

2nd orderdness => Existence of complementary autocorrelation function

□ Q If $X(t)$ is of second order, then $\tilde{R}_{xx}(t_1, t_2)$ exists $\forall t_1, t_2$?

A. Yes!

(sol)

Let $X(t) = X_r(t) + j X_i(t)$. Then

$$0 \leq E\{|X(t)|^2\} = E\{|X_r(t)|^2\} + E\{|X_i(t)|^2\} < \infty \text{ by } E\{|X(t)|^2\} < \infty, \forall t$$

So, we have $E\{|X_r(t)|^2\} < \infty$ & $E\{|X_i(t)|^2\} < \infty$ ☺

$$\Rightarrow R_{x_r x_r}, R_{x_i x_i} \text{ exists } \forall t_1, t_2$$

$$\text{Now } (E\{X_r(t_1) X_i(t_2)\}) \leq E\{X_r^2(t_1)\} E\{X_i^2(t_2)\} < \infty \text{ ☺}$$

$$\Rightarrow R_{x_r x_i} \text{ exists } \forall t_1, t_2$$

Consider $E\{X(t_1) X(t_2)\} = E\{(X_r(t_1) + j X_i(t_1)) (X_r(t_2) + j X_i(t_2))\}$

$$\tilde{R}_{xx}(t_1, t_2) \triangleq$$

$$= R_{x_r x_r} + j R_{x_r x_i} + j R_{x_i x_r} - R_{x_i x_i}$$

$\tilde{R}_{xx}$ exists because the 4 terms exist. Q.E.D.

Existence of the Complementary Autocorrelation Function for Complex Second-Order Processes

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1 Proof: Existence of $\tilde{R}_{XX}(t_1, t_2)$

This section proves that if a Complex Random Process $X(t)$ is a **Second-Order Process**, its **Complementary Autocorrelation Function** $\tilde{R}_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}$ must exist (i.e., be finite) for all t_1 and t_2 .

1.1 Definitions and Premise

The complex random process $X(t)$ is defined by its real part $X_R(t)$ and imaginary part $X_I(t)$:

$$X(t) = X_R(t) + jX_I(t)$$

Second-Order Condition: $X(t)$ is a second-order process if its second moment is finite for all time t :

$$E\{|X(t)|^2\} < \infty, \quad \forall t \quad (1)$$

Complementary Autocorrelation Function:

$$\tilde{R}_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

1.2 Derivation

1. **Existence of Real Component Second Moments:** From the second-order condition (Equation 1), we expand the squared magnitude:

$$0 \leq E\{|X(t)|^2\} = E\{|X_R(t) + jX_I(t)|^2\} = E\{X_R(t)^2\} + E\{X_I(t)^2\}$$

Since the sum is finite, both components must be individually finite:

$$E\{X_R(t)^2\} < \infty \quad \text{and} \quad E\{X_I(t)^2\} < \infty$$

This confirms that $X_R(t)$ and $X_I(t)$ are both second-order processes.

2. **Existence of Real Cross-Correlation Functions:** Since $X_R(t)$ and $X_I(t)$ are second-order, the Cauchy-Schwarz inequality guarantees the existence and finiteness of all four real-valued cross-correlation functions:

$$\begin{aligned} R_{RR}(t_1, t_2) &= E\{X_R(t_1)X_R(t_2)\} < \infty \\ R_{RI}(t_1, t_2) &= E\{X_R(t_1)X_I(t_2)\} < \infty \\ R_{IR}(t_1, t_2) &= E\{X_I(t_1)X_R(t_2)\} < \infty \\ R_{II}(t_1, t_2) &= E\{X_I(t_1)X_I(t_2)\} < \infty \end{aligned}$$

$$\begin{aligned} R_{IR}(t_1, t_2) &= E\{X_I(t_1)X_R(t_2)\} < \infty \\ R_{II}(t_1, t_2) &= E\{X_I(t_1)X_I(t_2)\} < \infty \end{aligned}$$

1

3. **Existence of the Complementary Autocorrelation $\tilde{R}_{XX}$:** We expand $\tilde{R}_{XX}(t_1, t_2)$ using the component processes:

$$\begin{aligned} \tilde{R}_{XX}(t_1, t_2) &= E\{X(t_1)X(t_2)\} \\ &= E\{(X_R(t_1) + jX_I(t_1))(X_R(t_2) + jX_I(t_2))\} \\ &= E\{X_R(t_1)X_R(t_2) - X_I(t_1)X_I(t_2) + j[X_R(t_1)X_I(t_2) + X_I(t_1)X_R(t_2)]\} \\ &= [E\{X_R(t_1)X_R(t_2)\} - E\{X_I(t_1)X_I(t_2)\}] \\ &\quad + j[E\{X_R(t_1)X_I(t_2)\} + E\{X_I(t_1)X_R(t_2)\}] \\ &= [R_{RR}(t_1, t_2) - R_{II}(t_1, t_2)] + j[R_{RI}(t_1, t_2) + R_{IR}(t_1, t_2)] \end{aligned}$$

Since all four component real correlation functions ($R_{RR}, R_{II}, R_{RI}, R_{IR}$) are finite, their linear combination, $\tilde{R}_{XX}(t_1, t_2)$, must also be **finite and therefore exists** for all t_1 and t_2 .

Q.E.D.

Mean-Square (MS) Periodic vs. Wide-Sense Periodic (WSP) Random Processes

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Abstract

This document explores the properties and implications of a **Mean-Square (MS) Periodic Random Process** within the framework of **second-order stochastic processes**. An MS periodic process $\{X(t), t \in \mathbb{R}\}$ with period T is defined by the stringent condition that the mean-square error between $X(t+T)$ and $X(t)$ is zero, implying $X(t+T) = X(t)$ **almost surely**. This strong constraint ensures that the process's realizations are virtually deterministic periodic functions. We rigorously demonstrate that MS periodicity guarantees the periodicity of its fundamental statistical moments: the **mean function** $m_X(t)$ and the **autocorrelation function** $R_X(t_1, t_2)$, with the same period T . Finally, we clarify that the MS periodic condition is a significantly stronger requirement than, yet is a sufficient condition for, the **Wide-Sense Periodic (WSP)** property, highlighting the consistency and structural rigidity imposed on the process's sample paths.

1 Prerequisites: Second-Order Random Process

A random process $\{X(t), t \in \mathbb{R}\}$ must be a **second-order random process** to satisfy the conditions for MS periodicity.

1.1 Second-Order Condition

The process $X(t)$ is second-order if its second moment is finite for all t :

□ "Spectrally-Related" 2nd-order
Complex-valued RPS.

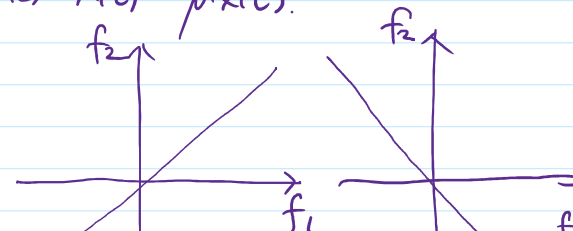
↘ Spectrally-unrelated.

$$R_{XX}(t_1, t_2) = \underline{C_{XX}(t_1, t_2)} + \mu_X(t_1)\mu_X^*(t_2)$$

↕ (2-D) Symplectic FT

$$S_{XX}(f_1, f_2), \quad \tilde{S}_{XX}(f_1, f_2)$$

$$\hat{X}(t) \triangleq X(t) - \mu_X(t)$$



1.1 Second-Order Condition

The process $X(t)$ is second-order if its second moment is finite for all t :

$$\mathbb{E}[|X(t)|^2] < \infty \quad \text{for all } t \in \mathbb{R}$$

This condition ensures that the mean function $m_X(t)$ and the autocorrelation function $R_X(t_1, t_2)$ are well-defined and finite.

2 Definition of an MS Periodic Random Process

A second-order random process $\{X(t), t \in \mathbb{R}\}$ is said to be **periodic in the Mean-Square (MS) sense** with period $T > 0$ if the following strong condition holds for all time t .

2.1 Mean-Square Convergence Condition

The mean-square error between the random variable at time $t + T$ and the random variable at time t is zero.

