CM17v251105wed

2025년 11월 5일 수요일 15:30

□ MS Periodicity

Motivated by $x(t) = x(t+T), \forall t$

$\Leftrightarrow x(t) - x(t+T) = 0, \forall t$

$\Leftrightarrow |x(t) - x(t+T)|^2 = 0, \forall t$

Def. $E\{|X(t) - X(t+T)|^2\} = 0, \forall t$

Lemma. $X(t) = X(t+T), \forall t$ in the MS sense

$$\Rightarrow \begin{cases} \text{(i)} & \mu_X(t) = \mu_X(t+T), \\ \text{(ii)} & r_{XX}(t,s) = r_{XX}(t+T, s), \forall(t,s) \\ & = r_{XX}(t, s+T), \forall(t,s) \\ & = r_{XX}(t+mT, s+nT), \forall(t,s), \forall(m,n) \end{cases}$$

proof. (i) By $E\{|X|^2\} \geq |E\{X\}|^2$.

$$\int_{-\infty}^{\infty} x^2 f_X(x)dx \geq \left(\int_{-\infty}^{\infty} x f_X(x)dx\right)^2$$

We have
$$0 \leq |E\{X(t) - X(t+T)\}|^2 \leq E\{|X(t) - X(t+T)|^2\} = 0, \forall t$$

$$\Rightarrow E\{X(t) - X(t+T)\} = 0, \forall t \Leftrightarrow \mu_X(t) = \mu_X(t+T), \forall t$$

(ii) By $|E\{XY^*\}|^2 \leq E\{|X|^2\}E\{|Y|^2\}$ $\qquad \langle a,b\rangle \leq \sqrt{\langle a,a\rangle\langle b,b\rangle}$

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$\langle a,b\rangle \le \sqrt{\langle a,a\rangle\langle b,b\rangle}$

$|a^H b|^2 \le \|a\|^2\|b\|^2$.

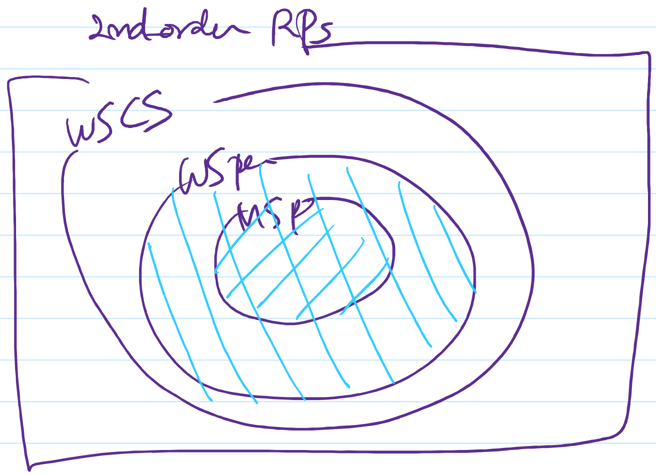
we have

$$0 \le \underbrace{\left|E\left[X(t)\{X(s)-X(s+T)\}^*\right]\right|^2}_{=0,\ \forall (t,s)} \le \underbrace{E\{|X(t)|^2\}\underbrace{E\{|X(s)-X(s+T)|^2\}}_{=0,\ \forall s}}_{=0\ \forall (t,s)},\ \forall (t,s)$$

$\Rightarrow E\{X(t)(X(s)-X(s+T))^*\}=0$

$\Leftrightarrow r_{xx}(t,s)=r_{xx}(t,s+T),\ \forall(t,s)$

$= r_{xx}(t+T,s),\ \forall(t,s)$ $\Rightarrow r_{xx}(t,s)=r_{xx}(t+mT,s+nT),\ \forall(t,s)\ \forall(m,n)$



