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Jensen's Inequality: Special Case Application

A Note on Expected Value and Variance

November 5, 2025

The Inequality: $|E(X)|^2 \leq E(|X|^2)$

This inequality is a special form of **Jensen's Inequality** or a direct consequence of the **Cauchy-Schwarz Inequality**. It does not have a unique, widely recognized, special name.

1. As a Result of Jensen's Inequality

Jensen's Inequality states that for a convex function ϕ and a random variable X :

$$\phi(E[X]) \leq E[\phi(X)]$$

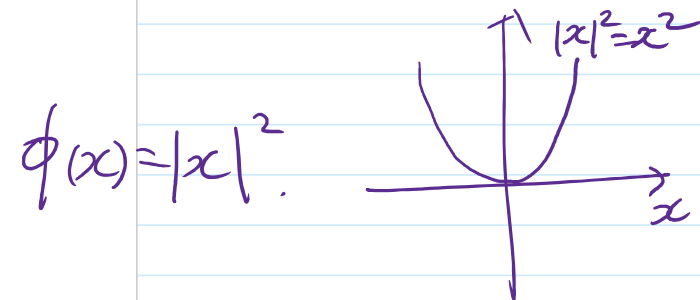
Applying this with the **convex function** $\phi(x) = x^2$ (since $\phi''(x) = 2 \geq 0$):

$$(E[X])^2 \leq E[X^2]$$

Since $(E[X])^2 = |E[X]|^2$ and $E[X^2] = E[|X|^2]$, this directly yields:

$$|E(X)|^2 \leq E(|X|^2)$$

2. As a Consequence of Non-Negative Variance



$$|E(X)|^2 \leq E(|X|^2)$$

2. As a Consequence of Non-Negative Variance

The inequality is fundamentally equivalent to the statement that the **variance** of a random variable is non-negative ($\text{Var}(X) \geq 0$).

real $\text{Var}(X) = E[X^2] - (E[X])^2 \geq 0$

Rearranging the terms confirms the relationship:

$$(E[X])^2 \leq E[X^2]$$

3. As a Special Case of Cauchy-Schwarz

The Cauchy-Schwarz Inequality for expected values is:

real $(E[XY])^2 \leq E[X^2]E[Y^2]$

1

complex $\text{Var}(X) = E[|X|^2] - |E[X]|^2 \geq 0$

complex $|E[XY^*]|^2 \leq E[|X|^2]E[|Y|^2]$

$\downarrow Y=1$

$$|E[X]|^2 \leq E[|X|^2] \cdot 1$$

Setting the random variable $Y = 1$ (a constant) results in:

$$(E[X \cdot 1])^2 \leq E[X^2]E[1^2]$$

$$(E[X])^2 \leq E[X^2] \cdot 1$$

$$(E[X])^2 \leq E[X^2]$$

Second-Order Random Processes: Definitions, Implications, and Significance

Gemini AI Assistant and Joen Ho Cho

November 6, 2025

Abstract

This document rigorously defines the class of **second-order random processes** $X(t)$ by the condition of finite second moment, $\mathbb{E}\{|X(t)|^2\} < \infty$. We demonstrate that this fundamental requirement is both necessary and sufficient to guarantee the existence and finiteness of the process's essential statistical descriptors: the **mean function** $\mu_X(t)$, the **autocorrelation function** $R_{XX}(t_1, t_2)$, and the **autocovariance function** $C_{XX}(t_1, t_2)$. The finiteness of the autocorrelation function is proven using the **Schwarz Inequality**. Furthermore, the document discusses the significance of second-order processes as the foundation for defining and analyzing **Wide-Sense Stationary (WSS)** processes, which are crucial for signal processing and communication theory.

1 Definition of a Second-Order Random Processes

A **random process** (or stochastic process) $X(t)$ is said to be a **second-order random process** if the expected value of the squared magnitude of $X(t)$ is finite for every time instant t in its domain $\mathcal{I}$.

1.1 Formal Definition

The process $X(t)$ is a second-order random process if

$$0 \leq \mathbb{E}\{|X(t)|^2\} < \infty \text{ for all } t \in \mathcal{I} \quad (1)$$

where $\mathcal{I}$ is the index set (or domain) of the process.

1.2 Implications: Existence of Mean and Variance

The condition in Equation (1) ensures that the process has finite **power** at every time t . This automatically implies that the **mean** and **variance** of $X(t)$ are also finite:

- The **Mean** $\mu_X(t) = \mathbb{E}\{X(t)\}$ is finite.
- The **Variance** $\sigma_X^2(t) = \mathbb{E}\{|X(t) - \mu_X(t)|^2\}$ is finite.

2 Key Statistical Functions

For a second-order random process $X(t)$, the following statistical functions are well-defined and essential for its analysis.

$\mathbb{E}\{|X(t)|^2\}$ existence.

$$\left(\mathbb{E}\{X(t)\} \right)^2 \leq \mathbb{E}\{|X(t)|^2\} < \infty, \forall t$$

$$= \mathbb{E}\{|X(t)|^2\} - (\mu_X(t))^2 \text{ exists } \forall t$$

$$\Leftrightarrow \mathbb{E}\{X(t)\} \text{ exists } \&$$

2 Key Statistical Functions

- $\mathbb{E}\{X(t)\}$ (-1) $\mu_X(t)$

$\Leftrightarrow \mathbb{E}\{X(t)\}$ exists & .

For a second-order random process $X(t)$, the following statistical functions are well-defined and essential for its analysis.

2.1 Mean Function

The mean function describes the average behavior of the process over time.

$$\mu_X(t) = \mathbb{E}\{X(t)\} \quad (2)$$

1

2.2 Autocorrelation Function

The autocorrelation function measures the statistical dependence between the process values at two different time instants, t_1 and t_2 .

$$R_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X^*(t_2)\} \quad (3)$$

where $X^*(t_2)$ denotes the complex conjugate of $X(t_2)$. For real-valued processes, $X^*(t_2) = X(t_2)$.