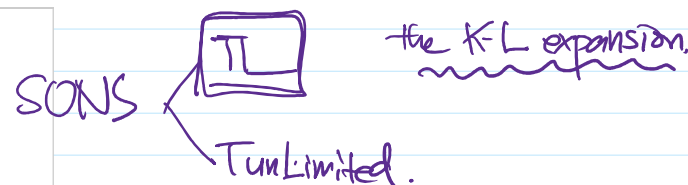
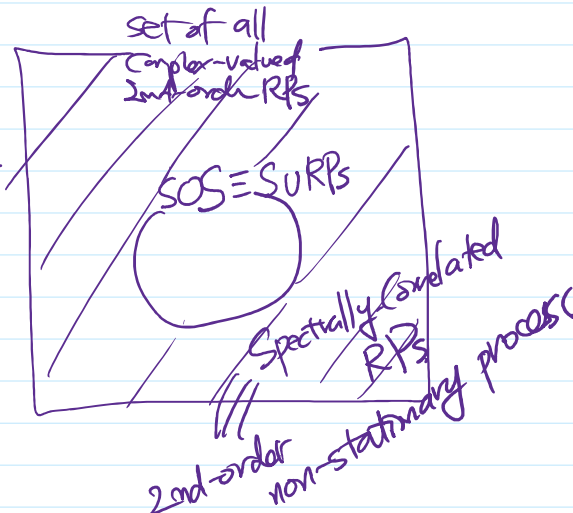
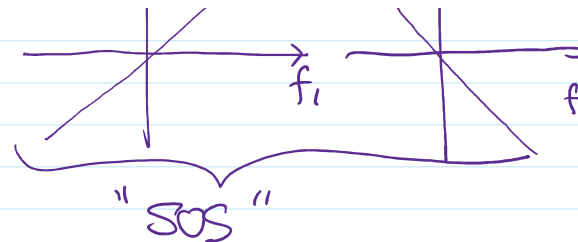
$$\mathbb{E}[|X(t+T) - X(t)|^2] = 0 \quad \text{for all } t \in \mathbb{R}$$

Interpretation: Almost Sure Equality The MS condition $\mathbb{E}[|X(t)|^2] = 0$ for a random variable Y implies that Y is equal to zero **with probability 1** (almost surely). Therefore, the MS periodic condition is equivalent to:

$$X(t+T) = X(t) \quad \text{with probability 1 (a.s.)}$$

This is a very strong requirement: almost every sample path (realization) of an MS periodic process is a **deterministic periodic function** with period T .

$X(t) = X(t+T)$, in the MS sense
 Q. $\hat{X}(t) = \begin{cases} X(t) & , t \neq mT \\ \text{i.i.d. RV} & , t = mT \end{cases}$



3 Key Statistical Properties (Consequences of MS Periodicity)

The MS condition forces all key statistical functions of the process to be periodic with the same period T .

3.1 Periodicity of the Mean Function

The mean function $m_X(t) = \mathbb{E}[X(t)]$ is periodic with period T .

$$m_X(t) = m_X(t+T)$$

Proof Sketch: This result follows directly from the MS condition by applying the Schwarz inequality (or Jensen's inequality for the absolute value):

$$|\mathbb{E}[X(t+T)] - \mathbb{E}[X(t)]|^2 = |\mathbb{E}[X(t+T) - X(t)]|^2 \leq \mathbb{E}[|X(t+T) - X(t)|^2] = 0$$

3.2 Periodicity of the Autocorrelation Function

The autocorrelation function $R_X(t_1, t_2) = \mathbb{E}[X(t_1)X^*(t_2)]$ is periodic with period T in both arguments, t_1 and t_2 .

$$R_X(t_1, t_2) = R_X(t_1+T, t_2) = R_X(t_1, t_2+T)$$

Derivation: Since $X(t_1+T) = X(t_1)$ almost surely, substituting this into the expectation yields:

$$R_X(t_1+T, t_2) = \mathbb{E}[X(t_1+T)X^*(t_2)] = \mathbb{E}[X(t_1)X^*(t_2)] = R_X(t_1, t_2)$$

A similar argument holds for the second argument t_2 .

$$R_X(t_1, t_2) = R_X(t_1 + T, t_2) = R_X(t_1, t_2 + T)$$

Derivation: Since $X(t_1 + T) = X(t_1)$ almost surely, substituting this into the expectation yields:

$$R_X(t_1 + T, t_2) = \mathbb{E}[X(t_1 + T)X^*(t_2)] = \mathbb{E}[X(t_1)X^*(t_2)] = R_X(t_1, t_2)$$

A similar argument holds for the second argument t_2 .

3.3 Periodicity of the Autocovariance Function

The autocovariance function $C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)m_X^*(t_2)$ is also periodic in both arguments:

$$C_X(t_1, t_2) = C_X(t_1 + T, t_2) = C_X(t_1, t_2 + T)$$

4 Relation to Wide-Sense Periodic (WSP) Processes

The MS periodic condition is a much stronger requirement than that for a WSP process.

- **Wide-Sense Periodic Process (WSP):** A second-order process $X(t)$ is WSP if it satisfies the periodicity of the mean and autocorrelation functions:

1. $m_X(t) = m_X(t + T)$
2. $R_X(t_1, t_2) = R_X(t_1 + T, t_2)$

- **Key Distinction (MS vs. WSP):** An MS periodic process is always a WSP process. However, a WSP process is **not** necessarily MS periodic. A WSP random process only requires the mean function to be periodic and the autocorrelation function to be periodic in t_1 and t_2 . The fundamental difference lies in the constraint on sample paths.

- For instance, consider replacing the values of an MS periodic process $X(t)$ at $t = nT$ (where n is an integer) with ****independent and identically distributed (i.i.d.) random variables**** Z_n .
- The resulting process remains ****WSP**** because its statistical averages (mean and autocorrelation) are preserved over the period T .
- However, the individual sample paths are no longer periodic because $X(nT)$ is not equal to $X((n+1)T)$ (i.e., $Z_n \neq Z_{n+1}$ with high probability). This violation of periodicity in the sample paths confirms that the new process **fails to satisfy the MS periodic condition**.

- **Conclusion:** An MS periodic process is always a WSP process. However, a WSP process is **not** necessarily MS periodic, as the WSP condition does not impose any constraints on the individual sample paths, allowing them to be non-periodic, provided their statistical averages satisfy the periodic rules. The MS condition guarantees a high degree of consistency across the realizations. The MS condition guarantees a high degree of consistency across the realizations by requiring almost sure equality, while the WSP condition only requires periodic statistical averages.

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Abstract

This document addresses the spectral analysis of a **second-order random process** $X(t)$ defined over a finite interval $[0, T]$. We examine the limitations of the traditional Power Spectral Density (PSD) definition for such **finite-energy signals** and introduce the **Karhunen-Loève (K-L) Expansion** as the primary analytical tool, especially for non-stationary processes. We also clarify the optimal conditions under which the K-L basis simplifies to the Fourier basis.

1 The Challenge of Spectral Density for Time-Limited Processes

A random process $X(t)$ confined to the interval $[0, T]$ is characterized by sample paths that are **finite-energy signals**, meaning $\int_0^T |X(t)|^2 dt < \infty$.

1.1 Limitation of Power Spectral Density (PSD)

The classical PSD is defined for signals with finite **time-averaged power** P . For a finite-duration signal, the time-averaged power approaches zero as the averaging time approaches infinity ($P \rightarrow 0$ as $T \rightarrow \infty$). Consequently, the traditional PSD is ill-defined.

1.2 Alternative: Energy Spectral Density (ESD)

Instead of PSD, the concept of **Energy Spectral Density (ESD)** $S_E(f)$ is often utilized for finite-energy signals:

$$S_E(f) = E \left[\left| \int_0^T X(t) e^{-j2\pi f t} dt \right|^2 \right]$$

The ESD characterizes the distribution of the average energy in the frequency domain.

2 Karhunen-Loève (K-L) Expansion

For a time-limited second-order random process, the K-L expansion provides the optimal basis for representation based on the **covariance function**.