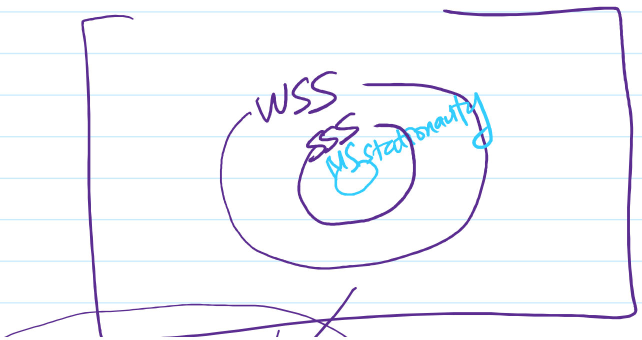
□. Recall Def. $X(t)$ is WSCS w/ period T if

(i) $\mu_x(t)=\mu_x(t+T)$, $\forall t$

(ii) $r_{xx}(t,s)=r_{xx}(t+T,s+T)$, $\forall(t,s)$

MS periodicity $\Rightarrow$ WS periodicity $\Rightarrow$ WSCS

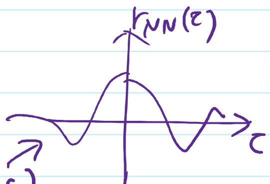
□ cf)



[$N(t)$ WSS Gaussian
$S_{NN}(f)$

$\mu_N(t)=a\ \forall t$

$r_{NN}(t,s)=r_{NN}(t-s)$



□Q MS Stationarity

$E\{|X(t)-X(t-\tau)|^2\}=0$, $\forall t, \forall \tau$

MS periodicity

$E\{|X(t)-X(t+T)|^2\}=0$

$E\{|X(t)-X(t-\tau)|^2\}=0$

WS stationarity

(i) $\mu_X(t)=c, \forall t$

(ii) $r_{XX}(t,s)=r_{XX}(t-s,0), \forall (t,s)$,

WS periodicity

(i) $\mu_X(t)=\mu_X(t+T)$

(ii) $r_{XX}(t,s)=r_{XX}(t+mT, s+nT), \forall (t,s)$ $\forall m,n$

/ex/ $X(t)$ WSS Gaussian

$S_{NN}(f)$

$\mu_N(t)=a\ \forall t$

$r_{NN}(t,s)=r_{NN}(t-s)$

$x(t)=x(t-\tau), \forall t, \forall \tau$

$=c.\ \forall t$

$x(t)=x(t-T), \forall t, \exists T_0$

periodicity

$x(t)=x(t-\tau), \forall (t,\tau)$

$=c, \forall t$

$x(t)-x(t-\tau)=0, \forall (t,\tau)$

$|x(t)-x(t-\tau)|^2=0 \quad \forall (t,\tau)$

$E\{|X(t)-X(t+\tau)|^2\}=0, \forall (t,\tau)$.

$=E\{X(t)X^*(t)\}$

$=E\{X(t-\tau), X^*(0)\}$

$E\{X(t-\tau)X^*(t-\tau)\}$

$=E\{X(t)X^*(t)\}$

$R_{XX}(t,t)-R_{XX}(t,t-\tau)-R_{XX}(t-\tau,t)+R_{XX}(t-\tau,t-\tau)=0 \quad \forall t, \forall \tau.$

$\| \; R_{XX}(t,t)$

almost surely

$X(t)=X(t-\tau)$ for all t,τ

$2R_{XX}(t,$

□ Lemma. $X(t) = X(t+T), \forall t$ in the MS sense $\Rightarrow$

(i) $\mu_X(t) = \mu_X(t+T), \forall t$ &

(ii) $R_{XX}(t,s) = R_{XX}(t+mT, s+nT), \forall (t,s), \forall (m,n)$

(proof)

(i) $0 \le |E\{X(t) - X(t+T)\}|^2 \le E\{|X(t) - X(t+T)|^2\} = 0, \forall t.$

$\Rightarrow E\{X(t) - X(t+T)\} = 0, \forall t$

$\Leftrightarrow \mu_X(t) = \mu_X(t+T), \forall t$

(ii) $|E\{X(t)(X(s) - X(s+T))^*\}|^2 \le E\{|X(t)|^2\} \underbrace{E\{|X(s) - X(s+T)|^2\}}_{=0,\ \forall s} = 0 \quad \forall (t,s)$

$\Rightarrow E\{X(t)(X(s) - X(s+T))^*\} = 0, \forall (t,s)$

$\Leftrightarrow R_{XX}(t,s) = R_{XX}(t, s+T).$

Similarly

$$R_{xx}(t,s) = R_{xx}(t+T, s)$$

$$\therefore \; R_{xx}(t,s) = R_{xx}(t, s+T) = R_{xx}(t, s+nT)$$
$$= R_{xx}(t+T, s) = R_{xx}(t+mT, s)$$
$$= R_{xx}(t+mT, s+nT) \qquad \forall (t,s) \;\; \forall (m,n)$$

☐ MS stationarity ?

