3 in Section IV, we have

$$\mathbf{E}_{2M,N} \bar{\mathbf{W}}_{MN} \bar{\mathbf{r}} = \begin{bmatrix} \mathbf{E}_1 \hat{s}^* \\ \mathbf{E}_2 \hat{s} \end{bmatrix}. \quad (50)$$

By using (43) and (50), we can rewrite $\hat{T}_{\text{LRT}}^{(N)}$ defined in (49) as $\hat{T}_{\text{LRT}}^{(N)} = (\hat{s}^T \bar{\Sigma}^* \hat{s}^* + \hat{s}^H \bar{\Sigma} \hat{s}) / (2MN) = T_p^{(N)}$ because $\hat{s}^H \bar{\Sigma} \hat{s}$ is real-valued and (43) leads

$$\mathbf{E}_{2M,N} \bar{\Omega} \mathbf{E}_{2M,N}^T = \begin{bmatrix} \mathbf{E}_1 \bar{\Sigma}^* \mathbf{E}_1^T & \mathbf{O}_{MN} \\ \mathbf{O}_{MN} & \mathbf{E}_2 \bar{\Sigma} \mathbf{E}_2^T \end{bmatrix}. \quad (51)$$

Therefore, the conclusion follows. $\square$

The computation of the proposed test statistic $T_p^{(N)}$ requires the pre-processing of $\mathbf{r}$ to obtain $\hat{s}$, the matrix inversion of $(\sigma^2 \mathbf{I}_{MN} + \Sigma)$, and the matrix-vector multiplication to compute the quadratic form. As shown in Section IV, the asymptotic FRESH properization of the length- MN vector $\mathbf{r}$ and the matrix inversion of $(\sigma^2 \mathbf{I}_{MN} + \Sigma)$ require $\mathcal{O}(MN \log(MN))$ and $\mathcal{O}(M^3 N)$ complex-valued scalar multiplications, respectively. In addition, the matrix-vector multiplication requires $\mathcal{O}(M^2 N)$ complex-valued scalar multiplications, because $(\sigma^2 \mathbf{I}_{MN} + \Sigma)$ is an (MN) -by- (MN) block matrix with diagonal blocks of size $(N/2)$ -by- $(N/2)$. Since we consider the block processing where M is a fixed small number and N is much larger than M , the overall complexity in computing the exact LRT statistic and the proposed test statistic as functions only of N can be rewritten now as $\mathcal{O}(N^2)$ and $\mathcal{O}(N \log N)$, respectively. Thus, the computational complexity of the proposed test statistic is much lower than that of the exact LRT statistic.

B. Numerical Results

In this subsection, numerical results are provided that show only the asymptotic optimality of the proposed test statistic because its computational efficiency can be similarly shown

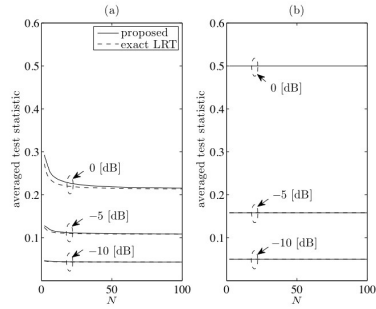


Fig. 9. Statistical averages of the exact LRT and the proposed test statistics versus N for $E_s/N_0 = 0, -5$, and -10 [dB], under (a) $\mathcal{H}_0$ and (b) $\mathcal{H}_1$.

as Fig. 7. Throughout this subsection, the zero-mean improper-complex SOCS Gaussian random process to be detected is obtained by uniformly sampling the QPSK-like signal generated in a similar way to that described in Section IV-B.