2.1 K-L Decomposition for Non-Zero Mean Processes

Assume the process $X(t)$ has a non-zero mean $\mu(t) = E[X(t)]$. We first decompose $X(t)$ into its mean and zero-mean component, $\tilde{X}(t)$:

$$X(t) = \mu(t) + \tilde{X}(t), \quad 0 \leq t \leq T$$

The K-L expansion applies to the zero-mean component $\tilde{X}(t)$:

$$\tilde{X}(t) = \sum_{k=1}^{\infty} \tilde{Z}_k \phi_k(t)$$

The overall expansion is:

$$X(t) = \mu(t) + \sum_{k=1}^{\infty} \tilde{Z}_k \phi_k(t)$$

2.2 The Eigen-Problem and Mercer's Theorem

The optimal deterministic basis functions $\phi_k(t)$ and the corresponding eigenvalues λ_k are obtained by solving the **Fredholm Integral Equation** involving the **Covariance Function** $C_{XX}(t_1, t_2) = E[\tilde{X}(t_1)\tilde{X}^*(t_2)]$ over $[0, T]$:

$$\int_0^T C_{XX}(t_1, t_2) \phi_k(t_2) dt_2 = \lambda_k \phi_k(t_1), \quad 0 \leq t_1 \leq T$$

Mercer's Theorem ensures that the covariance function, being continuous and non-negative definite, can be expressed by the sum of its eigenvalues and eigenfunctions:

$$C_{XX}(t_1, t_2) = \sum_{k=1}^{\infty} \lambda_k \phi_k(t_1) \phi_k^*(t_2)$$

2.3 Properties of the K-L Coefficients ($\tilde{Z}_k$)

The random variables $\tilde{Z}_k$ (coefficients) are calculated as:

$$\tilde{Z}_k = \int_0^T \tilde{X}(t) \phi_k^*(t) dt$$

- **Zero Mean:** $E[\tilde{Z}_k] = 0$.
- **Uncorrelatedness:** $\tilde{Z}_k$ are **uncorrelated** for $k \neq l$, meaning their covariance is zero:

$$\text{Cov}(\tilde{Z}_k, \tilde{Z}_l) = E[\tilde{Z}_k \tilde{Z}_l^*] - E[\tilde{Z}_k] E[\tilde{Z}_l^*] = \lambda_k \delta_{kl}$$

- **Orthogonality:** Since $E[\tilde{Z}_k] = 0$, the uncorrelatedness condition is equivalent to the **orthogonality** condition: $E[\tilde{Z}_k \tilde{Z}_l^*] = \lambda_k \delta_{kl}$.
- **Variance:** The variance is given by the eigenvalue: $\text{Var}(\tilde{Z}_k) = \lambda_k$.

3 K-L Expansion as a Replacement for Fourier Analysis in Non-Stationary Processes

For **non-stationary** processes, where the statistical properties change with time, the K-L expansion is the superior analytical tool, effectively replacing traditional Fourier analysis.

3.1 Limitations of Fourier Analysis for Non-Stationary Processes

- **Inefficient Basis:** The Fourier basis functions, $e^{j2\pi kt/T}$, are optimized for time-invariant (WSS) properties. When applied to a non-stationary process, the process's energy is smeared across many frequency components, resulting in a large number of required coefficients for accurate approximation.
- **Correlated Coefficients:** The resulting Fourier coefficients are generally **correlated**, making it difficult to analyze the process as a sum of statistically independent components.

3.2 K-L's Advantages for Non-Stationary Processes

The K-L expansion's strength lies in its ability to adapt its basis to the specific statistical structure of the process.

3.2.1 A. Optimal Energy Compaction

The K-L expansion is optimal in the sense of minimizing the **Mean Square Error (MSE)** when truncating the series to N terms.

- **Minimal Coefficients:** The eigenvalues λ_k (variance of $\tilde{Z}_k$) typically decrease rapidly, meaning the process's energy is highly concentrated in the first few terms. This allows for the most **efficient** data compression and modeling.
- **Discrete Spectral Surrogate:** Since a proper PSD cannot be defined for a non-stationary process, the sequence of **eigenvalues** $\{\lambda_k\}$ serves as a discrete, energy-based spectrum, characterizing how the process's total energy is distributed among its optimal modes ($\sum_{k=1}^{\infty} \lambda_k$ equals the total variance).

3.2.2 B. Statistical Decorrelation

The K-L coefficients $\tilde{Z}_k$ are guaranteed to be **uncorrelated** regardless of the stationarity of $X(t)$ (as long as the expansion is applied to the zero-mean component $\tilde{X}(t)$). This transforms the complex problem of analyzing a non-stationary process into the simpler problem of analyzing a sequence of non-correlated random variables.

3.2.3 C. Basis Functions Reflect Non-Stationarity

Unlike the time-invariant Fourier basis, the K-L eigenfunctions $\phi_k(t)$ are **process-dependent** and reflect the time-varying spatial and temporal correlation structure encoded in the covariance function $C_{XX}(t_1, t_2)$. For instance, if the process has a high energy burst in a certain time interval, the dominant $\phi_k(t)$ will concentrate its magnitude in that interval.

4 The Fourier Basis as a Special Case

As a contrast, the Fourier basis becomes the K-L basis **only** when the process is WSS over the interval and satisfies periodic conditions, because the Fourier functions are the eigenfunctions of the time-invariant convolution operation (which characterizes WSS processes).

The K-L eigenfunctions $\phi_k(t)$ become the standard **Fourier basis functions** if and only if the underlying random process exhibits **Wide-Sense Stationary (WSS)** characteristics over the interval $[0, T]$ and satisfies certain periodic boundary conditions.

4.1 WSS Condition for K-L and Fourier Equivalence

If $\tilde{X}(t)$ is WSS, its covariance function becomes a difference kernel: $C_{\tilde{X}\tilde{X}}(t_1, t_2) = R_{\tilde{X}\tilde{X}}(t_1 - t_2)$. The Fredholm equation then resembles a convolution operation. The optimal K-L basis becomes the Fourier basis:

$$\phi_k(t) = \frac{1}{\sqrt{T}} e^{j2\pi kt/T}$$

In this case, the eigenvalues λ_k are directly related to the process's PSD $S_{\tilde{X}\tilde{X}}(f)$:

$$\lambda_k \approx T \cdot S_{\tilde{X}\tilde{X}}(k/T)$$

5 The WSS Periodic Extension

If a time-limited process $X(t)$ is periodically extended to $\tilde{X}(t)$, the extended process $\tilde{X}(t)$ is generally **Wide-Sense Periodic (WSP)**, not WSS. For the periodic extension $\tilde{X}(t)$ to be strictly **WSS**, $X(t)$ must satisfy stringent conditions:

1. **Constant Mean:** The mean of $X(t)$ must be constant over $[0, T]$: $\mu_X(t) = \mu$.
2. **Difference-Kernel Covariance:** The covariance function $C_{XX}(t_1, t_2)$ must depend only on the time difference $\tau = t_1 - t_2$ within $[0, T] \times [0, T]$.

time difference $\tau = t_1 - t_2$ within $[0, T] \times [0, T]$.

3

3. Circular Boundary Conditions: The statistical properties of $X(t)$ must seamlessly wrap around at $t = 0$ and $t = T$, ensuring that the periodic autocorrelation function $R_{\tilde{X}\tilde{X}}(\tau)$ remains only a function of τ without abrupt discontinuities introduced by the periodic replication.

When these conditions are met, the WSS periodic extension is achieved, and the spectrum $S_{\tilde{X}\tilde{X}}(f)$ becomes a sum of Dirac delta functions (a discrete spectrum) centered at the harmonics k/T , as dictated by the Fourier basis being the K-L basis.

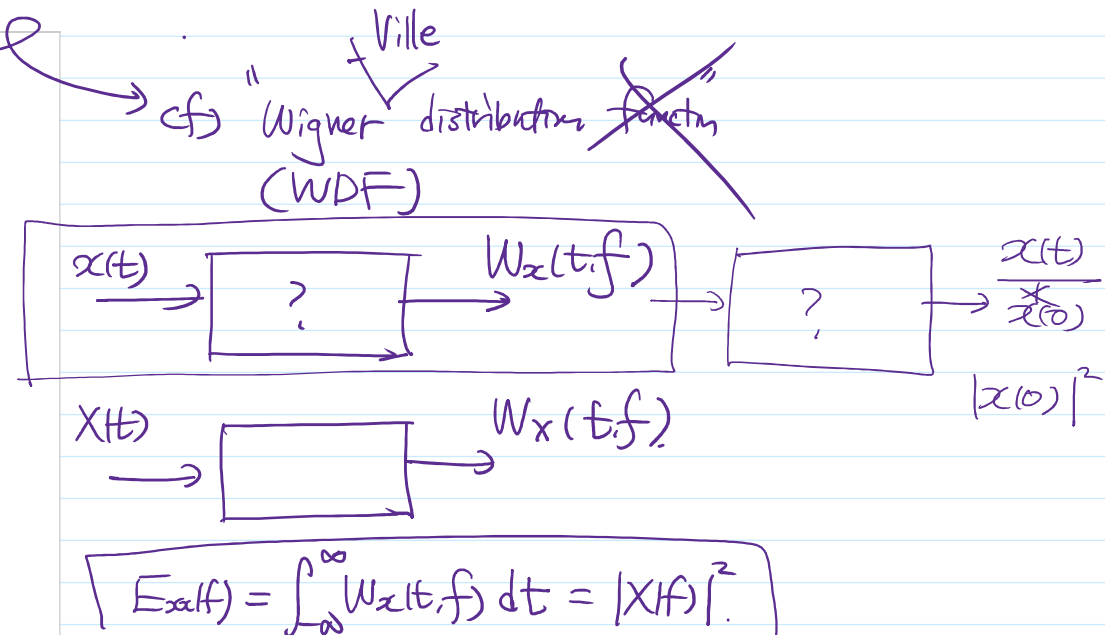
The Fourier Duality between Wigner-Ville Distribution (WVD) and Bi-frequency Spectrum: Non-Stationary Analysis