☐ Def. $\boxed{E\{|X(t) - X(t-\tau)|^2\} = 0, \quad \forall (t,\tau)}$

☐ Lemma. MSS $\Rightarrow$ WSS

(i) $0 \le |E\{X(t) - X(t-\tau)\}|^2 \le E\{|X(t) - X(t-\tau)|^2\} = 0, \; \forall (t,\tau)$

$$\Rightarrow E\{X(t) - X(t-\tau)\} = 0, \quad \forall (t,\tau)$$

$$\Leftrightarrow \mu_x(t) = \mu_x(t-\tau), \; \forall (t,\tau)$$

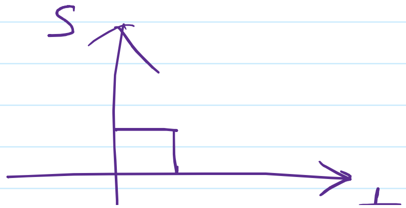
$$\Leftrightarrow \mu_x(t) = c, \; \forall t$$

(ii) $|E\{X(t)(X(s) - X(s-\tau))^*\}|^2 \le E\{|X(t)|^2\} \boxed{E\{|X(s) - X(s-\tau)|^2\} = 0}, \forall (t,s,\tau)$

$= 0, \; \forall (s,\tau)$

$$\Rightarrow \boxed{R_{xx}(t,s) = R_{xx}(t, s-\tau)}, \quad \forall (t,s,\tau)$$

$$\Leftrightarrow R_{xx}(t,s) = R_{xx}(t,s') \quad \forall (t,s,s')$$



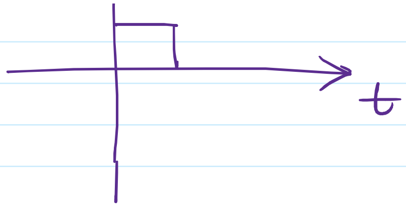
Similarly

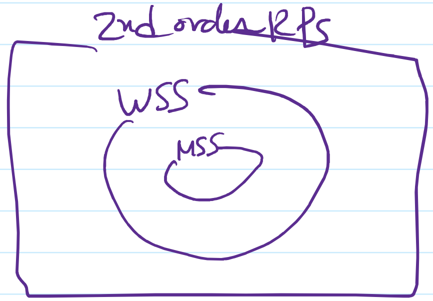
Similarly

$R_{XX}(t,s) = R_{XX}(t',s') \quad \forall (t,s,t',s')$

$= \sigma^2 \quad \forall (t,s)$

$\Rightarrow R_{XX}(t,s) = R_{XX}(t-s,0). \quad \forall (t,s)$





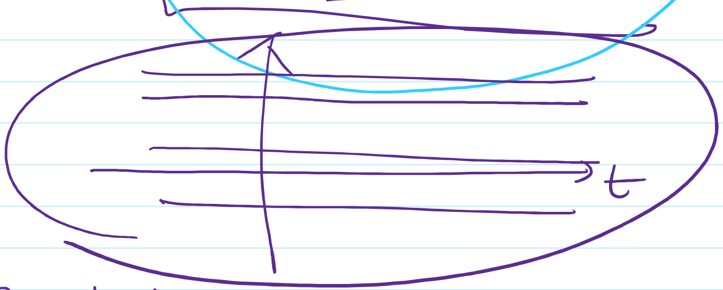
"No meaning!"

