

Corrections

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Mean-Square (MS) Periodic vs. Wide-Sense Periodic (WSP) Random Processes

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Abstract

This document explores the properties and implications of a **Mean-Square (MS) Periodic Random Process** within the framework of **second-order stochastic processes**. An MS periodic process $\{X(t), t \in \mathbb{R}\}$ with period T is defined by the stringent condition that the mean-square error between $X(t+T)$ and $X(t)$ is zero, implying $X(t+T) = X(t)$ **almost surely**. This strong constraint ensures that the process's realizations are virtually deterministic periodic functions. We rigorously demonstrate that MS periodicity guarantees the periodicity of its fundamental statistical moments: the **mean function** $m_X(t)$ and the **autocorrelation function** $R_X(t_1, t_2)$, with the same period T . Finally, we clarify that the MS periodic condition is a significantly stronger requirement than, yet is a sufficient condition for, the **Wide-Sense Periodic (WSP)** property, highlighting the consistency and structural rigidity imposed on the process's sample paths.

1 Prerequisites: Second-Order Random Process

A random process $\{X(t), t \in \mathbb{R}\}$ must be a **second-order random process** to satisfy the conditions for MS periodicity.

1.1 Second-Order Condition

The process $X(t)$ is second-order if its second moment is finite for all t :

$$\mathbb{E}\{|X(t)|^2\} < \infty \quad \text{for all } t \in \mathbb{R}$$

This condition ensures that the mean function $m_X(t)$ and the autocorrelation function $R_X(t_1, t_2)$ are well-defined and finite.

2 Definition of an MS Periodic Random Process

A second-order random process $\{X(t), t \in \mathbb{R}\}$ is said to be **periodic in the Mean-Square (MS) sense** with period $T > 0$ if the following strong condition holds for all time t .

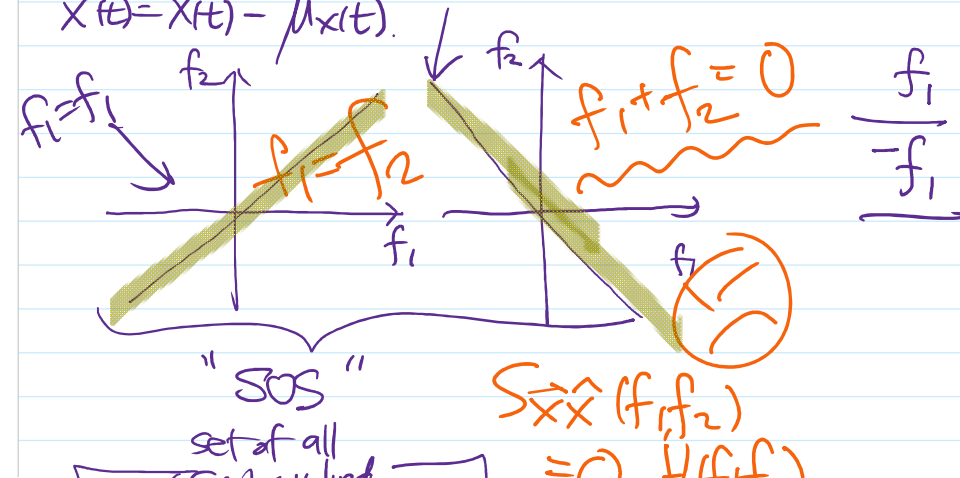
□ "Spectrally-Related" 2nd-order
Complex-valued RPs.
Spectrally-unrelated.

$$R_{XX}(t_1, t_2) = C_{XX}(t_1, t_2) + \mu_X(t_1)\mu_X^*(t_2)$$

↑ (2-D) Symplectic FT

$$S_{\hat{X}\hat{X}}(f_1, f_2), \quad \tilde{S}_{XX}(f_1, f_2)$$

$$\hat{X}(t) \triangleq X(t) - \mu_X(t)$$



Time-Domain Analysis of Second-Order Cyclostationary (SOCS) Processes under Frequency Up- and Down-Conversions

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Abstract

This document presents a **time-domain analysis** of how frequency **up-conversion** (to a real passband signal $Y(t)$) and **down-conversion** (to a complex baseband signal $X(t)$) affect the properties of a **Second-Order Cyclostationary (SOCS)** process $X(t)$. By inspecting the periodicity of the mean, autocorrelation (ACF), and complementary autocorrelation (CACF) functions, we derive the strict frequency conditions required to preserve cyclostationarity. We demonstrate that the ACF of the down-converted signal $\hat{X}(t)$ **always** maintains T -periodicity, regardless of the frequency offset Δf . However, the SOCS property is generally lost unless the synchronization condition is met. Specifically, for a non-zero mean process, the carrier f_c (for $Y(t)$) or frequency error Δf (for $\hat{X}(t)$) must be an integer multiple of the cyclic frequency $(1/T)$. For a zero-mean, non-proper (improper) SOCS process, this condition is relaxed to $f_c = m/(2T)$ or $\Delta f = k'/(2T)$. We also note that for zero-mean, proper-complex processes, cyclostationarity is unconditionally preserved.

1. Cyclostationary Signal Definitions

1.1. Real Wide-Sense Cyclostationary (WSCS) Process $Y(t)$

A real random process $Y(t)$ is defined as Wide-Sense Cyclostationary (WSCS) with cyclic period T if:

- **Mean Function:** $\mu_Y(t) = E[Y(t)]$ is periodic with period T , i.e., $\mu_Y(t) = \mu_Y(t + T)$.
- **Autocorrelation Function (ACF):** $R_{YY}(t, s) = E[Y(t)Y(s)]$ satisfies $R_{YY}(t, s) = R_{YY}(t + T, s + T)$.