The first result is to show the convergence of the proposed test statistic to the exact LRT statistic. Fig. 9 shows the statistical average of the exact LRT and that of the proposed test statistics for symbol energy per noise density $E_s/N_0 = 0, -5$, and -10 [dB]. It can be seen that the statistical averages of the exact LRT and the proposed test statistics approach the same non-zero value as the number of samples tends to infinity under both hypotheses. In particular, they coincide under $\mathcal{H}_1$ because the trace of $\bar{\Sigma}$ are the same as that of $\bar{\mathbf{R}}_{\bar{x}}$. Since the convergence w.p. 1 is hard to show by using Monte-Carlo simulations, we instead show the convergence in probability that is implied by the convergence w.p. 1 [39]. Fig. 10 shows the empirical cumulative distribution function (CDF) of the difference of the proposed test statistic from the exact LRT statistic for symbol energy per noise density $E_s/N_0 = -5$ [dB] and $N = 100, 200$, and 400 , where the empirical CDF is obtained by 10^5 Monte-Carlo runs. It can be seen that, as implied by Theorem 4, the empirical CDF of the difference quickly converges to the unit step function as the number of samples increases under both hypotheses.

The second result is to compare the receiver operating characteristic (ROC) curves of the optimal detector that uses the exact LRT statistic (45) and the proposed detector that uses the proposed test statistic (47). As a common practice in computing the PDF of a quadratic function of Gaussian random vectors [40], we approximate the statistics by gamma random variables to obtain the probability of miss $P_M \triangleq \Pr(\text{declare } \mathcal{H}_0 | \mathcal{H}_1)$ and the probability of false alarm $P_{FA} \triangleq \Pr(\text{declare } \mathcal{H}_1 | \mathcal{H}_0)$. The CDF $F_X(x; a, b)$ of the gamma random variable X with two parameters a and b is given by

$$F_X(x; a, b) = 1 - \left(\frac{ax}{\sqrt{b}} \right)^{b-1}, \quad \forall x > 0, \quad (52)$$

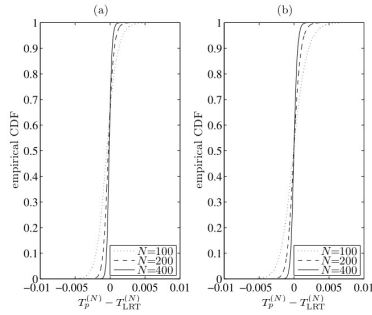


Fig. 10. Empirical CDFs of the difference of the proposed test statistic from the exact LRT statistic for $N = 100, 200$, and 400 , under (a) $\mathcal{H}_0$ and (b) $\mathcal{H}_1$.

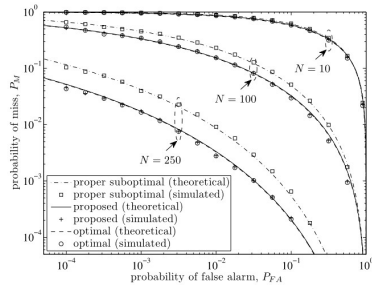


Fig. 11. ROC curves of the optimal and the proposed detectors.

where the Pearson's form of incomplete gamma function $I(u, p)$ is defined by

$$I(u, p) \triangleq \frac{1}{\Gamma(p+1)} \int_0^u t^p e^{-t} dt, \quad (53)$$

and where the gamma function $\Gamma(x)$ is defined by $\Gamma(x) \triangleq \int_0^\infty t^{x-1} e^{-t} dt$ [39]. Note that the mean and the variance of the random variable whose CDF is $F_X(x; a, b)$ are given by ab and ab^2 , respectively. Thus, P_M and P_{FA} of the optimal and the proposed detectors can be computed by using the conditional means and variances of the statistics under $\mathcal{H}_0$ and $\mathcal{H}_1$. Fig. 11 shows the ROC curves of the optimal and the proposed detectors for symbol energy per noise density $E_b/N_0 = -5$ [dB] and $N = 10, 100$, and 250 , where the theoretical results are evaluated by using (52) while the simulated results are obtained from 10^5 Monte-Carlo runs.

It also shows the ROC curve of a suboptimal detector that utilizes the LRT statistic obtained under the assumption of a proper-complex observed signal. It can be seen that the performance of the detectors is accurately approximated by using the gamma distributions. It can be also seen that the proposed detector performs almost the same as the optimal detector does even when $N = 10$.