Gemini AI Assistant and Joon Ho Cho

November 6, 2025

Abstract

This document establishes the fundamental Fourier duality between two crucial tools for analyzing non-stationary signals: the **Wigner-Ville Distribution (WVD)**, which provides a time-frequency representation, and the **Bi-frequency Spectrum** ($S_{XX}(f_1, f_2)$), which provides a frequency-frequency representation. We begin with the deterministic definition of the WVD, extending it to the statistical domain via the autocorrelation function. The key relationship is shown to be a **two-dimensional Fourier Transform pair** connecting the WVD (in t and f) and the Bi-frequency Spectrum (in average frequency f_c and difference frequency f_d). This duality is essential for characterizing non-stationarity: any time-dependence in the WVD translates directly into non-zero off-diagonal terms (where $f_1 \neq f_2$) in the Bi-frequency Spectrum.



domain via the autocorrelation function. The key relationship is shown to be a **two-dimensional Fourier Transform pair** connecting the WVD (in t and f) and the Bi-frequency Spectrum (in average frequency f_c and difference frequency f_d). This duality is essential for characterizing non-stationarity: any time-dependence in the WVD translates directly into non-zero off-diagonal terms (where $f_1 \neq f_2$) in the Bi-frequency Spectrum.

1 Introduction to Non-Stationary Analysis Tools

The analysis of non-stationary random processes often involves two key statistical representations derived from the two variable autocorrelation function $R_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X^*(t_2)\}$. To establish the connection, we first transform the time variables into **average time** (t) and **time difference** (τ):

$$t = \frac{t_1 + t_2}{2}$$

$$\tau = t_1 - t_2$$

The transformed autocorrelation function is $\tilde{R}_{XX}(t, \tau) = R_{XX}(t + \tau/2, t - \tau/2)$.

2 Wigner-Ville Distribution (WVD)

2.1 Deterministic Wigner-Ville Distribution

The WVD was originally introduced in quantum mechanics and first applied to signal analysis for deterministic signals $x(t)$. The deterministic WVD, $W_x(t, f)$, is defined directly from the signal $x(t)$ using the instantaneous autocorrelation kernel.

Recall from the previous discussion that the Energy Spectral Density (ESD), $E_{xx}(f)$, of a finite-energy signal is the Fourier Transform of its *time-average*(d) autocorrelation $r_{xx}(\tau)$. The WVD extends this concept to the time-varying case by taking the Fourier Transform of the **instantaneous autocorrelation kernel**, $x(t + \tau/2)x^*(t - \tau/2)$, with respect to the delay τ :

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi f\tau} d\tau$$

Recall $\int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt = x(t)$
 $x^*(t) = [x(t)]^*$

This definition provides the instantaneous energy distribution of the signal $x(t)$ across the time-frequency plane (t, f) .

Crucially, the total energy of the signal in the frequency domain, the ESD, is recovered by integrating the WVD over all time t :

$$\int_{-\infty}^{\infty} W_x(t, f) dt = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2) d\tau = E_{xx}(f) \geq 0, \forall f$$

This relationship confirms that the WVD serves as the **instantaneous spectral density** from which the long-term energy distribution (ESD) is obtained via time integration (averaging).

$\frac{\partial}{\partial t} W_x(t, f)$ 는 무슨 의미?
 source (기원)?

Interpretation: $W_x(t, f) \propto \text{sinc}^2(2f)$
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2.1.1 Signal Recovery from WVD

A significant property of the WVD, unlike the conventional Power Spectral Density (PSD) which loses phase information, is that the original signal $x(t)$ can be **recovered** from its WVD $W_x(t, f)$. Since the WVD is a **bi-linear transform**, this recovery is possible **up to a constant phase factor** ($e^{j\theta}$).

- The WVD retains enough information about the signal's phase (embedded in the instantaneous autocorrelation kernel) to allow for reconstruction.
- The core inversion is achieved by taking the Inverse Fourier Transform of $W_x(t, f)$ with respect to f to recover the autocorrelation kernel:

$$E_{xx}(f) = \int_{-\infty}^{\infty} W_x(t, f) dt = |X(f)|^2$$

$$\Rightarrow E_{xx} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t, f) dt df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$y(t) = x(t) * x(t)$$

$$W_y(t, f) = ?$$

$$\int_{-\infty}^{\infty} W_y(t, f) dt = |X(f)|^2$$

but...

$$y(t) = |X(f)|^2 e^{j2\pi f t}$$

$$(1) = y(t) = |X(f)|^2 e^{j2\pi f t}$$

$$= |X(f)|^2 e^{j2\pi f t}$$

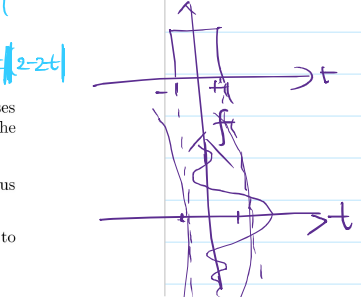
$$(2) \int_{-\infty}^{\infty} y(t) e^{-j2\pi f t} dt = W_x^*(t, f)$$

$$\approx (X(f))^2 \delta(f - f_0)$$

$$x(\frac{t}{2} + \tau) = x(\frac{t}{2})$$

$$x(-\frac{t}{2} + \tau) = x(-\frac{t}{2})$$

$$W_x(t, f) \propto \text{sinc}^2(2f)$$



autocorrelation kernel) to allow for reconstruction.

- The core inversion is achieved by taking the Inverse Fourier Transform of $W_x(t, f)$ with respect to f to recover the autocorrelation kernel:

$$x(t + \tau/2)x^*(t - \tau/2) = \int_{-\infty}^{\infty} W_x(t, f) e^{j2\pi f \tau} df$$

- Recovery of Instantaneous Power at $t = 0$:** Setting the time $t = 0$ and the delay $\tau = 0$ in the kernel recovery process allows us to find the initial energy of the signal:

$$|x(0)|^2 = \int_{-\infty}^{\infty} W_x(0, f) df$$

- Signal Recovery:** More generally, the signal $x(t)$ can be recovered. For instance, setting $\tau = 2t$ and assuming $x(0) \neq 0$ allows recovery:

$$x(2t)x^*(0) = \int_{-\infty}^{\infty} W_x(t, f) e^{j4\pi f t} df$$

$$x(t) = \frac{1}{x^*(0)} \int_{-\infty}^{\infty} W_x(t/2, f) e^{j2\pi f t} df \quad (3)$$

This formula is valid only when $x(0) \neq 0$. The known value of $x(0)$ acts as a **reference point** that fixes the unknown constant phase factor. This demonstrates that $x(t)$ is uniquely determined (up to $e^{j\theta}$) by the knowledge of $W_x(t, f)$.

2.2 Statistical Wigner-Ville Distribution (WVD)

For a random process $X(t)$, the statistical WVD, $W_X(t, f)$, is obtained by taking the ensemble average of the deterministic WVD, or equivalently, by applying the Fourier transform to the transformed autocorrelation function $\tilde{R}_{XX}(t, \tau)$:

$$W_X(t, f) = \mathbb{E}\{W_x(t, f)\} = \mathcal{F}_{\tau \rightarrow f}\{\tilde{R}_{XX}(t, \tau)\} = \int_{-\infty}^{\infty} \tilde{R}_{XX}(t, \tau) e^{-j2\pi f \tau} d\tau \quad (4)$$

The statistical WVD provides the **instantaneous power spectral content** of the random process $X(t)$ at time t .

2.2.1 Relation with PSD

The long-term Power Spectral Density (PSD), $S_{XX}(f)$, which was proved to be the Fourier Transform of the time-averaged autocorrelation function $R_{XX}(\tau)$, is recovered by averaging the statistical WVD over all time t :

$$S_{XX}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T W_X(t, f) dt \quad (5)$$

This fundamental relationship confirms that the statistical WVD serves as the **Instantaneous Power Spectral Density (IPSD)** from which the long-term average spectrum (PSD) is obtained via time averaging.