For a real process, the relationship between the Autocorrelation Function (ACF), Autocovariance Function (ACVF), and Mean Function is:

- **Autocovariance Function (ACVF):**

$$C_{YY}(t, s) = E[(Y(t) - \mu_Y(t))(Y(s) - \mu_Y(s))]$$

- **ACF and ACVF Relation:**

$$R_{YY}(t, s) = C_{YY}(t, s) + \mu_Y(t)\mu_Y(s)$$

1.2. Complex Wide-Sense Cyclostationary (WSCS) Process $X(t)$

A complex random process $X(t)$ is defined as Wide-Sense Cyclostationary (WSCS) with cyclic period T if:

- **Mean Function:** $\mu_X(t) = E[X(t)]$ is periodic with period T , i.e., $\mu_X(t) = \mu_X(t + T)$.
- **Autocorrelation Function:** $R_{XX}(t, s) = E[X(t)X^*(s)]$ satisfies $R_{XX}(t, s) = R_{XX}(t + T, s + T)$.

1.3. Second-Order Cyclostationary (SOCS) Process

$X(t)$ is an SOCS RP with cyclic period T if it is WSCS and satisfies the following **SOCS additional condition** for its Complementary Autocorrelation Function (CACF):

- **CACF Condition:** $\tilde{R}_{XX}(t, s) = E[X(t)X(s)]$ satisfies $\tilde{R}_{XX}(t, s) = \tilde{R}_{XX}(t + T, s + T)$.

#

1.4. Proper-Complex Random Process

A complex random process $X(t)$ is defined as a **proper-complex random process** (or circular symmetric process) if its Complementary Autocovariance Function (CACVF) is zero for all t and s :

$$\tilde{C}_{XX}(t, s) = E[(X(t) - \mu_X(t))(X(s) - \mu_X(s))] = 0 \quad (1)$$

For a proper-complex process, the CACF is solely determined by the mean function:

$$\tilde{R}_{XX}(t, s) = \tilde{C}_{XX}(t, s) + \mu_X(t)\mu_X(s) = \mu_X(t)\mu_X(s)$$

If a proper-complex process $X(t)$ is WSCS, then $\mu_X(t) = \mu_X(t + T)$. Consequently, $\tilde{R}_{XX}(t + T, s + T) = \mu_X(t + T)\mu_X(s + T) = \mu_X(t)\mu_X(s) = \tilde{R}_{XX}(t, s)$. Thus, for a proper-complex process, the **WSCS condition implies the SOCS condition**.

2. ~~WSCS~~ Analysis of Up-converted Signal $Y(t)$

We analyze the conditions under which the up-converted real signal $Y(t)$, which is related to the complex baseband signal $X(t)$ via the carrier frequency f_c , ~~remains WSCS with period T~~ .

The relationship is defined as:

$$Y(t) = \text{Re} \left\{ X(t) e^{j2\pi f_c t} \right\} = \frac{1}{2} (X(t) e^{j2\pi f_c t} + X^*(t) e^{-j2\pi f_c t}) \quad (2)$$

2.1 Mean Function Analysis of $Y(t)$

The periodic mean function $\mu_X(t)$ can be expanded using a Fourier series:

$$\mu_X(t) = \sum_k c_k e^{j2\pi(k/T)t} \quad (3)$$

The periodicity of the mean function $\mu_Y(t)$ requires the carrier frequency f_c to be synchronized with the cyclic frequency $1/T$.

$$\mu_Y(t) = \text{Re} \left\{ \mu_X(t) e^{j2\pi f_c t} \right\} = \text{Re} \left\{ \sum_k c_k e^{j2\pi(f_c + k/T)t} \right\} \quad (4)$$

For $\mu_Y(t)$ to be periodic with period T (assuming $\mu_X(t) \neq 0$), we require f_c to be an integer multiple of $1/T$:

$$f_c = \frac{k'}{T}, \quad k' \in \mathbb{Z} \quad (5)$$

2.2. Autocorrelation Function Analysis of $Y(t)$ (Detailed Derivation)

For $Y(t)$ to be WSCS, its autocorrelation function $R_{YY}(t, s) = E[Y(t)Y(s)]$ must satisfy $R_{YY}(t, s) = R_{YY}(t + T, s + T)$.

Using the expansion of $Y(t)$ from Eq. (2), $R_{YY}(t, s)$ is derived as follows:

$$R_{YY}(t, s) = E \left[\frac{1}{2} (X(t) e^{j2\pi f_c t} + X^*(t) e^{-j2\pi f_c t}) \frac{1}{2} (X(s) e^{j2\pi f_c s} + X^*(s) e^{-j2\pi f_c s}) \right]$$

$$\begin{aligned}
 R_{YY}(t, s) &= E \left[\frac{1}{2} (X(t)e^{j2\pi f_c t} + X^*(t)e^{-j2\pi f_c t}) \frac{1}{2} (X(s)e^{j2\pi f_c s} + X^*(s)e^{-j2\pi f_c s}) \right] \\
 &= \frac{1}{4} E \left[X(t)X(s)e^{j2\pi f_c(t+s)} + X(t)X^*(s)e^{j2\pi f_c(t-s)} \right. \\
 &\quad \left. + X^*(t)X(s)e^{-j2\pi f_c(t-s)} + X^*(t)X^*(s)e^{-j2\pi f_c(t+s)} \right]
 \end{aligned}$$