2.3 Autocovariance Function

The autocovariance function measures the correlation between the process values at t_1 and t_2 , after removing the effect of the mean.

$$C_{XX}(t_1, t_2) = \text{Cov}[X(t_1), X(t_2)] = \mathbb{E}\{[X(t_1) - \mu_X(t_1)][X^*(t_2) - \mu_X^*(t_2)]\} \quad (4)$$

The autocorrelation and autocovariance functions are related by:

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X^*(t_2) \quad (5)$$

3 Existence of the Autocorrelation Function

The **Autocorrelation Function** $R_{XX}(t_1, t_2)$ is defined as:

$$R_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X^*(t_2)\} \quad (6)$$

3.1 Proof of Finiteness using the Schwarz Inequality

To prove that $R_{XX}(t_1, t_2)$ is finite, we use the **Schwarz Inequality** (or Cauchy-Schwarz inequality) for random variables U and V :

3.1 Proof of Finiteness using the Schwarz Inequality

To prove that $R_{XX}(t_1, t_2)$ is finite, we use the **Schwarz Inequality** (or Cauchy-Schwarz inequality) for random variables U and V :

$$|\mathbb{E}\{UV^*\}|^2 \leq \mathbb{E}\{|U|^2\} \mathbb{E}\{|V|^2\} \quad (7)$$

Let $U = X(t_1)$ and $V = X(t_2)$. Substituting these into the inequality yields:

$$|R_{XX}(t_1, t_2)|^2 = |\mathbb{E}\{X(t_1)X^*(t_2)\}|^2 \leq \mathbb{E}\{|X(t_1)|^2\} \mathbb{E}\{|X(t_2)|^2\} \quad (8)$$

Since $X(t)$ is a second-order random process, the condition in (11) guarantees that:

- $\mathbb{E}\{|X(t_1)|^2\} < \infty$
- $\mathbb{E}\{|X(t_2)|^2\} < \infty$

Therefore, the product on the right-hand side is finite:

$$\mathbb{E}\{|X(t_1)|^2\} \mathbb{E}\{|X(t_2)|^2\} < \infty \quad (9)$$

This implies that $|R_{XX}(t_1, t_2)|^2$ is finite, and consequently, the **Autocorrelation Function** $R_{XX}(t_1, t_2)$ is **finite and exists** for all $t_1, t_2 \in T$.

4 Existence of the Autocovariance Function

The **Autocovariance Function** $C_{XX}(t_1, t_2)$ is related to the autocorrelation function and the mean function $\mu_X(t) = \mathbb{E}\{X(t)\}$ by:

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X^*(t_2) \quad (10)$$

The existence of $C_{XX}(t_1, t_2)$ relies on the finiteness of the terms on the right-hand side of Equation (10):

1. **Finiteness of $R_{XX}(t_1, t_2)$:** As shown above, this is guaranteed by the second-order condition.
2. **Finiteness of $\mu_X(t)$:** The second-order condition, $\mathbb{E}\{|X(t)|^2\} < \infty$, implies the finiteness of the first moment, $\mathbb{E}\{|X(t)|\} < \infty$ (a result also stemming from the Schwarz Inequality). Since the first moment is finite, the mean function $\mu_X(t)$ is finite for all t .

Since $C_{XX}(t_1, t_2)$ is the difference between two finite quantities, it must also be finite. Thus, the **Autocovariance Function** $C_{XX}(t_1, t_2)$ is **finite and exists** for all $t_1, t_2 \in T$.

5 Significance

The class of second-order random processes is crucial in engineering and physics, especially in signal processing and communication theory, because it is the prerequisite for defining and analyzing **Wide-Sense Stationary (WSS)** processes and performing **linear filtering** operations. A random process $X(t)$ is a **second-order random process** if the expected value of its squared magnitude is finite for

A 2nd-order RP

Def $\checkmark X(t) \equiv \underline{\text{WSS}}$ if

(i) $\mu_X(t) = c, \forall t$

(ii) $R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2, 0)$
 $= R_{XX}(t_1 - t_2), \forall t_1, t_2$

Def. A 2nd-order RP $X(t)$ is SOS if

(i) $\mu_X(t) = c, \forall t$ Second-order stationary

(ii) $R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2, 0) = R_{XX}(t_1 - t_2), \forall t_1, t_2$

(iii) $\tilde{R}_{XX}(t_1, t_2) = \tilde{R}_{XX}(t_1 - t_2, 0) = \tilde{R}_{XX}(t_1 - t_2), \forall t_1, t_2$

ef) Stationary to order 2.

(i) $F_{X(t)}(x) = F_{X(t+\Delta)}(x), \forall (x, \Delta, t)$

(ii) $F_{X(t_1), X(t_2)}(x_1, x_2)$
 $= F_{X(t_1+\Delta), X(t_2+\Delta)}(x_1, x_2)$
 $\forall (t_1, t_2, \Delta, x_1, x_2).$

Def A 2nd-order RP

Def $\checkmark X(t) \equiv \underline{\text{WSS}}$ if WSS w/ cycle period T

(i) $\mu_X(t) = c, \forall t$ ~~$\mu_X(t+T)$~~

(ii) $R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2, 0)$

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(i) $\mu_X(t) = c, \forall t$ Second-order stationary ~~$\mu_X(t+T)$~~

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$$(i) \quad \mu_X(t_1, t_2) = \mu_X(t_1, 0) \mu_X(t_2, 0)$$

$$(ii) \quad R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2, 0) \\ = R_{XX}(t_1 - t_2), \quad \forall(t_1, t_2) \\ = R_{XX}(t_1 - T, t_2 + T), \quad \forall(t_1, t_2)$$

$$(i) \quad R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2, 0) \mu_X(t_1, t_2)$$

$$(ii) \quad \tilde{R}_{XX}(t_1, t_2) = \tilde{R}_{XX}(t_1 - t_2, 0) = \tilde{R}_{XX}(t_1 - t_2) \\ \tilde{R}_{XX}(t_1 + T, t_2 + T), \quad \forall(t_1, t_2)$$

The Origin of Fourier Transform with Respect to Delay in Spectral Analysis

Gemini AI Assistant and Joon Ho Cho

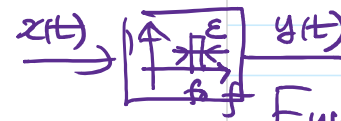
November 6, 2025

Abstract

This document explores the fundamental origin of the **Fourier Transform with respect to the delay variable** (τ) in classical spectral analysis. We establish this principle by examining the analysis of a deterministic, finite-energy signal $x(t)$. The **Energy Spectral Density (ESD)**, $E_{xx}(f)$, is initially defined in the frequency domain as the magnitude squared of the signal's Fourier Transform. Through the application of the **Convolution Theorem** and the properties of time reversal and conjugation, we rigorously prove the equivalence of this definition to the Fourier Transform of the **time-averaged autocorrelation function**, $r_{xx}(\tau)$. This relationship, known as the **Deterministic Wiener-Khinchine Theorem**, provides the essential logical link: spectral content ($E_{xx}(f)$) is inherently dual to time correlation ($r_{xx}(\tau)$), thereby dictating the necessary transformation from the time-lag domain to the frequency domain for all subsequent spectral definitions, including the Power Spectral Density (PSD) for random processes.