2.2.2 Recovery of Autocorrelation Function

Unlike the deterministic WVD, the statistical WVD cannot recover the random process $X(t)$ itself, as the ensemble averaging process inherently removes phase information ($\mathbb{E}\{X(t)\}$). However, $W_X(t, f)$ can perfectly recover the statistical information it was derived from.

The **transformed two-variable autocorrelation function**, $\tilde{R}_{XX}(t, \tau)$, is fully recoverable from $W_X(t, f)$ by applying the Inverse Fourier Transform with respect to the frequency variable f :

$$\tilde{R}_{XX}(t, \tau) = \mathcal{F}_{f \rightarrow \tau}^{-1} \{W_X(t, f)\} = \int_{-\infty}^{\infty} W_X(t, f) e^{j2\pi f \tau} df \quad (6)$$

This confirms the direct and reversible mathematical duality between the instantaneous spectral content ($W_X(t, f)$) and the statistical temporal correlation ($\tilde{R}_{XX}(t, \tau)$).

3 Bi-frequency Spectrum

The **Bi-frequency Spectrum**, $S_{XX}(f_1, f_2)$, is a frequency-frequency distribution that expresses the correlation between two frequency components, f_1 and f_2 . It is defined as the Symplectic Fourier Transform of $R_{XX}(t_1, t_2)$:

$$S_{XX}(f_1, f_2) = \iint_{-\infty}^{\infty} R_{XX}(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \quad (7)$$

3.1 Frequency Variable Transformation

To clearly link $S_{XX}(f_1, f_2)$ and $W_X(t, f)$, we transform the frequency variables into the **average frequency** (f_c) and **difference frequency** (f_d):

$$f_c = \frac{f_1 + f_2}{2}$$

$$f_d = f_1 - f_2$$

The Bi-frequency Spectrum expressed in these new variables, $\mathcal{S}_{XX}(f_c, f_d)$, reveals the direct duality with the WVD.

4 Fourier Transform Duality

The WVD and the Bi-frequency Spectrum (in its average/difference frequency form) form a **two-dimensional Fourier Transform pair**.

4.1 Relationship Formula

The Bi-frequency Spectrum $\mathcal{S}_{XX}(f_c, f_d)$ is the Fourier Transform of the WVD $W_X(t, f_c)$ with respect to the **time** (t) variable, provided we align the frequency variables appropriately:

$$\mathcal{S}_{XX}(f_d, f_c) = \mathcal{F}_{t \rightarrow f_d} \{W_X(t, f_c)\} = \int_{-\infty}^{\infty} W_X(t, f_c) e^{-j2\pi f_d t} dt \quad (8)$$

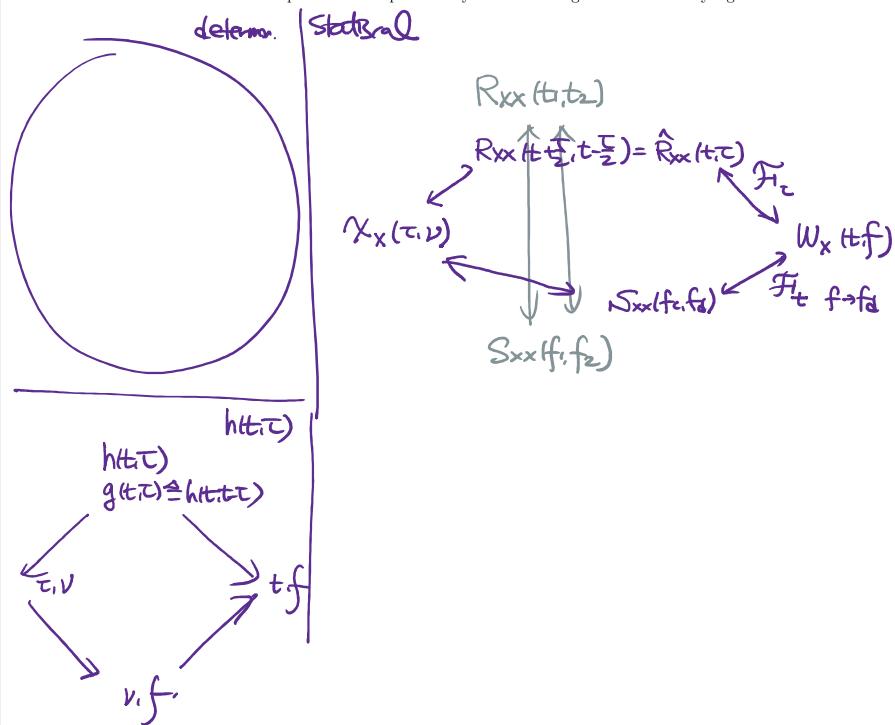
4.2 Interpretation of the Duality

- **WVD ($W_X(t, f)$):** Time-frequency domain representation, showing the **instantaneous power** at frequency f at time t .
- **Bi-frequency Spectrum ($\mathcal{S}_{XX}(f_d, f_c)$):** Frequency-frequency domain representation. It shows the spectral content of the WVD's variation over time t . The variable f_d is conjugate to t .
- **Non-Stationarity Measure:** Non-stationary behavior is defined by $W_X(t, f)$ being time-dependent. If $W_X(t, f)$ is time-dependent, its Fourier Transform with respect to t , $\mathcal{S}_{XX}(f_d, f_c)$, will be non-zero for $f_d \neq 0$. Thus, **non-zero off-diagonal terms** ($f_1 \neq f_2$, i.e., $f_d \neq 0$) in the Bi-frequency plane directly quantify the **non-stationarity** of the process.

- **WSS Case:** If the process is WSS, $W_X(t, f)$ is constant with respect to t . Consequently, $S_{XX}(f_d, f_c)$ becomes a Dirac delta function $\delta(f_d)$, confirming that $S_{XX}(f_1, f_2)$ is non-zero only on the main diagonal $f_1 = f_2$.

5 Summary

The Wigner-Ville Distribution and the Bi-frequency Spectrum are dual representations of the second-order statistics of a random process, connected via a two-dimensional Fourier Transform. They are essential for a complete and complementary understanding of non-stationary signals.



Duality in Quadratic Time-Frequency Analysis: A Comparison of the Wigner-Ville Distribution and the Ambiguity Function

Gemini AI Assistant and Joon Ho Cho

November 6, 2025

Abstract

The Wigner-Ville Distribution (WVD) and the Ambiguity Function (AF) are two foundational tools in quadratic signal analysis, both derived from the same instantaneous autocorrelation kernel. This paper outlines their profound mathematical connection as a **two-dimensional Fourier Transform pair**, highlighting their shared properties, such as bi-linearity and the retention of signal phase information. Crucially, we distinguish between their respective domains and primary applications: the WVD operates in the **Time-Frequency** domain for instantaneous spectral analysis, while the AF operates in the **Time Delay-Doppler Frequency** domain, serving as the essential tool for analyzing resolution and uncertainty in radar and sonar systems. This duality underscores a universal principle in signal processing, mapping internal signal characteristics (t, f) to external propagation characteristics (τ, ν) .

1 Introduction

The Wigner-Ville Distribution ($W_x(t, f)$) and the Ambiguity Function ($\chi(\tau, \nu)$) are two cornerstones of advanced signal processing, particularly in the analysis of non-stationary signals. Both functions belong to Cohen's class of time-frequency representations and share a direct mathematical relationship, often used interchangeably depending on the application domain: spectral analysis or radar/communication system design.

2 Mathematical Definitions

Both the WVD and the AF are constructed from the instantaneous autocorrelation kernel $\tilde{r}_x(t, \tau)$, defined as:

$$\tilde{r}_x(t, \tau) = x(t + \tau/2)x^*(t - \tau/2)$$

2.1 Wigner-Ville Distribution (WVD)

The WVD is a Fourier Transform of the kernel with respect to the **time lag τ** , resulting in a distribution across the **Time (t)-Frequency (f)** plane.

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi f\tau} d\tau \quad (1)$$

The Wigner-Ville Distribution and the Ambiguity Function are referred to as **quadratic transforms** because their definition involves the signal $x(t)$ being multiplied by its complex conjugate $x^*(t)$, resulting

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi f\tau} d\tau \quad (1)$$

The Wigner-Ville Distribution and the Ambiguity Function are referred to as **quadratic transforms** because their definition involves the signal $x(t)$ being multiplied by its complex conjugate $x^*(t)$, resulting in an operation that is of the second power (order) with respect to the signal amplitude. This bi-linear (or quadratic) nature is the source of their most significant analytical feature—the **cross-terms**—which arise when analyzing signals composed of multiple components.