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Applying the expectation operator $E[\cdot]$ and using the definitions $R_{XX}(t, s) = E[X(t)X^*(s)]$ and $\tilde{R}_{XX}(t, s) = E[X(t)X(s)]$:

$$\begin{aligned}
 R_{YY}(t, s) &= \frac{1}{4} \left[\tilde{R}_{XX}(t, s)e^{j2\pi f_c(t+s)} + R_{XX}(t, s)e^{j2\pi f_c(t-s)} \right. \\
 &\quad \left. + R_{XX}^*(s, t)e^{-j2\pi f_c(t-s)} + \tilde{R}_{XX}^*(s, t)e^{-j2\pi f_c(t+s)} \right] \quad (6)
 \end{aligned}$$

Using the Hermitian property $R_{XX}^*(s, t) = R_{XX}(t, s)$ and the symmetry of $\tilde{R}_{XX}$ for proper-complex signals, $R_{YY}(t, s)$ can be rewritten as:

$$R_{YY}(t, s) = \frac{1}{2} \operatorname{Re} \left\{ R_{XX}(t, s)e^{j2\pi f_c(t-s)} + \tilde{R}_{XX}(t, s)e^{j2\pi f_c(t+s)} \right\} \quad (7)$$

Q. X is SOCS (2)
WSCS (4)

2.2.1 WSCS Periodicity Condition for $R_{YY}(t, s)$

For $R_{YY}(t, s)$ to be periodic with period T , we require $R_{YY}(t+T, s+T) = R_{YY}(t, s)$.

$R_{XX}(t, s)$ has period T when $X(t)$ is **WSCS**, but $\tilde{R}_{XX}(t, s)$ requires $X(t)$ to be **SOCS** to have period T . This analysis proceeds by assuming $X(t)$ satisfies the **SOCS** condition, so both functions ($R_{XX}, \tilde{R}_{XX}$) are T -periodic. We examine the phase terms in Eq. (7):

- Term 1: Autocorrelation R_{XX}** The phase term is $e^{j2\pi f_c(t-s)}$, $t-s$ is the time difference τ , which is independent of the absolute time t . Since $R_{XX}(t, s) = R_{XX}(t+T, s+T)$, this term ****always**** satisfies the periodicity condition, regardless of f_c .
- Term 2: Complementary Autocorrelation $\tilde{R}_{XX}$** The phase term is $e^{j2\pi f_c(t+s)}$. When shifting $t \rightarrow t+T$ and $s \rightarrow s+T$, the new phase is:

$$e^{j2\pi f_c((t+T)+(s+T))} = e^{j2\pi f_c(t+s)} \cdot e^{j2\pi f_c(2T)}$$

For this term to be periodic with period T , the factor $e^{j2\pi f_c(2T)}$ must equal 1:

$$e^{j2\pi(2f_c)T} = 1$$

This requires the doubled carrier frequency $2f_c$ to be an integer multiple of the cyclic frequency $1/T$:

$$2f_c = \frac{m}{T}, \quad m \in \mathbb{Z} \quad \text{or} \quad f_c = \frac{m}{2T} \quad (8)$$

2.2.2 Overall Conclusion for $Y(t)$ WSCS

- If $X(t)$ is general SOCS with $\mu_X(t) \neq 0$:** Both mean function (μ_Y) and autocorrelation (R_{YY}) must be T -periodic. The dominant condition is $f_c = k'/T$. **Result:** $Y(t)$ is WSCS **only if** $f_c = k'/T$.

2.2.2 Overall Conclusion for $Y(t)$ WSCS

- If $X(t)$ is **general SOCS** with $\mu_X(t) \neq 0$: Both mean function (μ_Y) and autocorrelation (R_{YY}) must be T -periodic. The dominant condition is $f_c = k'/T$. **Result:** $Y(t)$ is WSCS **only** if $f_c = k'/T$.
- If $X(t)$ is **general SOCS** with $\mu_X(t) = 0$: Only the autocorrelation condition is required. Since $\hat{R}_{XX}(t, s) \neq 0$ (for general SOCS), Term 2 must be periodic. **Result:** $Y(t)$ is WSCS **only** if $f_c = m/2T$.
- If $X(t)$ is **Proper-Complex**: If $X(t)$ is Proper-Complex and $\mu_X(t) \neq 0$, the mean condition $f_c = k'/T$ dominates. If $X(t)$ is Proper-Complex and $\mu_X(t) = 0$, then $\hat{R}_{XX}(t, s) = 0$. Term 2 is zero, and R_{YY} is always T -periodic regardless of f_c . **Result:** $Y(t)$ is WSCS **only** if $f_c = k'/T$ (if $\mu_X(t) \neq 0$) or **always** (if $\mu_X(t) = 0$).

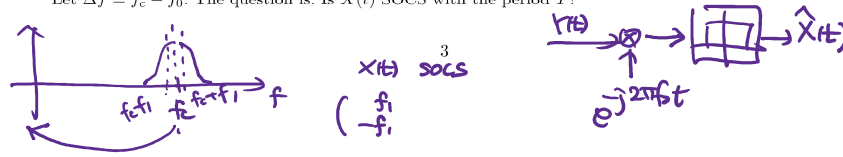
$Y(t)$
WSCS w/ T
 $\Rightarrow$
 $X(t)$
SOCS w/ T

3. Analysis of Down-Conversion with Incorrect Frequency

We analyze the down-converted complex signal $\hat{X}(t)$, defined by:

$$\hat{X}(t) = X(t)e^{j2\pi(f_c - f_0)t} \quad (9)$$

Let $\Delta f = f_c - f_0$. The question is: Is $\hat{X}(t)$ SOCS with the period T ?