VI. DISCUSSIONS

As mentioned earlier, there are two types of WL block processing of a DT improper-complex SOCS random process. So far, we only have considered processing the signal vector augmented by its complex conjugate. In this section, we consider the alternative approach of processing the real part of the signal vector augmented by the imaginary part.

Assume again as in Section III-B that $\mathbf{x}$ consists of a finite number of consecutive samples of a zero-mean SOCS random process $X[n]$ with cycle period $M \in \mathbb{N}$. Following the notation used in [41], let $\bar{\mathbf{x}}_{\mathbb{R}}$ be the length- $2MN$ augmented vector defined as

$$\bar{\mathbf{x}}_{\mathbb{R}} \triangleq \begin{bmatrix} \text{Re}\{\mathbf{x}\} \\ \text{Im}\{\mathbf{x}\} \end{bmatrix}, \quad (54)$$

where $\text{Re}\{\cdot\}$ and $\text{Im}\{\cdot\}$ denote the real and the imaginary parts. Then, the covariance matrix $\mathbf{R}_{\bar{\mathbf{x}}_{\mathbb{R}}} \triangleq \mathbf{E}\{\bar{\mathbf{x}}_{\mathbb{R}} \bar{\mathbf{x}}_{\mathbb{R}}^T\} = \mathbf{E}\{\bar{\mathbf{x}}_{\mathbb{R}} \bar{\mathbf{x}}_{\mathbb{R}}^H\}$ of $\bar{\mathbf{x}}_{\mathbb{R}}$ is given by a block matrix

$$\mathbf{R}_{\bar{\mathbf{x}}_{\mathbb{R}}} = \begin{bmatrix} \mathbf{E}\{\text{Re}\{\mathbf{x}\} \text{Re}\{\mathbf{x}\}^T\} & \mathbf{E}\{\text{Re}\{\mathbf{x}\} \text{Im}\{\mathbf{x}\}^T\} \\ \mathbf{E}\{\text{Im}\{\mathbf{x}\} \text{Re}\{\mathbf{x}\}^T\} & \mathbf{E}\{\text{Im}\{\mathbf{x}\} \text{Im}\{\mathbf{x}\}^T\} \end{bmatrix}. \quad (55)$$

Note that each (MN) -by- (MN) submatrix in (55) is a block Toeplitz matrix with block size M -by- M because the cycle period of $X[n]$ is M .

Consider the same estimation and detection problems examined in Sections IV and V but now by using real-valued augmented vectors. First, in the estimation problem, the observation model (26) can be rewritten as

$$\bar{\mathbf{r}}_{\mathbb{R}} = \bar{\mathbf{x}}_{\mathbb{R}} + \bar{\mathbf{v}}_{\mathbb{R}}, \quad (56)$$

where $\bar{\mathbf{r}}_{\mathbb{R}}$ and $\bar{\mathbf{v}}_{\mathbb{R}}$ are defined similarly to $\bar{\mathbf{x}}_{\mathbb{R}}$ from $\mathbf{r}$ and $\mathbf{v}$ in (26), respectively. Note that, since $\mathbf{v}$ is modeled by an additive proper-complex white noise vector with covariance matrix $\sigma^2 \mathbf{I}_{MN}$, $\bar{\mathbf{v}}_{\mathbb{R}}$ is an additive real-valued white noise vector with covariance matrix $(\sigma^2/2) \mathbf{I}_{2MN}$. Thus, the WLMSE estimator $\mathbf{x}_{\text{opt}}$ of $\mathbf{x}$ is obtained as $\mathbf{x}_{\text{opt}} = [\mathbf{I}_{MN} \mathbf{J} \mathbf{I}_{MN}] \bar{\mathbf{x}}_{\mathbb{R}, \text{opt}}$, where