1 Equivalence of ESD and Autocorrelation Fourier Transform

For a deterministic signal $x(t)$ with finite energy $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$, the equivalence between the Energy Spectral Density (ESD) and the Fourier Transform of the Time-Averaged Autocorrelation Function forms



$$E_{yy} \approx |X(f)|^2 \epsilon_0$$

$$Y(f) = X(f) H(f)$$

$$|Y(f)|^2 = |X(f)|^2 |H(f)|^2$$

$$E_{yy} = \int_{-\infty}^{\infty} |Y(f)|^2 df \approx |X(f)|^2 \epsilon_0$$

the ESD of $x(t)$.

$$= \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt$$

1 Equivalence of ESD and Autocorrelation Fourier Transform

For a deterministic signal $x(t)$ with finite energy $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$, the equivalence between the Energy Spectral Density (ESD) and the Fourier Transform of the Time-Averaged Autocorrelation Function forms the foundation for the Power Spectral Density (PSD) extension.

1.1 Definition of Energy Spectral Density (ESD)

The ESD, $E_{xx}(f)$, is defined as the magnitude squared of the signal's Fourier Transform $X(f) = \mathcal{F}\{x(t)\}$:

$$E_{xx}(f) = |X(f)|^2 = X(f)X^*(f) = \left| \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \right|^2 \quad (1)$$

1.2 Definition of Time-Averaged Autocorrelation Function

The time-averaged autocorrelation function, $r_{xx}(\tau)$ measures the self-similarity of the signal $x(t)$ at a time lag τ :

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)x^*(t)dt = \int_{-\infty}^{\infty} x(\lambda)x^*(\lambda-\tau)d\lambda = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)dt \quad (2)$$

1.3 Derivation of the Equivalence (Deterministic Wiener-Khinchine Theorem)

We use the Fourier Transform property that multiplication in the frequency domain corresponds to convolution in the time domain.

1. **Apply Inverse Fourier Transform to ESD:** The inverse Fourier Transform of the ESD $E_{xx}(f) = X(f)X^*(f)$ must be the convolution of $x(t)$ with the time-reversed conjugate of $x(t)$. We use the property $\mathcal{F}^{-1}\{X^*(f)\} = x^*(-t)$.

$$\mathcal{F}^{-1}\{E_{xx}(f)\} = \mathcal{F}^{-1}\{X(f)X^*(f)\} = x(t) * x^*(-t) =$$

1

2. **Evaluate the Convolution:** The convolution $x(t) * x^*(-t)$ is defined as:

$$x(t) * x^*(-t) = \int_{-\infty}^{\infty} x(\lambda)x^*(-(t-\lambda))d\lambda = \int_{-\infty}^{\infty} x(\lambda)x^*(\lambda-t)d\lambda = r_{xx}(t)$$

3. **Identify the Autocorrelation:** By comparing the result of the convolution with the definition of the time-averaged autocorrelation $r_{xx}(\tau)$ (Equation 2), we see that the convolution result is $r_{xx}(t)$.

$$\mathcal{F}^{-1}\{E_{xx}(f)\} = r_{xx}(t)$$

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$$E_{xx} \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt \geq 0$$

$$E_{xy} \triangleq \int_{-\infty}^{\infty} x(t)y^*(t)dt \in \mathbb{C}$$

$$E_{xx}, r_{xx} = E_{xx} + 2\text{Re}\{E_{xy}\} + E_{yy}$$

3. **Identify the Autocorrelation:** By comparing the result of the convolution with the definition of the time-averaged autocorrelation $r_{xx}(\tau)$ (Equation 2), we see that the convolution result is $r_{xx}(t)$.

$$\boxed{\mathcal{F}^{-1}\{E_{xx}(f)\} = r_{xx}(t)} = r_{xx}(t)$$

4. **Conclusion of Equivalence:** Since $E_{xx}(f)$ is the Fourier Transform of $r_{xx}(t)$, we establish the fundamental equivalence:

$$\boxed{|X(f)|^2 = E_{xx}(f) = \mathcal{F}\{r_{xx}(\tau)\} = \int_{-\infty}^{\infty} r_{xx}(\tau) e^{-j2\pi f\tau} d\tau} \quad (3)$$

1.4 Summary

The result demonstrates that for deterministic, finite-energy signals, the Energy Spectral Density (ESD) defined by the magnitude squared of the Fourier Transform is equivalent to the Fourier Transform of the time-averaged autocorrelation function. This is the deterministic precursor to the Wiener-Khinchine Theorem for random processes.

Power Spectral Density of General Non-Stationary Processes :Derivation and Limitations of the Wiener-Khinchine Extension

Gemini AI Assistant and Joon Ho Cho

November 6, 2025

Abstract

This document presents a step-by-step rigorous derivation of the Power Spectral Density (PSD) for a general, second-order non-stationary random process, building upon the definition via the time-truncated Fourier transform. The derivation validates the **Wiener-Khinchine Theorem**, and reveals that the resulting PSD, $S_{XX}(f)$, is the Fourier transform of the **Time-Averaged Autocorrelation Function**, $\mathcal{R}_{XX}(\tau)$, which is obtained by averaging the two-variable autocorrelation function, $R_{XX}(t_1, t_2)$, over time $t = (t_1 + t_2)/2$ with $\tau = t_1 - t_2$. We conclude that while this definition provides a mathematically sound measure of average power distribution, the mandatory time-averaging step inherently **removes the process's non-stationary information**. Consequently, the standard PSD fails to capture the evolution of frequency content over time, effectively reducing the analysis to that of an equivalent Wide-Sense Stationary (WSS) process. This limitation highlights the necessity of time-frequency distributions for true non-stationary analysis.