$X(t) = X, \forall t. \quad E[X] = c$

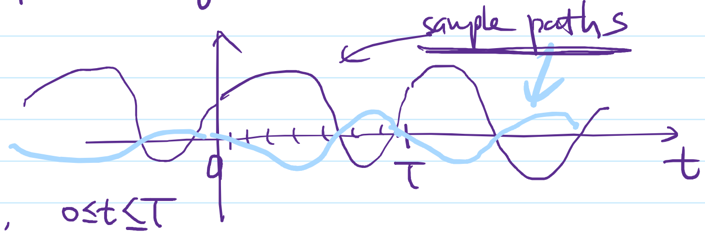
MS sense. $\quad E[X^2] = \sigma^2$

← A RV is generated & repeats for all t.

→ No much meaning.



☐ MS Periodicity



For $[0,T)$, we have uncountably infinite # of correlated RVs.

Then, this period repeats to make the sample path a periodic function.

$X(t), \; 0 \le t \le T$

$\rightarrow Y(t) = \sum_{n=-\infty}^{\infty} X(t-nT), \; -\infty < t < \infty$

→ Sometimes, we only have a finite-interval observation.

For convenience, we may periodically extend it to make it

an MS periodic RP, where we may apply Fourier basis to

For convenience, we may periodically extend it to make it an MS periodic RP, where we may apply Fourier basis for analysis. For example, KL expansion may be possible w/ Fourier basis.

$C_{xx}\underline{v}_i = \lambda_i \underline{v}_i$ ⟺ $\int_0^T C_{xx}(t,s)\, v_i(s)\, ds = \lambda_i v_i(t), \quad 0 \le t,s < T$

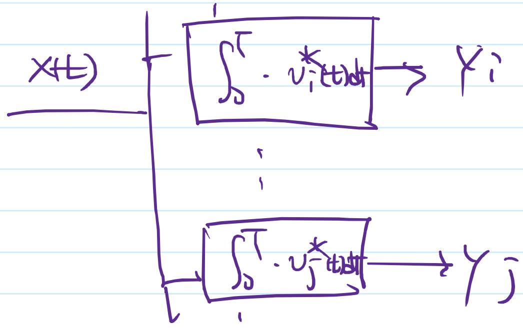
$\Rightarrow C_{xx} = \sum \lambda_i \underline{v}_i \underline{v}_i^H$ ⟺ $C_{xx}(t,s) = \sum_i \lambda_i v_i(t) v_i^*(s), \quad 0 \le t,s < T$

$V = [\underline{v}_1 \cdots \underline{v}_N]$

where $\int_0^T v_i(t) v_j^*(t)\, dt = \delta_{ij}$

$\Rightarrow C_{xx} = V \Lambda V^H$

$\Rightarrow \underline{X} \to \boxed{V^H} \to \underline{Y}$ ⟺



$\Rightarrow \underline{Y} = V^H \underline{X}$

$\Leftrightarrow \underline{X} = V \underline{Y}$

$Cov(\underline{Y}) = \Lambda$.

$= diag(\underline{\lambda})$ ⟹ $X(t) = \sum_i Y_i v_i(t), \quad 0 \le t < T$

where $Cov(Y_i, Y_j) = \lambda_i \delta_{ij}$ i.e. uncorrelated RVs.

$v_k(t)$ can be $e^{j2\pi \frac{k}{T}t}$, then Y_k is the Fourier Coefficient

$x(t) = \sum Y_k e^{j2\pi \frac{kt}{T}} \quad 0 < t < T$

$X(t) = \sum_k Y_k e^{j2\pi \frac{kt}{T}}, \quad 0 \le t < T.$

$Cov(Y_i, Y_j)$

□ A time-limited 2nd-order RP $X(t)$, $0 \le t \le T$.

$E\{X(t)\} = \mu_X(t). \quad 0 \le t \le T$

$R^H = R.$

$E\{X(t)X^*(s)\} = r_{xx}(t,s) \quad 0 \le t, s \le T \quad \to = r^*_{xx}(s,t) \cdots$ (i)

$= E\{X^*(s)X(t)\}$

(ii) $\int_0^T\int_0^T a^*(t)\, r_{xx}(t,s)\, a(s)\, dt\, ds = E\left\{\left|\int_0^T a^*(t)X(t)\, dt\right|^2\right\} \ge 0.$

$E\left\{\left(\int_0^T a^*(t)X(t)dt\right)\left(\int_0^T a^*(t)X(t)dt\right)^*\right\}$