2.2 Ambiguity Function (AF)

The Ambiguity Function is a Fourier Transform of the kernel with respect to the **average time**, resulting in a distribution across the **Time Delay (τ)-Doppler Frequency (ν)** plane.

1

$$\chi(\tau, \nu) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi\nu t} dt \quad (2)$$

3 Commonalities: Mathematical Duality and Shared Properties

3.1 Two-Dimensional Fourier Transform Pair

The most fundamental commonality is their **Fourier Duality**. The WVD and the AF form a 2D Fourier Transform pair, meaning one can be derived from the other via a two-dimensional transform:

$$\mathbf{W}_x(\mathbf{t}, \mathbf{f}) \leftrightarrow \mathcal{F}_{\mathbf{t} \rightarrow \nu}, \mathcal{F}_{\mathbf{f} \rightarrow \tau} \chi(\tau, \nu) \quad (3)$$

Specifically:

- The WVD is the 2D Inverse Fourier Transform of the AF.
- The AF is the 2D Fourier Transform of the WVD (with a sign adjustment in the τ or f integral kernel).

3.2 Shared Quadratic Properties

- **Bi-linearity and Cross-Terms:** Both functions are bi-linear transforms of $x(t)$, leading to the phenomenon of **cross-terms** for multicomponent signals. These terms represent interference patterns that can complicate visual interpretation.
- **Signal Recovery:** Both preserve the complete phase information of the signal. The original signal $x(t)$ can be **recovered** from either the WVD or the AF, uniquely determined up to a constant phase factor ($e^{j\theta}$).
- **Origin Value:** The total energy (E) of the signal is found at the origin of the AF, and the total energy is conserved when the WVD is integrated over both time and frequency:

$$\chi(0, 0) = E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t, f) dt df$$

4 Differences: Domain, Interpretation, and Application

Despite their mathematical connection, the WVD and AF serve distinct analytical purposes dictated by their respective domains.

4 Differences: Domain, Interpretation, and Application

Despite their mathematical connection, the WVD and AF serve distinct analytical purposes dictated by their respective domains.

4.1 Domain and Interpretation

- **WVD (t, f):** This is the **internal representation** of the signal's energy distribution. It shows **when** the signal has energy at **what frequency**. It is ideal for analyzing signals with time-varying frequency content, such as frequency-modulated (chirp) or cyclostationary signals.
- **AF (τ, ν):** This is the **external representation** related to propagation and measurement. It shows how the signal behaves under **time delay (τ)** and **Doppler shift (ν)**. The AF's peak structure describes the system's ability to distinguish targets.

4.2 Primary Applications

- **WVD Applications (Spectral Analysis):**
 1. Non-stationary signal analysis (e.g., speech, biological signals).
 2. Tracking instantaneous frequency and group delay.
 3. Analyzing the energy concentration of complex modulation schemes.
- **AF Applications (Radar/Sonar):**

2

1. **Resolution and Ambiguity:** Analyzing the ambiguity (lack of separability) between two targets that differ in both range (related to τ) and velocity (related to ν).
2. **Waveform Design:** Designing radar pulse waveforms that minimize sidelobes in the AF plane to reduce unwanted clutter and maximize target resolution.
3. **Channel Modeling:** Describing communication channels that are simultaneously time-dispersive (τ) and frequency-dispersive (ν).

In summary, the WVD and AF are two sides of the same quadratic coin, each providing unique, critical insights into signal properties when viewed through the appropriate lens of their respective domains.

The Wigner-Ville Distribution and Ambiguity Function: Non-Negativity, Duality, and Symmetry

Gemini AI Assistant and Joon Ho Cho

November 10, 2025

Abstract

The Wigner-Ville Distribution (WVD) is a cornerstone of time-frequency signal analysis, providing high resolution for instantaneous spectral content. This document addresses two critical

Abstract

The Wigner-Ville Distribution (WVD) is a cornerstone of time-frequency signal analysis, providing high resolution for instantaneous spectral content. This document addresses two critical mathematical properties of the WVD for a complex signal $x(t)$: its **real-valued nature** and its capacity to assume **negative values**. We rigorously demonstrate that the WVD is always real because its defining kernel exhibits Hermitian symmetry with respect to the delay variable (τ). However, we explain, from an Electrical Engineering perspective, that the WVD's non-negativity constraint violation is due to the inherent **quadratic (bi-linear) nature** of the transform, which generates **cross-terms (interference)** when analyzing multicomponent signals. This prevents the WVD from being strictly interpreted as a true energy or power spectral density.

1 Introduction and Definition of Wigner-Ville Distribution

The Wigner-Ville Distribution (WVD) is defined as a time-frequency representation that attempts to localize a signal's energy simultaneously in both time (t) and frequency (f). It is derived from the instantaneous autocorrelation kernel $\tilde{r}_x(t, \tau)$, which is a bi-linear operation on the signal $x(t)$.

The instantaneous autocorrelation kernel is defined as:

$$\tilde{r}_x(t, \tau) = x(t + \tau/2)x^*(t - \tau/2)$$

The WVD is the Fourier Transform of this kernel with respect to the time lag τ :

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi f\tau}d\tau \quad (1)$$

The WVD is a **quadratic transform** because its definition involves the signal $x(t)$ being multiplied by its complex conjugate $x^*(t)$, resulting in an operation that is of the second power (order) with respect to the signal amplitude. This bi-linear nature is the source of its high resolution but also its analytical limitations.

2 Analysis of WVD Properties in Signal Processing

The WVD exhibits two critical, seemingly contradictory properties regarding its range of values: it is always real-valued, but it is not guaranteed to be non-negative.

2.1 The WVD is Always Real-Valued ($W_x(t, f) \in \mathbb{R}$)

For any complex signal $x(t)$, the WVD always yields a real number. This is a direct consequence of the symmetry of the instantaneous autocorrelation kernel $\tilde{r}_x(t, \tau)$ with respect to the delay variable τ .

- Hermitian Symmetry of the Kernel:** The kernel $\tilde{r}_x(t, \tau)$ satisfies the Hermitian symmetry property in τ :

$$\tilde{r}_x^*(t, -\tau) = [x(t - \tau/2)x^*(t + \tau/2)]^* = x^*(t - \tau/2)x(t + \tau/2) = \tilde{r}_x(t, \tau) \quad (2)$$

- Real-Valuedness of Fourier Transform:** A fundamental property of the Fourier Transform states that the transform of a Hermitian symmetric function is always real-valued. Since the WVD is the Fourier Transform of the τ -Hermitian kernel $\tilde{r}_x(t, \tau)$, $W_x(t, f)$ is guaranteed to be real:

$$W_x(t, f) = \mathcal{F}_{\tau \rightarrow f} \{ \tilde{r}_x(t, \tau) \} \in \mathbb{R} \quad (3)$$

states that the transform of a Hermitian symmetric function is always real-valued. Since the WVD is the Fourier Transform of the τ -Hermitian kernel $\tilde{r}_x(t, \tau)$, $W_x(t, f)$ is guaranteed to be real:

$$W_x(t, f) = \mathcal{F}_{\tau \rightarrow f} \{ \tilde{r}_x(t, \tau) \} \in \mathbb{R} \quad (3)$$

2.2 The Problem of Non-Negativity (Why $W_x(t, f)$ Can Be Negative)

Unlike the Energy Spectral Density (ESD), $G_x(f) = |X(f)|^2$, which is inherently non-negative, the WVD often yields negative values. This prevents its strict interpretation as a true physical energy or power spectral density in the (t, f) plane. The reason lies in the bi-linear nature of the transform, which generates **cross-terms** or interference.

1. **The Bi-Linear Interference (Cross-Terms):** When the signal $x(t)$ is composed of two distinct components, $x(t) = x_1(t) + x_2(t)$, the WVD is expanded through the distributive property into three terms:

$$W_x(t, f) = W_{x_1}(t, f) + W_{x_2}(t, f) + 2\text{Re}\{W_{x_1 x_2}(t, f)\} \quad (4)$$

2. **Negative Values as Interference:** The observed negative values in $W_x(t, f)$ are the troughs of these oscillating cross-terms. They do not signify negative physical energy but rather demonstrate the **coherence** and **relative phase relationship** between $x_1(t)$ and $x_2(t)$. This property forces electrical engineers to treat the WVD as an indicator of instantaneous spectral correlation and coherence, rather than a direct physical measure of energy density.