3.1. Mean Function (WSCS Condition)

The mean function $\mu_{\hat{X}}(t) = \mu_X(t)e^{j2\pi\Delta f t}$ is periodic with period T (WSCS condition) **only** if:

$$\Delta f = f_c - f_0 = \frac{k}{T}, \quad k \in \mathbb{Z}$$

3.2. Autocorrelation Function (ACF) Analysis

For $\hat{X}(t)$ to be WSCS, its Autocorrelation Function $R_{\hat{X}\hat{X}}(t, s) = E[\hat{X}(t)\hat{X}^*(s)]$ must be periodic with period T .

The ACF is given by:

$$\begin{aligned} R_{\hat{X}\hat{X}}(t, s) &= E[X(t)e^{j2\pi\Delta f t} \cdot X^*(s)e^{-j2\pi\Delta f s}] \\ &= E[X(t)X^*(s)]e^{j2\pi\Delta f(t-s)} \\ &= R_{XX}(t, s)e^{j2\pi\Delta f(t-s)} \end{aligned} \quad (10)$$

We check the periodicity condition $R_{\hat{X}\hat{X}}(t + T, s + T) = R_{\hat{X}\hat{X}}(t, s)$:

$$\begin{aligned} R_{\hat{X}\hat{X}}(t + T, s + T) &= R_{XX}(t + T, s + T)e^{j2\pi\Delta f((t+T)-(s+T))} \\ &= R_{XX}(t, s)e^{j2\pi\Delta f(t-s)} \\ &= R_{\hat{X}\hat{X}}(t, s) \end{aligned}$$

Since $R_{XX}(t, s)$ is T -periodic (due to $X(t)$ being WSCS), and the phase term $e^{j2\pi\Delta f(t-s)}$ depends only on the time difference $\tau = t - s$ (which is independent of the absolute time shift T), the ACF is ****always**** T -periodic, regardless of the frequency offset Δf .

Conclusion (ACF): The autocorrelation condition for $\hat{X}(t)$ to be WSCS is **always** satisfied if $X(t)$ is WSCS.

only on the time difference $\tau = t - s$ (which is independent of the absolute time shift T), the ACF is ****always**** T -periodic, regardless of the frequency offset Δf .

Conclusion (ACF): The autocorrelation condition for $\hat{X}(t)$ to be WSCS is **always** satisfied if $X(t)$ is WSCS.

3.3. Complementary Autocorrelation (SOCS Additional Condition)

For $\hat{X}(t)$ to be SOCS, its CACF $\tilde{R}_{\hat{X}\hat{X}}(t, s) = E[\hat{X}(t)\hat{X}(s)]$ must be periodic with period T .

$$\tilde{R}_{\hat{X}\hat{X}}(t, s) = E[X(t)X(s)]e^{j2\pi\Delta f(t-s)} = \tilde{R}_{XX}(t, s)e^{j2\pi\Delta f(t-s)} \quad (11)$$

The periodicity condition $\tilde{R}_{\hat{X}\hat{X}}(t + T, s + T) = \tilde{R}_{\hat{X}\hat{X}}(t, s)$ requires the phase factor to satisfy:

$$e^{j2\pi\Delta f(2T)} = 1 \implies \boxed{\Delta f} = \frac{k'}{2T}, \quad k' \in \mathbb{Z}$$

Conclusion (SOCS/CACF):

- If $X(t)$ is **general SOCS** with $\mu_X(t) \neq 0$: $\hat{X}(t)$ is SOCS **only if** $\Delta f = k/T$.
- If $X(t)$ is **general SOCS** with $\mu_X(t) = 0$: $\hat{X}(t)$ is SOCS **only if** $\Delta f = k'/2T$.
- If $X(t)$ is **Proper-Complex** with $\mu_X(t) = 0$: $\tilde{R}_{XX}(t, s) = 0$, so $\tilde{R}_{\hat{X}\hat{X}}(t, s) = 0$. The SOCS condition is trivially satisfied. $\hat{X}(t)$ is SOCS **always**.

4. Summary of Cyclostationary Preservation under Frequency Conversion

The preservation of the cyclostationary property depends on precise frequency synchronization.

- **Up-conversion** $X(t) \rightarrow Y(t)$ (WSCS preservation):
 - If $X(t)$ is general SOCS and $\mu_X(t) \neq 0$: $Y(t)$ is WSCS **only if** $f_c = k'/T$.
 - If $X(t)$ is general SOCS and $\mu_X(t) = 0$: $Y(t)$ is WSCS **only if** $f_c = m/2T$.
 - If $X(t)$ is Proper-Complex and $\mu_X(t) = 0$: $Y(t)$ is WSCS **always**.

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- **Down-conversion** $X(t) \rightarrow \hat{X}(t)$ (SOCS preservation):
 - **ACF condition (WSCS/SOCS part 2):** Always satisfied regardless of Δf .
 - **Mean condition (WSCS/SOCS part 1):** If $\mu_X(t) \neq 0$, $\Delta f = k/T$.
 - **CACF condition (SOCS additional):** If $\tilde{R}_{XX} \neq 0$ (general SOCS, $\mu_X(t) = 0$), $\Delta f = k'/2T$.
 - **Proper-Complex case:** If $X(t)$ is Proper-Complex and $\mu_X(t) = 0$, $\hat{X}(t)$ is SOCS **always**.