1 Definition of PSD via Time-Truncated Fourier Transform

For a second-order random process $X(t)$, we define the time-truncated version $X_T(t)$ over the interval $[-T, T]$:

$$X_T(t) = \begin{cases} X(t) & \text{if } |t| \leq T \\ 0 & \text{if } |t| > T \end{cases}$$

The Fourier Transform of $X_T(t)$ is:

$$\mathcal{F}\{X_T(t)\} = \int_{-T}^T X(t) e^{-j2\pi f t} dt$$

The Power Spectral Density (PSD) $S_{XX}(f)$ is defined as the limit of the ensemble average of the magnitude squared of this Fourier transform, divided by the observation time $2T$:

The Power Spectral Density (PSD) $S_{XX}(f)$ is defined as the limit of the ensemble average of the magnitude squared of this Fourier transform, divided by the observation time $2T$:

$$\forall f, 0 \leq S_{XX}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \mathbb{E} \left[\left| \int_{-T}^T X(t) e^{-j2\pi f t} dt \right|^2 \right] = E_{XX}(f)$$

This definition is consistent with interpreting the PSD as the average power at frequency f .

2 Derivation using Two-Variable Autocorrelation Function

2.1 Expanding the Magnitude Squared Term

We expand the magnitude squared term in Equation 1:

$$\begin{aligned} \left| \int_{-T}^T X(t) e^{-j2\pi f t} dt \right|^2 &= \int_{-T}^T \int_{-T}^T X(t_1) e^{-j2\pi f t_1} X^*(t_2) e^{j2\pi f t_2} dt_1 dt_2 \\ &= \int_{-T}^T \int_{-T}^T X(t_1) X^*(t_2) e^{-j2\pi f (t_1 - t_2)} dt_1 dt_2 \end{aligned}$$

1

$e^{-j2\pi f (t_1 - t_2)}$ w/ $t_1 = t_2$

2.2 Applying Ensemble Average

Substituting this back into the PSD definition and exchanging the expectation and integration (since the bounds are finite):

$$S_{XX}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T \mathbb{E} [X(t_1) X^*(t_2)] e^{-j2\pi f (t_1 - t_2)} dt_1 dt_2$$

The ensemble average is the **two-variable autocorrelation function**, $R_{XX}(t_1, t_2)$.

$$S_{XX}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XX}(t_1, t_2) e^{-j2\pi f (t_1 - t_2)} dt_1 dt_2$$

2.3 Transformation to Average Time (t) and Delay (τ)

To separate the variables, we change the integration variables:

$$\begin{aligned} \tau &= t_1 - t_2 \quad (\text{Delay}) \\ t &= \frac{t_1 + t_2}{2} \quad (\text{Average Time}) \end{aligned}$$

$$\rightarrow E \left\{ X\left(t + \frac{\tau}{2}\right) X^*\left(t - \frac{\tau}{2}\right) \right\} = R_{XX}\left(t + \frac{\tau}{2}, t - \frac{\tau}{2}\right)$$

$\approx$

The autocorrelation function becomes $\tilde{R}_V(t, \tau) = R_{VV}(t + \tau/2, t - \tau/2)$. The integral bounds also

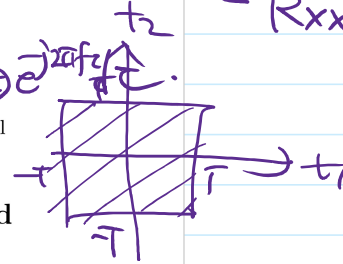
$$t = \frac{t_1 + t_2}{2} \quad (\text{Average Time})$$

$$\rightarrow E[X(t+\frac{\tau}{2})X(t-\frac{\tau}{2})] = R_{XX}(t+\frac{\tau}{2}, t-\frac{\tau}{2}) = \tilde{R}_{XX}(t, \tau)$$

The autocorrelation function becomes $\tilde{R}_X(t, \tau) = R_{XX}(t + \tau/2, t - \tau/2)$. The integral bounds also change, where t and τ are coupled.

$$S_{XX}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-2T}^{2T} e^{-j2\pi f\tau} \left(\int_{t \in D} \tilde{R}_X(t, \tau) dt \right) d\tau = \int_{-\infty}^{\infty} \tilde{R}_X(t, \tau) e^{-j2\pi f\tau} dt$$

where $D(\tau)$ denotes the restricted domain of t for a fixed τ to keep the integration within the original square $[-T, T] \times [-T, T]$.



3 Wiener-Khinchine Theorem for Non-Stationary Processes and Limitation

For a general non-stationary process, $\tilde{R}_X(t, \tau)$ depends on both t and τ . We define the term inside the parenthesis as the **Time-Averaged Autocorrelation Function**, $\mathcal{R}_{XX}(\tau)$:

$$\mathcal{R}_{XX}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{t \in D(\tau)} \tilde{R}_X(t, \tau) dt$$

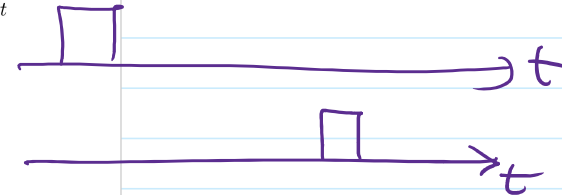
3.1 Conclusion of the Logical Flow

Substituting this definition back into the PSD equation yields:

$$S_{XX}(f) = \int_{-\infty}^{\infty} \mathcal{R}_{XX}(\tau) e^{-j2\pi f\tau} d\tau \quad (2)$$

Interpretation and Limitation:

- **General Result:** The PSD of a general non-stationary process is the **Fourier Transform** of its **Time-Averaged Autocorrelation Function** $\mathcal{R}_{XX}(\tau)$.
- **Loss of Time Resolution:** This result is structurally identical to the classic Wiener-Khinchine Theorem, but the PSD $S_{XX}(f)$ is now the spectrum of the time-averaged statistics. The $\lim_{T \rightarrow \infty} \frac{1}{2T} \int \dots dt$ operation **removes the time dependency** (t) from the autocorrelation function $\tilde{R}_X(t, \tau)$.
- **The Dilemma:** The standard PSD definition, while mathematically rigorous for power signals, cannot capture the **evolution of frequency content over time** for a non-stationary process. It only provides the overall, long-term average power distribution across frequencies, effectively reducing the non-stationary process to an equivalent Wide-Sense Stationary (WSS) process with autocorrelation $\mathcal{R}_{XX}(\tau)$.



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4 Wiener-Khinchine Theorem for WSS Processes

4.1 Variable Transformation and WSS Assumption

We now assume the process $X(t)$ is **Wide-Sense Stationary (WSS)**. This means $R_{XX}(t_1, t_2)$ depends only on the **time difference** $\tau = t_1 - t_2$. Thus, $R_{XX}(t_1, t_2) = R_{XX}(\tau)$.