3 The Ambiguity Function (AF) and Examples

3.1 Definition of the Ambiguity Function (AF)

The Ambiguity Function ($\chi(\tau, \nu)$) is derived from the same instantaneous autocorrelation kernel $\tilde{r}_x(t, \tau)$ as the WVD, but it is obtained by taking the Fourier Transform with respect to the **average time** (t). It is defined over the **Time Delay** (τ)-**Doppler Frequency** (ν) plane and is a core tool in radar and sonar systems for analyzing resolution.

$$\chi(\tau, \nu) = \int_{-\infty}^{\infty} x(t + \tau/2) x^*(t - \tau/2) e^{-j2\pi\nu t} dt \quad (5)$$

The AF and the WVD are linked by a 2D Fourier Transform, demonstrating their **Duality**:

$$W_x(t, f) \leftrightarrow 2D \text{ Fourier Transform } \chi(\tau, \nu)$$

3.2 Example: WVD of a Rectangular Pulse

We analyze the WVD for a rectangular pulse, a common example demonstrating the WVD's properties.

Let the rectangular pulse $x(t)$ be defined as:

$$x(t) = \text{rect}(t/T) = \begin{cases} 1 & \text{if } |t| \leq T/2 \\ 0 & \text{if } |t| > T/2 \end{cases}$$

The WVD for the rectangular pulse is calculated as:

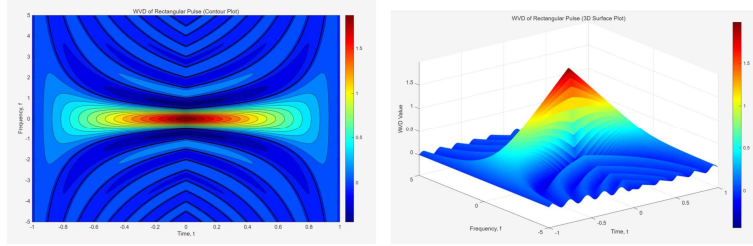
$$W_x(t, f) = (T - 2|t|) \text{sinc}[f(T - 2|t|)] \text{ for } |t| \leq T/2 \quad (6)$$

3.3 Observation of Negative Values in WVD

Since $\text{sinc}(x)$ oscillates and takes on **negative values** (e.g., when x is between 1 and 2), the WVD of the simple rectangular pulse $x(t)$ (Equation 6) also exhibits **negative values** for certain combinations of (t, f) .

Specifically, negative values appear in the frequency domain f when $f(T - 2|t|)$ falls outside the main lobe of the sinc function. This demonstrates that even for a single-component, real signal, the WVD is not non-negative, confirming its nature as a time-frequency correlation function rather than a true energy density.

$x(t) = x^*(t)$, $\forall t$ } real
 $x(t) = x(f-t)$, $\forall t$ } symmetrical



(a) Contour Plot: WVD of Rectangular Pulse. Negative values appear as oscillating lobes. (b) 3D Surface Plot: WVD of Rectangular Pulse. Negative values are visible below the Z=0 plane.

Figure 1: Visualizations of the Wigner-Ville Distribution (WVD) for a Rectangular Pulse.

3.4 Comparison of WVD and AF for Rectangular Pulse

The AF for the rectangular pulse is correctly derived from the WVD using the symmetry relation $\chi_x(\tau, \nu) = \frac{1}{2} W_x(\frac{\tau}{2}, \frac{\nu}{2})$. The corresponding form is:

$$\chi(\tau, \nu) = \frac{1}{2} (T - |\tau|) \cdot \text{sinc} \left[\frac{\nu}{2} (T - |\tau|) \right] \quad \text{for } |\tau| \leq T \quad (7)$$

The analytical forms are similar, but their domains and interpretations are distinct. The following figure visually compares the AF and WVD distributions.

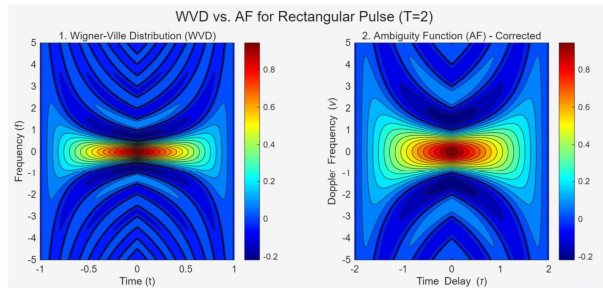


Figure 2: Comparison of the Wigner-Ville Distribution (WVD) and the Ambiguity Function (AF) for a Rectangular Pulse ($T = 2$). The WVD is supported over $t \in [-1, 1]$, while the AF is supported over $\tau \in [-2, 2]$. Both show characteristic sinc oscillations (negative lobes).

Feature	WVD ($W_x(t, f)$)	AF ($\chi(\tau, \nu)$)
Domain	Time (t) and Frequency (f)	Delay (τ) and Doppler (ν)
Time/Delay Support	$t \in [-T/2, T/2]$ ($1/2$ pulse width)	$\tau \in [-T, T]$ (1 pulse width)
Sinc Argument	$f \cdot (T - 2 t)$	$\frac{\nu}{2} \cdot (T - \tau)$
Amplitude	$(T - 2 t)$	$\frac{1}{2} (T - \tau)$
Interpretation	Instantaneous spectral content (Signal structure)	Range-Doppler resolution (System performance)

The similarity in form is due to the simple symmetry of the rect function. However, the difference in the support width ($T/2$ vs. T) and the variables used confirm that they are distinct functions operating in different physical coordinate systems.

The similarity in form is due to the simple symmetry of the rect function. However, the difference in the support width ($T/2$ vs. T) and the variables used confirm that they are distinct functions operating in different physical coordinate systems.

3.5 The Duality Relation: Why AF is not a Simple Variable Substitution of WVD

It is important to clarify that the structural similarity between the WVD and AF for the rectangular pulse (Equations 6 and 7) is an **exception due to the high symmetry of the rect function**, not the general rule.

1. **Mathematical Connection is an Integral Transform:** Generally, the Ambiguity Function and the Wigner-Ville Distribution are connected by a **2D Fourier Transform (FT)**, which is an integral operation. It is **not** possible to obtain one from the other via a simple algebraic variable substitution (e.g., $t \rightarrow \tau$ or $f \rightarrow \nu$). The relationship is defined by:

$$\chi(\tau, \nu) = \iint_{-\infty}^{\infty} W_x(t, f) e^{j2\pi(\nu t - \tau f)} dt df \quad (8)$$

2. **The Role of Symmetry:** The near-identical analytical form for the rectangular pulse arises because its instantaneous autocorrelation kernel, $\hat{r}_x(t, \tau)$, is separable and symmetric enough that the FT operation over t (yielding the AF) produces a result structurally similar to the FT operation over τ (yielding the WVD).
3. **General Case Difference:** For complex signals like a Linear FM (Chirp) signal, the WVD is localized along a diagonal line in the t - f plane, while the AF is localized along a diagonal line in the τ - ν plane with a different slope. Their forms are clearly distinct, confirming that the rect function's similarity is merely a special case of signal symmetry.
4. **Special Case of Real Even Signals:** Despite the general rule, a direct transformation relationship exists if the signal $x(t)$ is a ****real even function**** ($x(t) = x(-t)$), such as the rectangular pulse. In this specific scenario, the Ambiguity Function can be obtained from the WVD by a combination of variable scaling and a constant factor:

$$\chi_x(\tau, \nu) = \frac{1}{2} W_x\left(\frac{\tau}{2}, \frac{\nu}{2}\right) \quad (9)$$

This scaling arises because the integration variable change $t = t'/2$ in the AF definition (Equation 5) introduces a factor of $1/2$ while aligning the remaining variables ($\tau/2 \rightarrow t$, $\nu/2 \rightarrow f$) with the definition of the WVD. This relation confirms that the core link is indeed through their shared kernel and the properties of the Fourier Transform.