$= E\left[\int_0^T\int_0^T a^*(t)\, X(t)X(s)\, a(s)\, dt\, ds\right]$

$= \int_0^T\int_0^T a^*(t)\, r_{xx}(t,s)\, a(s)\, dt\, ds$

$\int_0^T \boxed{r_{xx}(t,s)}\, v_i(s)\, ds = \lambda_i v_i(t), \quad 0 \le t \le T.$ ← Fredholm Integral Equation

Kernel

$(\lambda_i, v_i(t))$;

$\int_0^T v_i(t)\, v_j^*(t)\, dt = \delta_{ij}$

$v_i(t) \to$ LTV filter w/ $r_{xx}(t,s)$ $\to \lambda_i v_i(t)$

$v_i(t) = 0, \quad t<0, \ t>T$

$r_{xx}(t,s) = \sum_i \lambda_i v_i(t) v_i^*(s)$

cf) $R = \sum_i \lambda_i \underline{v}_i \underline{v}_i^H$

$= V \Lambda V^H \quad \Lambda = diag(\lambda_1, \lambda_2 \cdots)$

$V = [\underline{v}_1, \underline{v}_2 \cdots]$

$\underline{X} \to \boxed{?} \to \underline{Y} \quad Cov(\underline{Y}) = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \\ & & & \sigma_N^2 \end{bmatrix}$

$E[\underline{X}] = \underline{\mu}$

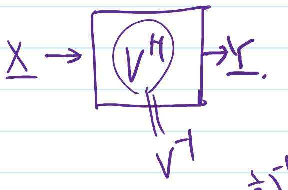
$Cov[\underline{X}] = C_{\underline{XX}} = C_{\underline{XX}}^H \succeq 0.$

$= V\Lambda V^H = V\Lambda^{\frac{1}{2}} V^H V \Lambda^{\frac{1}{2}} V^H$

$\underline{Y} = V^H \underline{X} \longrightarrow \underline{X} = V\underline{Y}$

$$= V\Lambda V^H = V\Lambda^{\frac{1}{2}} V^H V \Lambda^{\frac{1}{2}} V^H$$

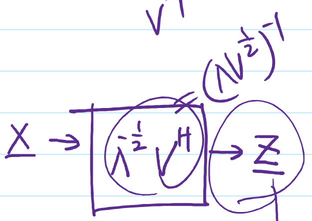
$$\boxed{Y = V^H X} \longrightarrow \quad X = VY$$



$E[Y] = V^H \mu_X$

$\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}} = \Lambda$

$$\mathrm{Cov}[Y] = \mathrm{Cov}[V^H X]$$
$$= E\left[V^H(X-\mu_X)\left(V^H(X-\mu_X)\right)^H\right]$$
$$= V^H \,\mathrm{Cov}[X]\, V = \cancel{V^H}(\cancel{V}\Lambda \cancel{V^H})V = \Lambda.$$



$E[Z] = \Lambda^{-\frac{1}{2}} V^H \mu_X$

$\mathrm{Cov}[Z] = I.$

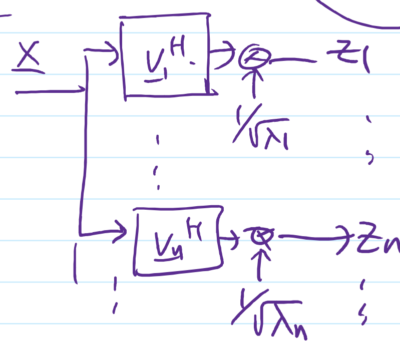
$$X = V\Lambda^{\frac{1}{2}} Z$$

$$= V \begin{bmatrix} \sqrt{\lambda_1}\, z_1 \\ \vdots \\ \sqrt{\lambda_N}\, z_N \end{bmatrix} = [v_1 v_2 \cdots v_N]\begin{bmatrix} \\ \\ \end{bmatrix}$$

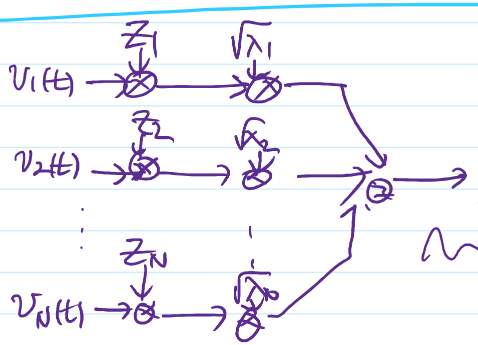
$$= \sum_i \sqrt{\lambda_i}\, z_i\, v_i$$

z_i's are uncorrelated RVs w/ same unit variance.