$$\bar{\mathbf{x}}_{\mathbb{R}, \text{opt}} = \mathbf{R}_{\bar{\mathbf{x}}_{\mathbb{R}}} \left(\frac{\sigma^2}{2} \mathbf{I}_{2MN} + \mathbf{R}_{\bar{\mathbf{x}}_{\mathbb{R}}} \right)^{-1} \bar{\mathbf{r}}_{\mathbb{R}} \quad (57)$$

is the LMMSE estimator of $\bar{\mathbf{x}}_{\mathbb{R}}$. Of course, it straightforward that the average MSE of this WLMSE estimator is the same as $\epsilon_{\text{opt}}^{(N)}$ of the WLMSE estimator in (29). Second, in the presence detection problem, the observation model (44) can be rewritten as

$$\begin{aligned} \mathcal{H}_0 : \bar{\mathbf{r}}_{\mathbb{R}} &= \bar{\mathbf{v}}_{\mathbb{R}} \\ \text{versus} \\ \mathcal{H}_1 : \bar{\mathbf{r}}_{\mathbb{R}} &= \bar{\mathbf{x}}_{\mathbb{R}} + \bar{\mathbf{v}}_{\mathbb{R}}, \end{aligned} \quad (58)$$

where $\bar{\mathbf{v}}_{\mathbf{R}}$ is an additive real-valued white Gaussian noise vector with covariance matrix $(\sigma^2/2)\mathbf{I}_{2MN}$. Then, it can be shown that the LRT statistic $T_{\text{LRT}}^{(N)}$ in (45) now as a function of $\bar{\mathbf{r}}_{\mathbf{R}}$ is obtained as

$$T_{\text{LRT}}^{(N)} = \frac{1}{2MN} \bar{\mathbf{r}}_{\mathbf{R}}^T \bar{\mathbf{R}}_{\bar{\mathbf{r}}_{\mathbf{R}}} \bar{\mathbf{r}}_{\mathbf{R}}, \quad (59)$$

where $\bar{\mathbf{R}}_{\bar{\mathbf{r}}_{\mathbf{R}}} \triangleq (2/\sigma^2)\mathbf{I}_{2MN} - ((\sigma^2/2)\mathbf{I}_{2MN} + \mathbf{R}_{\bar{\mathbf{r}}_{\mathbf{R}}})^{-1}$.

The common major burden in computing the LMMSE estimate $\bar{\mathbf{x}}_{\mathbf{R},\text{opt}}$ and the exact LRT statistic $T_{\text{LRT}}^{(N)}$ comes from the inversion of $((\sigma^2/2)\mathbf{I}_{2MN} + \mathbf{R}_{\bar{\mathbf{r}}_{\mathbf{R}}})$. Similar to Lemma 8, the covariance matrix $\mathbf{R}_{\bar{\mathbf{r}}_{\mathbf{R}}}$ can be converted into a $(2MN)$ -by- $(2MN)$ block Toeplitz matrix with block size $(2M)$ -by- $(2M)$, where the permutation matrix $\mathbf{E}_{2,MN}$ defined in the proof of the lemma is used again for the row and column permutations. Thus, the computationally efficient inversion technique in [35] for a block Toeplitz matrix again enables us to perform the inversion with $\mathcal{O}(M^3N^2)$ real-valued scalar multiplications. Therefore, the overall computational complexity is $\mathcal{O}(M^3N^2)$, which is the same as that of the optimal processing by using the complex-valued signal vector augmented by its complex conjugate, though the multiplications are now performed between real-valued scalars.

This computational complexity can be reduced to $\mathcal{O}(MN \log(MN) + M^3N)$ if an asymptotic property of $\mathbf{R}_{\bar{\mathbf{r}}_{\mathbf{R}}}$ is exploited in the frequency-domain. To proceed, we convert the real-valued augmented vector $\bar{\mathbf{x}}_{\mathbf{R}}$ into a complex-valued vector

$$\bar{\mathbf{W}}_{MN} \bar{\mathbf{x}}_{\mathbf{R}} = \begin{bmatrix} \mathbf{W}_{MN} \text{Re}\{\mathbf{x}\} \\ \mathbf{W}_{MN} \text{Im}\{\mathbf{x}\} \end{bmatrix} \quad (60)$$