We change variables:

- Delay (τ): $\tau = t_1 - t_2$
- Average Time (t): $t = t_2$ (or $t = (t_1 + t_2)/2$ can be used)

The double integral transforms, and the constant $R_{XX}(\tau)$ can be pulled out of the internal integral (which is an integral over t):

$$S_{XX}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-2T}^{2T} R_{XX}(\tau) e^{-j2\pi f\tau} \left(\int_{D(\tau)} dt \right) d\tau$$

where $D(\tau)$ represents the length of the internal integration domain, constrained by the original limits. For $|\tau| < 2T$, the length $\int_{D(\tau)} dt$ is exactly $2T - |\tau|$.

$$S_{XX}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-2T}^{2T} R_{XX}(\tau) e^{-j2\pi f\tau} (2T - |\tau|) d\tau$$

Dividing by $2T$:

$$S_{XX}(f) = \lim_{T \rightarrow \infty} \int_{-2T}^{2T} R_{XX}(\tau) e^{-j2\pi f\tau} \left(1 - \frac{|\tau|}{2T} \right) d\tau$$

4.2 Taking the Limit and Completing the Wiener-Khinchine Theorem

Taking the limit as $T \rightarrow \infty$, the term $\frac{|\tau|}{2T}$ goes to zero, and the integration bounds expand to infinity.

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi f\tau} d\tau$$

5 Summary of Logical Flow

The derivation follows this logical sequence:

- **PSD Definition:** Defined $S_{XX}(f)$ as the $T \rightarrow \infty$ limit of the ensemble average of the magnitude squared of the time-truncated Fourier spectrum, normalized by time $2T$.
- **Autocorrelation Introduction:** The $\mathbb{E}[\cdot]$ operation was moved inside the integral to introduce the **two-variable autocorrelation function** $R_{XX}(t_1, t_2)$.

The result demonstrates that the PSD is the Fourier Transform of the time-averaged autocorrelation function $\mathcal{R}_{XX}(\tau)$.

However, this approach of spectral analysis of a 2nd-order random process may lead to a serious limitation: **By necessity, the standard PSD loses all temporal information.** This is due to the inherent time-averaging process that must be performed for the PSD to be defined:

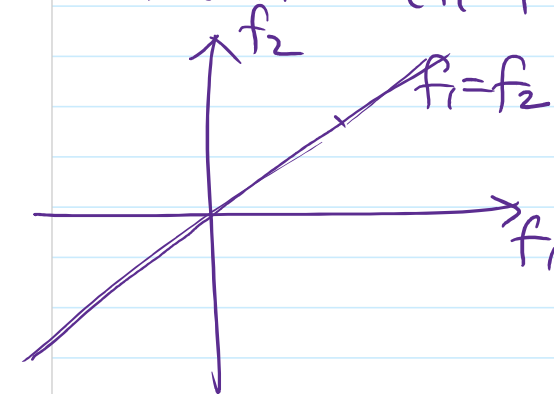
- **Core Operation:** The definition includes the time-averaging operation $(\lim_{T \rightarrow \infty} \frac{1}{2T} \int \cdots dt)$.
- **Data Collapse:** This operation collapses the **two-variable autocorrelation function** ($R_X(t_1, t_2)$) into a single function of delay, $R_{Avg}(\tau)$ (or $\mathcal{R}_{XX}(\tau)$).
- **Effect of Smoothing:** This time-averaging effectively **smooths out** any time-varying characteristics of the process, preventing the detection or analysis of the evolution of the signal's frequency content over time.

- **Non-Stationary Failure:** Since the evolution of spectral content is the defining feature of non-stationary behavior, the standard PSD is fundamentally incapable of capturing the transient or evolving nature of the signal's spectrum.

3

$$S_{xx}(f_1, f_2) = \mathcal{F}\{R_{xx}(t_1, t_2)\}$$

$$S_{xx}(f_p) = S_{xx}(f_1, f_2) \delta(f_1 - f_2)$$



The Bi-Frequency Spectrum of Non-Stationary Processes: Based on the Symplectic Fourier Transform

Gemini AI Assistant and Joon Ho Cho

November 6, 2025

Abstract

This document explores the **Bi-frequency Spectrum** ($S_{XX}(f_1, f_2)$), a critical analytical tool for characterizing **non-stationary random processes**. Defined as the two-dimensional Symplectic Fourier Transform of the two-variable autocorrelation function $R_{XX}(t_1, t_2)$, this spectrum serves as a measure of non-stationarity; significant non-zero values off the main diagonal ($f_1 \neq f_2$) directly indicate time-varying statistical properties. The spectrum is also known as the **Generalized Spectral Density (GSD)**.

We rigorously derive the general input-output relationship for a non-stationary process $X(t)$ propagating through a **Linear Time-Varying (LTV) system** with impulse response $h(t, \tau)$. The output spectrum $S_{YY}(f_1, f_2)$ is shown to be governed by a complex **integral relationship** involving the input spectrum $S_{XX}(f_1, f_2)$ and the system's frequency-time response $\mathcal{H}(f, \tau)$. Crucially, the simpler **multiplicative product form** ($S_{YY} = \mathcal{H} S_{XX} \mathcal{H}^*$) is generally **not valid** for arbitrary LTV systems, only holding under the constraint of a Linear Time-Invariant (LTI) system or within specialized analytical frameworks that adopt the product relation by definition.

ing the input spectrum $S_{XX}(f_1, f_2)$ and the system's frequency-time response $\mathcal{H}(f, \tau)$. Crucially, the simpler **multiplicative product form** ($S_{YY} = HS_{XX}H^*$) is generally **not valid** for arbitrary LTV systems, only holding under the constraint of a Linear Time-Invariant (LTI) system or within specialized analytical frameworks that adopt the product relation by definition.

1 Definition via Symplectic Fourier Transform

The analysis of non-stationary random processes utilizes the two-variable autocorrelation function, $R_{XX}(t_1, t_2)$, which is the expected value of the product of the process at two time instants: $R_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X^*(t_2)\}$.

The **Bi-frequency Spectrum**, $S_{XX}(f_1, f_2)$, is defined as the two-dimensional Fourier Transform of $R_{XX}(t_1, t_2)$, often expressed in a form related to the Symplectic Fourier Transform, which uses opposing signs in the exponent for the two time variables.

1.1 Formal Definition

The Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ is given by the following integral transform:

$$S_{XX}(f_1, f_2) = \iint_{-\infty}^{\infty} R_{XX}(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \quad (1)$$

This two-dimensional transform maps the time-domain correlation information into the frequency domain, where f_1 and f_2 are the two independent frequency variables.