$\square$ FRESH vec
 p -FRESH vec $\rightarrow$ all assumes f_c IS known! $\& N = \dots$

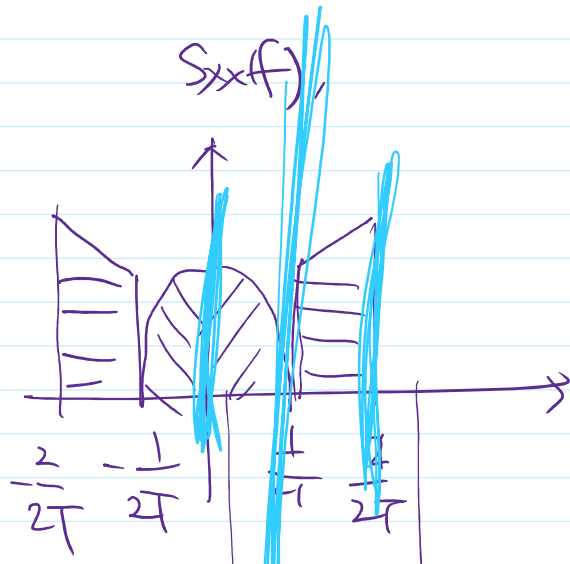
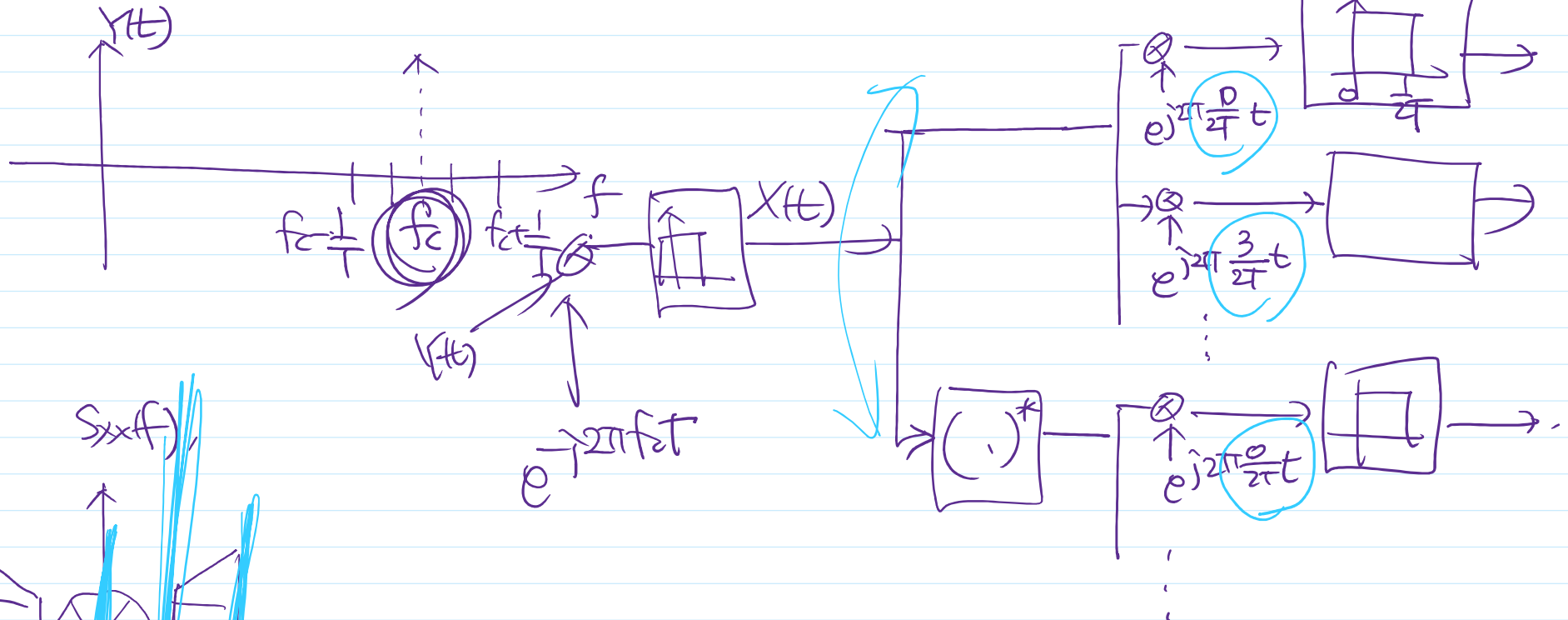
p-FRESH vec } all assumes f_c is known!
 FRESH-p

} No non-zero mean!
 assumed!



$\Rightarrow Y(t) = \text{Re}\{X(t)e^{j2\pi f_c t}\}$'s property is important!

□



2차 2차 1차 2차

복소 확률 과정의 Wigner 분포 함수, SCD/CSCD 분석 및 Up-converted 통과대역 신호의 스펙트럼 관계: SOCS 과정을 예시로

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November 14, 2025

Abstract

본 문서는 T -주기 2차 순환 정상 과정(SOCS)인 복소 확률 과정 $X(t)$ 의 스펙트럼 분석을 다룹니다. 먼저 $X(t)$ 의 2차 통계 함수인 자기 상관 함수(ACF)와 상보 자기 상관 함수(CACF) 커널을 정의합니다. 다음으로, 이 커널들을 이용하여 시간-주파수 영역의 표현인 Wigner 분포 함수(WDF)와 상보 WDF(CWDF)를 소개합니다. WDF 및 CWDF를 중심 시각 t 에 대해 푸리에 변환하여 순환 스펙트럼 밀도 함수(SCD)와 상보 순환 스펙트럼 밀도 함수(CSCD)를 엄밀하게 정의하고, 좌표 변환을 통해 Bi-frequency Spectrum 표현과의 동등한 관계를 확립합니다. 핵심 분석은 반송파 주파수 f_c 로 Up-conversion된 실수 통과대역 신호 $Y(t)$ 에 집중됩니다. $Y(t)$ 의 ACF 및 WDF에 대한 명시적인 공식이 유도됩니다. 가장 중요한 결과로, Up-converted 신호 $Y(t)$ 의 SCD $S_Y(\alpha, f)$ 가 기저대역 $X(t)$ 의 SCD $S_X(\alpha, f)$ 및 CSCD $\tilde{S}_X(\alpha, f)$ 의 선형 조합으로 표현되는 관계식을 제시합니다. 마지막으로, $X(t)$ 가 SOCS 과정일 경우의 스펙트럼 특성을 상세히 설명하며, SCD와 CSCD가 순환 전력 스펙트럼 밀도 함수(CPSD) 및 상보 CPSD(CCPSD) 형으로 구성되는 이산적인 특성을 보여줍니다.