by pre-multiplying the DFT matrix $\mathbf{W}_{MN}$ to $\text{Re}\{\mathbf{x}\}$ and $\text{Im}\{\mathbf{x}\}$. Then, similar to Proposition 1, we can derive the asymptotic property of the covariance matrix $\bar{\mathbf{W}}_{MN} \mathbf{R}_{\bar{\mathbf{r}}_{\mathbf{R}}} \bar{\mathbf{W}}_{MN}^H$ of $\bar{\mathbf{W}}_{MN} \bar{\mathbf{x}}_{\mathbf{R}}$ as follows. Since the submatrices of $\mathbf{R}_{\bar{\mathbf{r}}_{\mathbf{R}}}$ in (55) are (MN) -by- (MN) block Toeplitz matrices with block size M -by- M , each submatrix is asymptotically equivalent to a block matrix with diagonal blocks when pre- and post-multiplied by DFT and inverse DFT matrices, respectively. Thus, we have

$$\bar{\mathbf{W}}_{MN} \mathbf{R}_{\bar{\mathbf{r}}_{\mathbf{R}}} \bar{\mathbf{W}}_{MN}^H \sim \boldsymbol{\Omega}_{\mathbf{R}}, \quad (61)$$

where $\boldsymbol{\Omega}_{\mathbf{R}}$ is the $(2MN)$ -by- $(2MN)$ matrix given by

$$\boldsymbol{\Omega}_{\mathbf{R}} \triangleq (\bar{\mathbf{W}}_{MN} \mathbf{R}_{\bar{\mathbf{r}}_{\mathbf{R}}} \bar{\mathbf{W}}_{MN}^H) \odot (\mathbf{I}_{2M} \otimes \mathbf{I}_N). \quad (62)$$

Now, this asymptotic property (61) leads to the asymptotically optimal estimator and detector. First, in the estimation problem, we propose the suboptimal estimator $\mathbf{x}_{\text{approx}} = [\mathbf{I}_{MN} \mathbf{j} \mathbf{I}_{MN}] \bar{\mathbf{x}}_{\mathbf{R},\text{approx}}$ of $\mathbf{x}$, where

$$\bar{\mathbf{x}}_{\mathbf{R},\text{approx}} = \bar{\mathbf{W}}_{MN}^H \boldsymbol{\Omega}_{\mathbf{R}} \left(\boldsymbol{\Omega}_{\mathbf{R}} + \frac{\sigma^2}{2} \mathbf{I}_{2MN} \right)^{-1} \bar{\mathbf{W}}_{MN} \bar{\mathbf{r}}_{\mathbf{R}}. \quad (63)$$

Note that (63) is a suboptimal estimator of $\bar{\mathbf{x}}_{\mathbf{R}}$ obtained by using the property (61). It can be easily shown in a similar way to Theorem 3 that the average MSE performance of the proposed estimator approaches that of the WLMSE estimator $[\mathbf{I}_{MN} \mathbf{j} \mathbf{I}_{MN}] \bar{\mathbf{x}}_{\mathbf{R},\text{opt}}$, where $\bar{\mathbf{x}}_{\mathbf{R},\text{opt}}$ is given by (57). Thus, the proposed estimator $\mathbf{x}_{\text{approx}}$ is asymptotically optimal.