1.2 Bi-Frequency Spectrum under WSS Condition

The Bi-frequency Spectrum takes a remarkably simple form when the random process $X(t)$ is **Wide-Sense Stationary (WSS)**. The WSS condition stipulates that the autocorrelation function $R_{XX}(t_1, t_2)$ depends only on the time difference $\tau = t_1 - t_2$, i.e., $R_{XX}(t_1, t_2) = R_{XX}(\tau)$.

To demonstrate the relationship $S_{XX}(f_1, f_2) = S_{XX}(f_1)\delta(f_1 - f_2)$, we substitute the WSS condition into the formal definition (Eq. 1):

$$S_{XX}(f_1, f_2) = \iint_{-\infty}^{\infty} R_{XX}(t_1 - t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2$$

1

We introduce the variable change $\tau = t_1 - t_2$ (and $t_1 = t_2 + \tau$).

$$S_{XX}(f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi(f_1(t_2+\tau) - f_2 t_2)} d\tau dt_2$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi f_1 \tau - j2\pi(f_1 - f_2)t_2} d\tau dt_2$$

$$\begin{aligned}
S_{XX}(f_1, f_2) &= \iint_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi(f_1(t_2+\tau)-f_2t_2)} d\tau dt_2 \\
&= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi f_1 \tau} d\tau \cdot \int_{-\infty}^{\infty} e^{-j2\pi(f_1t_2-f_2t_2)} dt_2 \\
&= \left[\int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi f_1 \tau} d\tau \right] \cdot \left[\int_{-\infty}^{\infty} e^{-j2\pi(f_1-f_2)t_2} dt_2 \right]
\end{aligned}$$

We recognize the two bracketed terms:

- The first term is the standard Power Spectral Density (PSD) of the WSS process, $S_{XX}(f_1)$.
- The second term is the definition of the Dirac Delta function, $\delta(f_1 - f_2)$.

Therefore, for a WSS process, the Bi-frequency Spectrum simplifies to:

$$S_{XX}(f_1, f_2) = S_{XX}(f_1) \cdot \delta(f_1 - f_2) \quad (2)$$

This result confirms that for a WSS process, the spectrum is strictly non-zero only along the main diagonal where $f_1 = f_2$, implying that there is no correlation or power coupling between different frequency components.

2 Alternative Names

The Bi-frequency Spectrum is known by several names, particularly when discussing non-stationary processes:

- **Generalized Spectral Density (GSD):** This is perhaps the most common name, as it extends the concept of the standard Power Spectral Density (PSD) to non-stationary cases.
- **Two-Frequency Spectrum:** A descriptive name emphasizing the two independent frequency variables, f_1 and f_2 .
- **Complex Spectral Density:** Used in some literature, particularly when $X(t)$ is a complex-valued process.

3 Key Applications

The primary application of $S_{XX}(f_1, f_2)$ is the rigorous analysis and characterization of **non-stationary random processes**.

3.1 Characterizing Non-Stationarity

The Bi-frequency Spectrum serves as a direct measure of non-stationarity.

The bi-frequency spectrum serves as a direct measure of non-stationarity:

- For a **Wide-Sense Stationary (WSS)** process, $R_{XX}(t_1, t_2)$ depends only on the time difference $\tau = t_1 - t_2$. In this case, $S_{XX}(f_1, f_2)$ is non-zero only along the main diagonal where $f_1 = f_2$:

$$S_{XX}(f_1, f_2) = S_{XX}(f_1) \cdot \delta(f_1 - f_2)$$

where $S_{XX}(f_1)$ is the standard Power Spectral Density (PSD).

- The presence of significant non-zero values **off the main diagonal** ($f_1 \neq f_2$) indicates that the process is **non-(wide-sense)stationary**.

3.2 Linear Time-Varying (LTV) Systems

The Bi-frequency Spectrum is critical in analyzing how non-stationary signals propagate through LTV systems. The system's input spectrum $S_{XX}(f_1, f_2)$ is related to the output spectrum $S_{YY}(f_1, f_2)$ via the system's **Bi-frequency Transfer Function**, $H(f_1, f_2)$.



$$\begin{aligned} R_{YY}(t_1, t_2) &= R_{XX}(t_1, t_2) * h(t_1) * h^*(t_2) \\ R_{YX}(t_1, t_2) &= h(t_1) * R_{XX}(t_1, t_2) \\ R_{XY}(t_1, t_2) &= h(t_1) * R_{XX}(t_1, t_2) * h^*(t_2) \end{aligned}$$

$$S_{YY}(f_1, f_2) = H(f_1) S_{XX}(f_1, f_2) H^*(f_2)$$

If WSS

$$\begin{aligned} S_{YY}(f_1, f_2) &= H(f_1) S_{XX}(f_1) \delta(f_1 - f_2) H^*(f_2) \\ &= H(f_1) S_{XX}(f_1) H^*(f_1) \delta(f_1 - f_2) \end{aligned}$$

$$\therefore S_{YY}(f) = S_{XX}(f) |H(f)|^2$$

4 System and Process Definitions

Let $X(t)$ be a second-order random process, and let $Y(t)$ be the output of a **Linear Time-Varying (LTV) system** with time-varying impulse response $h(t, \tau)$.

4.1 LTV System Input-Output Relation

The output $Y(t)$ is given by the convolution integral:

$$Y(t) = \int_{-\infty}^{\infty} h(t, \tau) X(\tau) d\tau \quad (3)$$

4.2 Autocorrelation and Bi-Frequency Spectrum

- The autocorrelation function of the output $Y(t)$ is: $R_{YY}(t_1, t_2) = \mathbb{E}\{Y(t_1)Y^*(t_2)\}$.
- The Bi-frequency Spectrum of $Y(t)$ is defined as the Symplectic Fourier Transform of $R_{YY}(t_1, t_2)$:

$$S_{YY}(f_1, f_2) = \iint_{-\infty}^{\infty} R_{YY}(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \quad (4)$$

5 Derivation of the Output Autocorrelation Function (R_{YY})

Substitute the LTV input-output relation (Eq. 3) into the definition of $R_{YY}(t_1, t_2)$:

$$R_{YY}(t_1, t_2) = \mathbb{E} \left\{ \left[\int_{-\infty}^{\infty} h(t_1, \tau_1) X(\tau_1) d\tau_1 \right] \left[\int_{-\infty}^{\infty} h(t_2, \tau_2) X(\tau_2) d\tau_2 \right]^* \right\}$$