1. Wigner Distribution Function (WDF)와 Complementary WDF (CWDF) 정의

WDF는 ACF 커널을 중심 시각 t 와 시간차 τ 에 대한 함수로 사용하여 정의되며, 상보 WDF는 CACF 커널을 사용하여 정의된다.

1.1. 복소 기저대역 신호 $X(t)$ 의 WDF $W_X(t, f)$

$X(t)$ 의 WDF는 ACF 커널 $R_{XX}(t + \tau/2, t - \tau/2)$ 의 시간차 τ 에 대한 푸리에 변환으로 정의된다:

$$W_X(t, f) = \int_{-\infty}^{\infty} R_{XX}(t + \tau/2, t - \tau/2) e^{-j2\pi f \tau} d\tau$$

$$W_X(t, f) = \int_{-\infty}^{\infty} R_{XX}(t + \tau/2, t - \tau/2) e^{-j2\pi f \tau} d\tau$$

1.2. 상보 Wigner 분포 함수 (CWDF, Complementary Wigner Distribution Function) 정의

$X(t)$ 의 **CWDF** $\tilde{W}_X(t, f)$ 는 CACF 커널 $\tilde{R}_{XX}(t + \tau/2, t - \tau/2)$ 의 시간차 τ 에 대한 푸리에 변환으로 정의된다:

$$\tilde{W}_X(t, f) = \int_{-\infty}^{\infty} \tilde{R}_{XX}(t + \tau/2, t - \tau/2) e^{-j2\pi f \tau} d\tau$$

2. Spectral Correlation Density (SCD) 및 Complementary SCD (CSCD) 정의

SCD 및 CSCD는 각각 WDF $W_X(t, f)$ 와 CWDF $\tilde{W}_X(t, f)$ 를 중심 시각 t 에 대해 Fourier 변환하여 정의된다.

1

2.1. 순환 스펙트럼 밀도 함수 (SCD) 정의

$X(t)$ 의 **SCD** $S_X(\alpha, f)$ 는 다음과 같이 WDF $W_X(t, f)$ 의 시간 t 에 대한 Fourier 변환으로 정의된다:

$$S_X(\alpha, f) = \mathcal{F}_t \{W_X(t, f)\} = \int_{-\infty}^{\infty} W_X(t, f) e^{-j2\pi \alpha t} dt$$

여기서 α 는 **순환 주파수(Cyclic Frequency)**이다.

2.2. 상보 순환 스펙트럼 밀도 함수 (CSCD) 정의

$X(t)$ 의 **CSCD** $\tilde{S}_X(\alpha, f)$ 는 다음과 같이 CWDF $\tilde{W}_X(t, f)$ 의 시간 t 에 대한 Fourier 변환으로 정의된다:

$$\tilde{S}_X(\alpha, f) = \mathcal{F}_t \{\tilde{W}_X(t, f)\} = \int_{-\infty}^{\infty} \tilde{W}_X(t, f) e^{-j2\pi \alpha t} dt$$

3. Bi-frequency Spectrum과 순환 스펙트럼 밀도 함수(SCD/CSCD)의 관계

Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ 및 Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ 는 SCD $S_X(\alpha, f)$ 및 CSCD $\tilde{S}_X(\alpha, f)$ 와 주파수 변수 변환을 통해 연결되는 동등한 스펙트럼 표현이다.

3.1. Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ 와 SCD $S_X(\alpha, f)$ 의 관계

SCD는 Bi-frequency Spectrum을 $\alpha = f_1 - f_2$ 와 $f = (f_1 + f_2)/2$ 의 좌표계로 변환하여 얻을 수 있다:

$$S_X(\alpha, f) = S_{XX}(f + \alpha/2, f - \alpha/2)$$

SCD는 Bi-frequency Spectrum을 $\alpha = f_1 - f_2$ 와 $f = (f_1 + f_2)/2$ 의 좌표계로 변환하여 얻을 수 있다:

$$S_X(\alpha, f) = S_{XX}(f + \alpha/2, f - \alpha/2)$$

Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ 는 ACF $R_{XX}(t, s)$ 의 t 와 s 에 대한 2차원 푸리에 변환으로 정의된다.

3.1.1 SCD 변환의 기하학적 해석 (Geometric Interpretation)

Bi-frequency 평면 (f_1, f_2) 에서 Cyclic Spectrum 평면 (α, f) 로의 좌표 변환은 싹 변환으로 다음과 같이 표현된다:

$$\begin{bmatrix} \alpha \\ f \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

기하학적으로 이 변환은 **회전(Rotation)**과 **비균일 스케일링(Non-uniform Scaling)**의 조합으로 해석된다:

1. **회전 (Rotation):** (f_1, f_2) 평면은 효과적으로 45° **반시계 방향**으로 회전된다. α 축은 주 대각선 ($f_1 = f_2$)에 수직이 되고, f 축은 주 대각선과 평행하게 된다.
2. **스케일링 (Scaling):** 45° 시계 방향 회전 후의 스케일링 및 플립(Flip)으로 분해하면 다음과 같다:

$$\begin{bmatrix} f \\ \alpha \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

이는 회전 후 그래프가 f 방향(평균 주파수)으로는 $1/\sqrt{2}$ 배, α 방향(순환 주파수)으로는 $-\sqrt{2}$ 배 스케일링된다는 것을 의미하며, 이는 α 축에 대한 반전(Flip)과 $\sqrt{2}$ 배 스케일링을 포함한다.