Second, in the presence detection problem, we propose by using the property (61) the suboptimal test statistic $T_{\mathbf{R}}^{(N)}$

$$T_{\mathbf{R}}^{(N)} \triangleq \frac{1}{2MN} \bar{\mathbf{r}}_{\mathbf{R}}^T \bar{\mathbf{W}}_{MN}^H \boldsymbol{\Omega}_{\mathbf{R}} \bar{\mathbf{W}}_{MN} \bar{\mathbf{r}}_{\mathbf{R}}, \quad (64)$$

of $\bar{\mathbf{r}}_{\mathbf{R}}$, where $\boldsymbol{\Omega}_{\mathbf{R}} \triangleq (2/\sigma^2)\mathbf{I}_{2MN} - ((\sigma^2/2)\mathbf{I}_{2MN} + \boldsymbol{\Omega}_{\mathbf{R}})^{-1}$. It can be easily shown in a similar way to Theorem 4 that the proposed test statistic approaches the exact LRT statistic as the number of samples tends to infinity in the sense that

$$\lim_{N \rightarrow \infty} (T_{\mathbf{R}}^{(N)} - T_{\text{LRT}}^{(N)}) = 0, \text{ w.p. } 1, \quad (65)$$

under $\mathcal{H}_0$ and $\mathcal{H}_1$.

The common major burden in computing these asymptotically optimal estimate and test statistic comes from the multiplications of the DFT and the inverse DFT matrices and the inversion of $((\sigma^2/2)\mathbf{I}_{2MN} + \boldsymbol{\Omega}_{\mathbf{R}})$, as it can be seen from (63) and (64). As already mentioned, the multiplications of the (MN) -by- (MN) DFT and inverse DFT matrices to a vector of length MN can be efficiently performed with computational complexity of $\mathcal{O}(MN \log(MN))$ when the FFT algorithm is used. Since $\boldsymbol{\Omega}_{\mathbf{R}}$ is a $(2MN)$ -by- $(2MN)$ block matrix with diagonal blocks of size N -by- N , the inversion of $((\sigma^2/2)\mathbf{I}_{2MN} + \boldsymbol{\Omega}_{\mathbf{R}})$ requires $\mathcal{O}(M^3N)$ complex-valued scalar multiplications. Therefore, the overall computational complexity is $\mathcal{O}(MN \log(MN) + M^3N)$.

This order of complexity is not only lower than that of the optimal estimator $\mathbf{x}_{\text{opt}}$ and the exact LRT statistic $T_{\text{LRT}}^{(N)}$ but also the same as that of $\mathbf{x}_p$ and $T_p^{(N)}$ proposed in Sections IV and V by using the asymptotic FRESH properizer. However, as it is already advocated in [4], [5, Ch. 1], and [6], the WL processing using complex-valued representation of the signal is preferred to that using real-valued representation in that it provides mathematical elegance such as algebraic economies, evocative geometry, and probabilistic insights, and implementational advantage such as the efficient use of complex FFT. In addition, it is worth noting that the use of the asymptotic FRESH properizer results in complexity reduction roughly by half, which do not appear in the big O notation, compared to the processing proposed in this section by using the real part of the signal vector augmented by the imaginary part. This is because $\mathbf{x}_p$ and $T_p^{(N)}$ require the matrix inversion and the matrix-vector multiplications to be performed with (MN) -by- (MN) block matrices and length- MN vectors. On the other hand, $\bar{\mathbf{x}}_{\mathbf{R},\text{approx}}$ and $T_{\mathbf{R}}^{(N)}$ require the operations to be performed with $(2MN)$ -by- $(2MN)$ block matrices and length- $2MN$ vectors. Thus, the complexity in computing $\mathbf{x}_p$ and $T_p^{(N)}$ is approximately only a half of that in computing $\bar{\mathbf{x}}_{\mathbf{R},\text{approx}}$ and $T_{\mathbf{R}}^{(N)}$.

Of course, such complexity gaps in computing $\bar{\mathbf{x}}_{\mathbf{R},\text{approx}}$ and $T_{\mathbf{R}}^{(N)}$ against $\mathbf{x}_p$ and $T_p^{(N)}$, respectively, can be removed. Recall that the length of the output $\mathbf{y} = \mathbf{f}(\mathbf{x})$ of the asymptotic FRESH properizer is halved by removing the redundant information in $\bar{\mathbf{W}}_{MN} \bar{\mathbf{x}}$. We can do the same for $\bar{\mathbf{W}}_{MN} \bar{\mathbf{x}}_{\mathbf{R}}$. For the length- MN real-valued vector $\text{Re}\{\mathbf{x}\}$, its DFT $\mathbf{W}_{MN} \text{Re}\{\mathbf{x}\}$ has conjugate symmetry, i.e., $[\mathbf{W}_{MN} \text{Re}\{\mathbf{x}\}]_n = [\mathbf{W}_{MN} \text{Re}\{\mathbf{x}\}]_{MN-n+1}^*$. Similarly,