$\equiv$



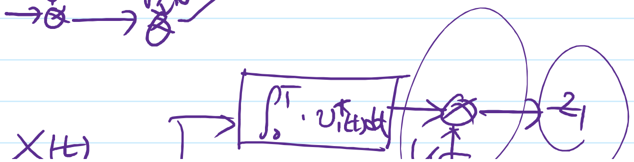
$$\begin{cases} Z \quad \mu_Z \\ \mathrm{Cov}(Z) = I \end{cases}$$



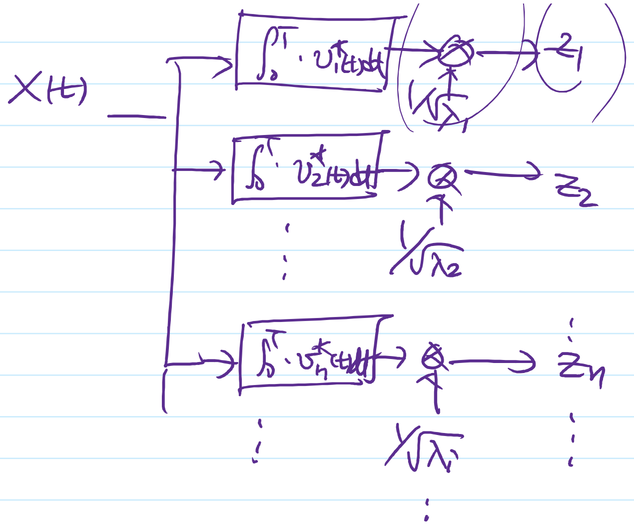
$$X(t) = \sum_i \sqrt{\lambda_i}\, z_i\, v_i(t)$$

$$\int_0^T v_i(t)\, v_j^*(t)\, dt = \delta_{ij}$$

where



where



SIS and Nyquist Criterion, Orthogonality vs. Bi-orthogonality, SNR, SIR, SINR, Zero-Forcing vs. MMSE

2025년 11월 5일 수요일
10:26

☐ Nyquist Criterion for zero ISI

intersymbol interference

Def. if $\int_{-\infty}^{\infty} p(t-mT)\, p^*(t-nT)\, dt = C\, \delta_{m,n}$, $C > 0$ $\forall m,n \in \mathbb{Z}$.

$\left(\begin{array}{l} m=n \Rightarrow \int_{-\infty}^{\infty} |p(t)|^2 dt = C \\ m \neq n \quad \int_{-\infty}^{\infty} p(t)\, p^*(t-nT)\, dt = \delta_{n,0} \end{array} \right)$.

p(t) satisfies a Nyquist criterion for T.

~~when the symbol rate is 1/T~~.

RP

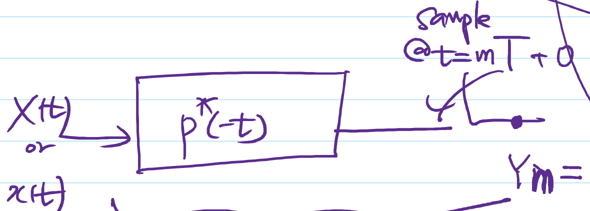
Caution!

Caution!

RP

$X(t)=\sum_{n=-\infty}^{\infty} d_n\, p(t-nT)$ (?) $(d_n)_n$ is a data symbol seq. w/ d_n's are RVs.

$x(t)=\sum_{n=-\infty}^{\infty} a_n\, p(t-nT)$ (?) $(a_n)_n$ " a deterministic sequence w/ finite energy.



$\underbrace{\qquad\qquad\qquad}\ Y_m =$ d_m / a_m

$\equiv$