Substitute the LTV input-output relation (Eq. 3) into the definition of $R_{YY}(t_1, t_2)$:

$$\begin{aligned} R_{YY}(t_1, t_2) &= \mathbb{E} \left\{ \left[\int_{-\infty}^{\infty} h(t_1, \tau_1) X(\tau_1) d\tau_1 \right] \left[\int_{-\infty}^{\infty} h(t_2, \tau_2) X(\tau_2) d\tau_2 \right]^* \right\} \\ &= \mathbb{E} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1, \tau_1) X(\tau_1) X^*(\tau_2) h^*(t_2, \tau_2) d\tau_1 d\tau_2 \right\} \end{aligned}$$

Since the expectation operator $\mathbb{E}\{\cdot\}$ is linear and the impulse response h is deterministic, we can interchange the expectation and integration:

$$\begin{aligned} R_{YY}(t_1, t_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1, \tau_1) \mathbb{E}\{X(\tau_1) X^*(\tau_2)\} h^*(t_2, \tau_2) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1, \tau_1) R_{XX}(\tau_1, \tau_2) d\tau_1 h^*(t_2, \tau_2) d\tau_2 \end{aligned} \quad (5)$$

6 Derivation of the Output Bi-Frequency Spectrum (S_{YY})

The Bi-frequency Spectrum $S_{YY}(f_1, f_2)$ is obtained by taking the Symplectic Fourier Transform of $R_{YY}(t_1, t_2)$ (Eq. 5):

$$\begin{aligned} S_{YY}(f_1, f_2) &= \iint_{t_1, t_2} R_{YY}(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \\ &= \iint_{t_1, t_2} \left[\iint_{\tau_1, \tau_2} h(t_1, \tau_1) R_{XX}(\tau_1, \tau_2) h^*(t_2, \tau_2) d\tau_1 d\tau_2 \right] e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \end{aligned}$$

Rearranging the terms to group related variables:

$$\begin{aligned} S_{YY}(f_1, f_2) &= \iint_{\tau_1, \tau_2} R_{XX}(\tau_1, \tau_2) \left[\iint_{t_1, t_2} h(t_1, \tau_1) e^{-j2\pi f_1 t_1} h^*(t_2, \tau_2) e^{j2\pi f_2 t_2} dt_1 dt_2 \right] d\tau_1 d\tau_2 \\ &= \iint_{\tau_1, \tau_2} \left[\int_{t_1} h(t_1, \tau_1) e^{-j2\pi f_1 t_1} dt_1 \right] R_{XX}(\tau_1, \tau_2) \left[\int_{t_2} h^*(t_2, \tau_2) e^{j2\pi f_2 t_2} dt_2 \right] d\tau_1 d\tau_2 \end{aligned}$$

Recall that the Bi-frequency Transfer Function $H(f_1, f_2)$ is the 2D Fourier Transform of the impulse response $h(t_1, t_2)$:

$$H(f_1, f_2) = \iint_{t_1, t_2} h(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2$$

3

Defining the generalized frequency response $\mathcal{H}(f, \tau)$ as the Fourier Transform of $h(t, \tau)$ with respect to the first variable t :

$$\mathcal{H}(f, \tau) = \int_t h(t, \tau) e^{-j2\pi f t} dt$$

Then the inner terms become

Defining the generalized frequency response $\mathcal{H}(f, \tau)$ as the Fourier Transform of $h(t, \tau)$ with respect to the first variable t :

$$\mathcal{H}(f, \tau) = \int_t h(t, \tau) e^{-j2\pi f t} dt$$

Then, the inner terms become:

- The first term is $\mathcal{H}(f_1, \tau_1)$.
- The third term is $\mathcal{H}^*(f_2, \tau_2)$ (due to the conjugate of $h(t_2, \tau_2)$ and the sign change in the exponential).

Substituting these back, we get:

$$S_{YY}(f_1, f_2) = \iint_{\tau_1, \tau_2} \mathcal{H}(f_1, \tau_1) R_{XX}(\tau_1, \tau_2) \mathcal{H}^*(f_2, \tau_2) d\tau_1 d\tau_2 \quad (6)$$

where $\mathcal{H}(f, \tau)$ is the system's **frequency-time response**, which is the Fourier Transform of $h(t, \tau)$ with respect to t . General

For a general Linear Time-Varying (LTV) system with impulse response $h(t, \tau)$ and a non-stationary input $X(t)$ with autocorrelation $R_{XX}(\tau_1, \tau_2)$, the output Bi-frequency Spectrum, $S_{YY}(f_1, f_2)$, is generally given by the above **integral relationship**:

This complex integral form arises because the LTV system mixes or couples the input's frequency components and correlation across different time instances (τ_1 and τ_2).

7 Conditions for the Multiplicative Relation

The simpler **multiplicative product form** often desired in spectral analysis is:

$$S_{YY}(f_1, f_2) = H(f_1, f_2) S_{XX}(f_1, f_2) H^*(f_1, f_2) \quad (7)$$

This product relation is generally **not valid** for arbitrary LTV systems and non-stationary inputs. It only holds under specific, restrictive constraints: Not in general

7.1 Constraint 1: The System Must Be LTI

The most straightforward case where the product form holds is when the LTV system reduces to a **Linear Time-Invariant (LTI) system**.

- **LTI Condition:** For an LTI system, the impulse response depends only on the time difference, $h(t, \tau) = h(t - \tau)$.
- **Spectral Consequence:** The Bi-frequency Transfer Function simplifies to:

$$H(f_1, f_2) = H(f_1) \delta(f_1 - f_2)$$

If the input $X(t)$ is also **Wide-Sense Stationary (WSS)**, then $S_{XX}(f_1, f_2)$ is also a delta function on the diagonal. Substituting these delta functions into the general integral relation (or the product relation) leads to the standard PSD filtering equation: $S_{YY}(f) = |H(f)|^2 S_{XX}(f)$, which is the LTI equivalent of the product form. **The multiplicative relation is therefore consistent with LTI/WSS analysis.**

7.2 Constraint 2: Adoption of a Specialized Framework (Generalized PSD)

In some non-stationary process literature, the multiplicative relation (Eq. 7) is adopted as a matter of definition or convenience within a specialized analytical framework known as the **Generalized Spectral Density (GSD)** approach.