$[W_{MN}\text{Im}\{x\}]_n = [W_{MN}\text{Im}\{x\}]_{MN-n+1}^*$. Thus, these redundancies can be removed to reduce the computational complexities of $\tilde{x}_{\text{R,approx}}$ and $I_{\text{R}}^{(M)}$ by half. This removal of redundancy in the frequency domain is nothing but what the asymptotic FRESH properizer does in the frequency domain. Note that, as already shown in [5], the real-valued augmented vector $\tilde{x}_{\text{R}}$ can be rewritten by using the complex-valued augmented vector $\tilde{x}$ in (16) as

$$\tilde{x}_{\text{R}} = \frac{1}{2} T_{MN}^H \tilde{x}, \quad (66)$$

where the $(2MN)$ -by- $(2MN)$ invertible matrix T_{MN} is defined as

$$T_{MN} \triangleq \begin{bmatrix} I_{MN} & jI_{MN} \\ I_{MN} & -jI_{MN} \end{bmatrix}. \quad (67)$$

Thus, the output $y = f(x)$ in (17) of the asymptotic FRESH properizer can be rewritten in terms of the DFTs of $\text{Re}\{x\}$ and $\text{Im}\{x\}$ as

$$\begin{aligned} y &= W_{MN}^H \bar{G}_{M,N} \bar{W}_{MN} T_{MN} \tilde{x}_{\text{R}}, \\ &= W_{MN}^H \left((I_{MN} + S_{M,N}) G_{M,N} W_{MN} \text{Re}\{x\} \right. \\ &\quad \left. + j(I_{MN} - S_{M,N}) G_{M,N} W_{MN} \text{Im}\{x\} \right). \end{aligned} \quad (68a, 68b)$$

Therefore, the removal of redundancy by the asymptotic FRESH properizer is equivalent to that by pre-multiplying $G_{M,N}$ to $W_{MN}\text{Re}\{x\}$ and $W_{MN}\text{Im}\{x\}$. However, it still needs to be noted that the use of the asymptotic FRESH properizer is more advantageous in that it enables the adoption of the conventional signal processing techniques and algorithms that are already developed for the block processing of proper-complex random processes.

VII. CONCLUSIONS AND FUTURE WORK

In this paper, the asymptotic FRESH properizer is proposed as a pre-processor for the block processing of a finite number of consecutive samples of a DT improper-complex SOCS random process. It turns out that the output of this pre-processor can be well approximated by a proper-complex random vector that has a highly-structured frequency-domain covariance matrix for sufficiently large block size. The asymptotic propriety of the output allows the direct application with negligible performance degradation of the conventional signal processing techniques and algorithms dedicated to the block processing of proper-complex random vectors. Moreover, the highly-structured frequency-domain covariance matrix of the output facilitates the development of low-complexity post-processors. By solving the signal estimation and the signal presence detection problems, it is demonstrated that the asymptotic FRESH properizer leads to the simultaneous achievement of computational efficiency and asymptotic optimality. The use of the real part of the signal augmented by the imaginary part instead of the signal augmented by its complex conjugate is discussed for the estimation and detection problems as well as for the asymptotic FRESH properization. Further research is warranted to apply this pre-processor to various communications and signal processing problems involving the block processing of a DT improper-complex SOCS random

process and to derive such almost optimal low-complexity post-processors. Since the cycle period is assumed known throughout this paper, the estimation of this parameter and the possible mismatch analysis are interesting future research directions. The convergence-rate analysis of systems employing the asymptotic FRESH properizer is also an interesting research direction.

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