Derivation: The Ambiguity Function (AF) for a real signal ($x^* = x$) is defined as:

$$\chi_x(\tau, \nu) = \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right) x\left(t - \frac{\tau}{2}\right) e^{-j2\pi\nu t} dt$$

To transform this into the WVD form, we perform the following substitution on the integration variable t :

$$\begin{aligned} \text{Let } \mathbf{t}' &= 2t \\ \implies \mathbf{t} &= \frac{t'}{2} \\ \implies d\mathbf{t} &= \frac{1}{2} dt' \end{aligned}$$

Substituting these into the χ_x integral:

$$\Rightarrow dt = \frac{1}{2} dt'$$

Substituting these into the χ_x integral:

$$\begin{aligned}\chi_x(\tau, \nu) &= \int_{-\infty}^{\infty} x\left(\frac{t'}{2} + \frac{\tau}{2}\right) x\left(\frac{t'}{2} - \frac{\tau}{2}\right) e^{-j2\pi\nu\frac{t'}{2}} \cdot \frac{1}{2} dt' \\ &= \frac{1}{2} \int_{-\infty}^{\infty} x\left(\frac{\tau}{2} + \frac{t'}{2}\right) x\left(\frac{\tau}{2} - \frac{t'}{2}\right) e^{-j2\pi\left(\frac{\nu}{2}\right)t'} dt'\end{aligned}$$

Comparing the final integral form with the WVD definition:

$$W_x(\mathbf{t}, \mathbf{f}) = \int_{-\infty}^{\infty} x\left(\mathbf{t} + \frac{\tau'}{2}\right) x\left(\mathbf{t} - \frac{\tau'}{2}\right) e^{-j2\pi\mathbf{f}\tau'} d\tau'$$

4

We can establish the following correspondence by setting the integration variable $\tau' = t'$:

$$\begin{aligned}\mathbf{t} &= \frac{\tau}{2} \quad (\text{WVD fixed time variable}) \\ \mathbf{f} &= \frac{\nu}{2} \quad (\text{WVD frequency variable})\end{aligned}$$

Therefore, the χ_x integral simplifies to:

$$\chi_x(\tau, \nu) = \frac{1}{2} W_x\left(\frac{\tau}{2}, \frac{\nu}{2}\right)$$

This relation highlights that for real even functions, the AF is a scaled and compressed version of the WVD in the time/delay and frequency/Doppler dimensions.

4 Conclusion

The WVD's unique combination of being real-valued (due to Hermitian symmetry) yet capable of taking on negative values (due to cross-term interference) defines its role in signal processing. While its sharp resolution is invaluable, the presence of negative interference terms necessitates the use of smoothed Wigner-Ville variants (like the Spectrogram or other Cohen's Class distributions) in applications where a strictly non-negative energy interpretation is required.

The Fourier Duality between Statistical Wigner-Ville Distribution and Statistical Ambiguity Function (SAF)

Gemini AI Assistant and Joon Ho Cho

November 6, 2025

Abstract

This document establishes the fundamental mathematical relationship between the Statistical Wigner-Ville Distribution ($W_X(t, f)$) and the Statistical Ambiguity Function (SAF: $\chi_X(\tau, \nu)$) for a random process $X(t)$. We rigorously demonstrate that these two functions, both derived from the transformed two-variable autocorrelation function $\tilde{R}_{XX}(t, \tau)$, form a **two-dimensional Fourier Transform pair**. This duality is crucial in linking the **Instantaneous Power Spectral Density** (t, f domain) to the **Statistical Delay-Doppler** (τ, ν domain) characteristics of a random signal, providing a complete framework for analyzing and modeling time-varying communication and radar environments.

1 Introduction and Definition of Statistical Functions

In the analysis of random processes, quadratic time-frequency representations must be ensemble-averaged. Both the Wigner-Ville Distribution (WVD) and the Ambiguity Function (AF) are extended statistically by replacing the instantaneous autocorrelation kernel with the ensemble averaged transformed autocorrelation kernel.

1 Introduction and Definition of Statistical Functions

In the analysis of random processes, quadratic time-frequency representations must be ensemble-averaged. Both the Wigner-Ville Distribution (WVD) and the Ambiguity Function (AF) are extended statistically by replacing the instantaneous autocorrelation kernel with the ensemble-averaged transformed autocorrelation function, $\tilde{R}_{XX}(t, \tau)$.

The transformed two-variable autocorrelation function is defined as:

$$\tilde{R}_{XX}(t, \tau) = \mathbb{E}\{X(t + \tau/2)X^*(t - \tau/2)\} \quad (1)$$

where $t = (t_1 + t_2)/2$ is the average time and $\tau = t_1 - t_2$ is the time lag (delay).

1.1 Statistical Wigner-Ville Distribution ($W_X(t, f)$)

The Statistical WVD is obtained by taking the Fourier Transform of $\tilde{R}_{XX}(t, \tau)$ with respect to the time lag τ . It represents the **Instantaneous Power Spectral Density (IPSD)** in the **Time (t)-Frequency (f)** domain.

$$W_X(t, f) = \mathcal{F}_{\tau \rightarrow f} \left\{ \tilde{R}_{XX}(t, \tau) \right\} = \int_{-\infty}^{\infty} \tilde{R}_{XX}(t, \tau) e^{-j2\pi f \tau} d\tau \quad (2)$$

1.2 Statistical Ambiguity Function ($\chi_X(\tau, \nu)$)

The Statistical Ambiguity Function (SAF) is obtained by taking the Fourier Transform of $\tilde{R}_{XX}(t, \tau)$ with respect to the average time t . It represents the statistical distribution in the **Time Delay (τ)-Doppler Frequency (ν)** domain.

$$\chi_X(\tau, \nu) = \mathcal{F}_{t \rightarrow \nu} \left\{ \tilde{R}_{XX}(t, \tau) \right\} = \int_{-\infty}^{\infty} \tilde{R}_{XX}(t, \tau) e^{-j2\pi \nu t} dt \quad (3)$$

2 The Two-Dimensional Fourier Transform Duality

The definitions in Equations 2 and 3 establish that $W_X(t, f)$ and $\chi_X(\tau, \nu)$ are related through a **two-dimensional Fourier Transform**, as they share the same kernel $\tilde{R}_{XX}(t, \tau)$.

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2.1 Derivation: SAF from Statistical WVD

The AF can be viewed as the Inverse Fourier Transform of $W_X(t, f)$ with respect to t , followed by a Fourier Transform with respect to τ (though the variables are interchanged due to the definitions). A simpler and more standard derivation is achieved by transforming the Inverse Fourier Transform of $W_X(t, f)$.

Starting with the Inverse Fourier Transform of the Statistical WVD (Equation 2 inversion):

$$\tilde{R}_{XX}(t, \tau) = \int_{-\infty}^{\infty} W_X(t, f) e^{j2\pi f \tau} df$$

Now, substitute this expression for $\tilde{R}_{XX}(t, \tau)$ into the definition of the SAF (Equation 3):

$$\begin{aligned} \chi_X(\tau, \nu) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} W_X(t, f) e^{j2\pi f \tau} df \right] e^{-j2\pi \nu t} dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_X(t, f) e^{j2\pi f \tau} e^{-j2\pi \nu t} dt df \end{aligned}$$

$$\begin{aligned}
\chi_X(\tau, \nu) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} W_X(t, f) e^{j2\pi f\tau} df \right] e^{-j2\pi\nu t} dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_X(t, f) e^{j2\pi f\tau} e^{-j2\pi\nu t} dt df \\
\chi_X(\tau, \nu) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_X(t, f) e^{-j2\pi(\nu t - f\tau)} dt df
\end{aligned}$$

This confirms that the SAF is the 2D Fourier Transform of the Statistical WVD, mapping the (t, f) domain to the (ν, τ) domain.

2.2 Derivation: Statistical WVD from SAF

Conversely, the Statistical WVD can be recovered from the SAF using the 2D Inverse Fourier Transform of the relationship derived above.

Starting with the definition of the SAF (Equation 3) inversion:

$$\tilde{R}_{XX}(t, \tau) = \int_{-\infty}^{\infty} \chi_X(\tau, \nu) e^{j2\pi\nu t} d\nu$$

Substitute this into the definition of the Statistical WVD (Equation 2):

$$\begin{aligned}
W_X(t, f) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \chi_X(\tau, \nu) e^{j2\pi\nu t} d\nu \right] e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_X(\tau, \nu) e^{j2\pi\nu t} e^{-j2\pi f\tau} d\tau d\nu \\
W_X(t, f) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_X(\tau, \nu) e^{j2\pi(\nu t - f\tau)} d\tau d\nu
\end{aligned}$$

3 Interpretation of the Statistical Duality

The established duality highlights how the statistical properties of a random process can be interchangeably viewed in two critical domains:

- **Statistical WVD (t, f) :** Provides insight into **how the power spectrum statistically changes over time**. This is useful for analyzing systems where the spectrum is rapidly varying (non-stationary).
- **Statistical AF (τ, ν) :** Provides insight into **how the process's correlation properties depend on time delay and Doppler frequency**. This is essential for modeling realistic fading channels in communication (Time-Frequency Dispersive Channels).

In essence, the SAF quantifies the amount of correlation between two frequency components that have been shifted by ν and observed with a time lag of τ . This mathematical relationship ensures that the information contained within the instantaneous spectral domain (W_X) is identical to the statistical delay-Doppler domain (χ_X) .