3.2. Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ 와 CSCD $\tilde{S}_X(\alpha, f)$ 의 관계

CSCD는 Complementary Bi-frequency Spectrum을 다음과 같이 변환하여 얻을 수 있다:

$$\tilde{S}_X(\alpha, f) = \tilde{S}_{XX}(f + \alpha/2, -f + \alpha/2)$$

Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ 는 CACF $\tilde{R}_{XX}(t, s)$ 의 t 와 s 에 대한 2차원 푸리에 변환으로 정의된다.

2

3.2.1 CSCD 변환의 기하학적 해석 (Geometric Interpretation)

Bi-frequency 평면 (f_1, f_2) 에서 Cyclic Spectrum 평면 (α, f) 로의 변환 행렬은 다음과 같이 정의된다:

$$\begin{bmatrix} \alpha \\ f \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

이 행렬은 표준 SCD 변환 행렬과 **반사(Reflection)** 행렬의 곱으로 분해될 수 있다:

$$\begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

기하학적으로 CSCD 변환은 $\tilde{S}_{XX}(f_1, f_2)$ 표면에 다음과 같은 순서의 연산을 적용하여 얻어진다:

1. **반사 (Reflection):** Complementary Bi-frequency Spectrum은 먼저 f_2 축에 대해 반사된다 (즉, $f_2 \rightarrow -f_2$). 이는 반사 행렬 $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ 로 처리하는 것과 같다.

기하학적으로 CSCD 변환은 $\tilde{S}_{XX}(f_1, f_2)$ 표면에 다음과 같은 순서의 연산을 적용하여 얻어진다:

1. **반사 (Reflection):** Complementary Bi-frequency Spectrum은 먼저 f_2 축에 대해 반사된다 (즉, $f_2 \rightarrow -f_2$). 이는 반사 행렬 $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ 을 적용하는 것과 같다.
2. **SCD 변환:** 반사된 스펙트럼에 표준 SCD 기하학적 변환 (45° 회전 및 비균일 스케일링/플립)이 적용된다.

4. Up-converted 신호 $Y(t)$ 의 ACF, WDF, 및 SCD 표현

$Y(t)$ 는 $X(t)$ 를 반송파 주파수 f_c 로 Up-conversion한 실수 통과대역 신호이다:

$$Y(t) = \text{Re} \{ X(t) e^{j2\pi f_c t} \} = \frac{1}{2} (X(t) e^{j2\pi f_c t} + X^*(t) e^{-j2\pi f_c t})$$

4.1. $Y(t)$ 의 Autocorrelation Function (ACF)

Zero-Mean 가정 하에 $Y(t)$ 의 ACF $R_{YY}(t, s) = E[Y(t)Y(s)]$ 는 기저대역 ACF R_{XX} 와 CACF $\tilde{R}_{XX}$ 로 표현된다:

$$R_{YY}(t + \tau/2, t - \tau/2) = \frac{1}{2} \text{Re} \left\{ R_{XX}(t + \tau/2, t - \tau/2) e^{j2\pi f_c \tau} + \tilde{R}_{XX}(t + \tau/2, t - \tau/2) e^{j4\pi f_c t} \right\}$$

4.2. $W_Y(t, f)$ 와 $W_X, \tilde{W}_X$ 의 관계

$Y(t)$ 의 WDF $W_Y(t, f)$ 는 $X(t)$ 의 WDF $W_X(t, f)$ 와 CWDF $\tilde{W}_X(t, f)$ 의 항으로 다음과 같이 표현된다:

$$W_Y(t, f) = \frac{1}{4} [W_X(t, f - f_c) + W_X^*(t, f + f_c)] + \frac{1}{4} [\tilde{W}_X(t, f) e^{j4\pi f_c t} + \tilde{W}_X^*(t, -f) e^{-j4\pi f_c t}]$$

4.3. $Y(t)$ 의 SCD 표현 (S_X 와 $\tilde{S}_X$ 의 관계)

$W_Y(t, f)$ 관계식에 시간 t 에 대한 Fourier 변환을 취하면, $Y(t)$ 의 SCD는 $X(t)$ 의 SCD 및 CSCD를 사용하여 다음과 같이 표현된다:

$$S_Y(\alpha, f) = \frac{1}{4} [S_X(\alpha, f - f_c) + S_X^*(-\alpha, f + f_c)] + \frac{1}{4} [\tilde{S}_X(\alpha - 2f_c, f) + \tilde{S}_X^*(-\alpha + 2f_c, -f)]$$

5. T -주기 SOCS 복소 확률 과정 $X(t)$ 의 특성 및 $Y(t)$ 의 특성

$X(t)$ 는 T -주기 SOCS 과정으로 정의되며, 그 2차 통계 함수는 T 주기를 만족한다. 이 특성은 Up-converted 신호 $Y(t)$ 의 특성을 결정한다.

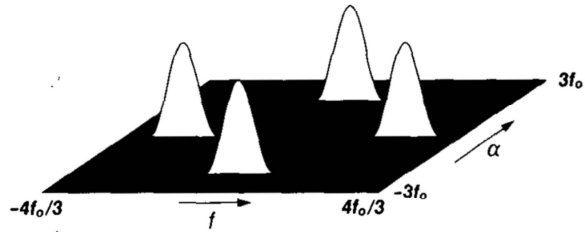


Fig. 1. Spectral correlation magnitude surface for an amplitude-modulated sine wave with carrier frequency f_0 .