- **Specialized Definition:** In this approach, $H(f_1, f_2)$ is not directly derived from the LTV impulse response $h(t, \tau)$ via simple Fourier Transform, but is rather defined as a **spectral operator** that acts multiplicatively on $S_{XX}(f_1, f_2)$.
- **Analytical Goal:** The purpose is often to retain the simplicity of the LTI spectral analysis (Output = Transfer Function $\times$ Input) while formally extending it to LTV systems. This requires **assuming** or **defining** the LTV system's spectral properties in a way that respects the multiplication rule, even though the underlying time-domain physics is described by the complex integral (Eq. 6).

8 Conclusion

The statement emphasizes that because LTV systems possess the general ability to **couple different frequency components** (i.e., non-zero off-diagonal terms in S_{XX} and H), the simple product $\mathbf{S}_{YY} = \mathbf{H}\mathbf{S}_{XX}\mathbf{H}^*$ is only rigorously valid when the system's LTV characteristics are highly constrained (i.e., when it approximates an LTI system) or when a simplified spectral framework is explicitly adopted.

8.1 Connection to Time-Frequency Analysis

While $S_{XX}(f_1, f_2)$ directly addresses frequency-domain correlation, it is fundamentally related to time-frequency representations. A common practice is to first transform $R_{XX}(t_1, t_2)$ into the variables of **average time** $t = (t_1 + t_2)/2$ and **time difference** $\tau = t_1 - t_2$ (yielding $\tilde{R}_{XX}(t, \tau)$). The Fourier Transform of $\tilde{R}_{XX}(t, \tau)$ with respect to τ results in the **Instantaneous Spectrum** $S_{XX}(t, f)$, which forms the basis for time-frequency distributions like the Wigner-Ville Distribution (WVD).

The Complementary Bi-Frequency Spectrum: Analysis of Complex-Valued Random Processes

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November 6, 2025

Abstract

This document introduces and defines the **Complementary Bi-frequency Spectrum** ($\tilde{S}_{XX}(f_1, f_2)$), also known as the Pseudo Bi-frequency Spectrum, which is essential for the comprehensive spectral analysis of **complex-valued random processes** $X(t)$. Unlike the standard Bi-frequency Spec-

Abstract

This document introduces and defines the **Complementary Bi-frequency Spectrum** ($\tilde{S}_{XX}(f_1, f_2)$), also known as the Pseudo Bi-frequency Spectrum, which is essential for the comprehensive spectral analysis of **complex-valued random processes** $X(t)$. Unlike the standard Bi-frequency Spectrum, which uses the Hermitian autocorrelation function, the complementary spectrum is based on the **pseudo autocorrelation function** $\tilde{R}_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X(t_2)\}$. The definition is derived via a standard double Fourier transform, and its primary significance lies in characterizing the **properness** (or circular symmetry) of the complex process and revealing **frequency coupling** information.

1 Introduction to Complex Correlation Functions

For a complex-valued random process $X(t)$, two second-order correlation functions are generally required to fully describe its statistical behavior:

1. The **Autocorrelation Function (ACF)**: $R_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X^*(t_2)\}$.
2. The **Pseudo Autocorrelation Function (PACF)** or **Complementary Autocorrelation Function**: $\tilde{R}_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X(t_2)\}$.

While the ACF is used to define the standard spectral density, the PACF is crucial for analyzing the complex process's statistical symmetry.

2 Definition of the Complementary Bi-Frequency Spectrum

The **Complementary Bi-frequency Spectrum** $\tilde{S}_{XX}(f_1, f_2)$ is defined as the two-dimensional Fourier Transform of the Pseudo Autocorrelation Function $\tilde{R}_{XX}(t_1, t_2)$.

2.1 Formal Definition

The Complementary Bi-frequency Spectrum is given by the following integral transform:

$$\tilde{S}_{XX}(f_1, f_2) = \iint_{-\infty}^{\infty} \tilde{R}_{XX}(t_1, t_2) e^{-j2\pi(f_1 t_1 + f_2 t_2)} dt_1 dt_2 \quad (1)$$

Note that this is a **standard double Fourier transform** (with a positive sign in the time domain, resulting in a sum in the frequency domain) and differs from the symplectic Fourier transform used for the standard Bi-frequency Spectrum (S_{XX}).

3 Significance and Applications

The $\tilde{S}_{XX}(f_1, f_2)$ spectrum provides unique information not contained in the standard spectrum $S_{XX}(f_1, f_2)$.

3.1 Characterizing Properness (Circular Symmetry)

The primary role of the Complementary Spectrum is to determine if the complex-valued random process $X(t)$ is **proper**.

- A process $X(t)$ is **proper** (i.e., its real and imaginary parts have no statistical correlation with each other) if and only if its pseudo autocorrelation function is zero for all time shifts: $\tilde{R}_{XX}(t_1, t_2) = 0$.
- Consequently, the process is proper if and only if the Complementary Bi-frequency Spectrum is zero for all frequencies:

$$X(t) \text{ is proper} \iff \tilde{S}_{XX}(f_1, f_2) = 0 \quad \text{for all } f_1, f_2$$

If $\tilde{S}_{XX}(f_1, f_2)$ is non-zero, the process is deemed **improper**. Improper processes require analysis of both $X(t)$ and its conjugate $X^*(t)$ for optimal signal processing.

3.2 Frequency Coupling and Summation Information

The structure of the transform integral (Eq. 1), featuring the term $e^{-j2\pi(f_1 t_1 + f_2 t_2)}$, highlights the spectrum's sensitivity to **frequency summation** or coupling:

- $\tilde{S}_{XX}(f_1, f_2)$ reveals statistical correlation between the frequency component at f_1 and the frequency component at f_2 .
- This information is particularly relevant in analyzing non-linear systems or mixing processes where frequency components are added (e.g., in spectral mixing or harmonic generation).

3.3 Relation to PSD (WSS Case)

For a **Wide-Sense Stationary (WSS)** process, $\tilde{R}_{XX}(t_1, t_2)$ depends only on the time difference $\tau = t_1 - t_2$. In this specialized case, the Complementary Bi-frequency Spectrum is non-zero only along the anti-diagonal where $f_1 = -f_2$:

$$\tilde{S}_{XX}(f_1, f_2) = \tilde{S}_{XX}(f_1) \cdot \delta(f_1 + f_2)$$

where $\tilde{S}_{XX}(f_1)$ is the single-variable **Pseudo Power Spectral Density (PPSD)**, which is the Fourier Transform of $\tilde{R}_{XX}(\tau)$.