Figure 1: Fig.1 in [1]

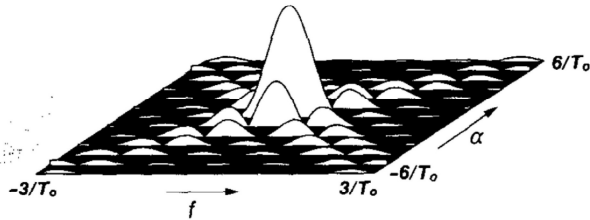


Fig. 2. Spectral correlation magnitude surface for a pulse-amplitude modulated pulse train with pulse rate $1/T_0$.

Figure 2: Fig.2 in [1]

5.1. $X(t)$ 의 2차 통계 및 스펙트럼 특성

5.1.1 Autocorrelation Function (ACF) 및 Complementary Autocorrelation Function (CACF)

$X(t)$ 의 ACF $R_{XX}(t, s) = E[X(t)X^*(s)]$ 와 CACF $\tilde{R}_{XX}(t, s) = E[X(t)X(s)]$ 는 모두 T 주기를 만족한다:

$$R_{XX}(t, s) = R_{XX}(t + T, s + T), \quad \tilde{R}_{XX}(t, s) = \tilde{R}_{XX}(t + T, s + T)$$

ACF 및 CACF 커널 $R_{XX}(t + \tau/2, t - \tau/2)$ 와 $\tilde{R}_{XX}(t + \tau/2, t - \tau/2)$ 는 중심 시각 t 에 대해 Joint Periodic (결합 주기적) 특성을 가진다.

5.1.2 ACF 및 CACF 커널의 Fourier 급수 전개

T -주기 SOCS 과정의 커널은 중심 시각 t 에 대해 주기 T 를 가지므로, 다음과 같이 Fourier 급수로 전개할 수 있다.

ACF 커널:

$$R_{XX}(t + \tau/2, t - \tau/2) = \sum_{k=-\infty}^{\infty} R_X^{(k)}(\tau) e^{j2\pi \frac{k}{T} t}$$

여기서 $R_X^{(k)}(\tau)$ 는 k 번째 순환 자기 상관 함수(CAF, Cyclic Autocorrelation Function)의 Fourier 계수이며, 이의 푸리에 변환이 순환 전력 스펙트럼 밀도 함수(CPSD, Cyclic Power Spectral Density) $S_X^{(k)}(f)$ 이다.

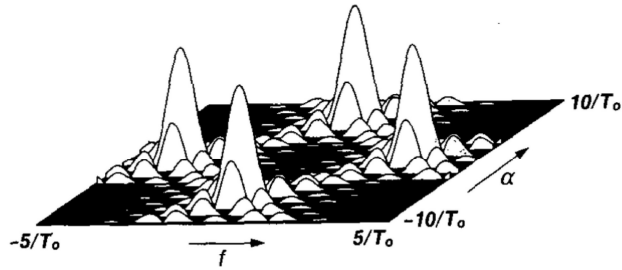


Fig. 3. Spectral correlation magnitude surface for amplitude-shift keying (and binary phase-shift keying) with carrier frequency $= 3.3/T_0$.

Figure 3: Fig.3 in [2]

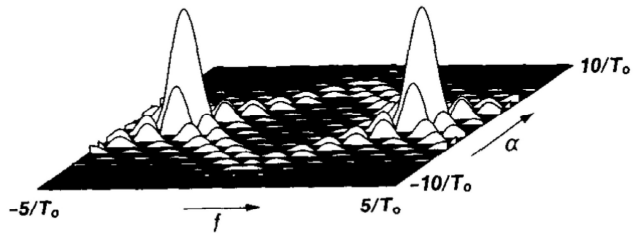


Fig. 4. Spectral correlation magnitude surface for balanced amplitude-phase-shift keying (and quaternary phase-shift keying) with carrier frequency $= 3.3/T_0$.

Figure 4: Fig. 4 in [2]

CACF 커널:

$$\tilde{R}_{XX}(t + \tau/2, t - \tau/2) = \sum_{k=-\infty}^{\infty} \tilde{R}_X^{(k)}(\tau) e^{j2\pi \frac{k}{T_0} t}$$

여기서 $\tilde{R}_X^{(k)}(\tau)$ 는 k 번째 상보 순환 자기 상관 함수(CCACF, Complementary Cyclic Autocorrelation Function)의 Fourier 계수이며, 이의 푸리에 변환이 상보 순환 전력 스펙트럼 밀도 함수(CCPSD, Complementary Cyclic Power Spectral Density) $\tilde{S}_X^{(k)}(f)$ 이다.

5.2. SOCS 기반 $W_Y(t, f)$ 유도 및 순환 스펙트럼 표현

$f_c = m_0/(2T)$ 조건이 충족되면, $Y(t)$ 는 T -주기 WSCS 과정을 만족하며, $Y(t)$ 의 WDF $W_Y(t, f)$ 는 다음과 같이 $\alpha_l = l/T$ 주파수를 갖는 Fourier 급수로 표현된다:

5.2. SOCS 기반 $W_Y(t, f)$ 유도 및 순환 스펙트럼 표현

$f_c = m_0/(2T)$ 조건이 충족되면, $Y(t)$ 는 T -주기 WSCS 과정을 만족하며, $Y(t)$ 의 WDF $W_Y(t, f)$ 는 다음과 같이 $\alpha_l = l/T$ 주파수를 갖는 Fourier 급수로 표현된다:

$$W_Y(t, f) = \sum_{l=-\infty}^{\infty} S_Y^{(l)}(f) e^{j2\pi \frac{l}{T} t}$$

5

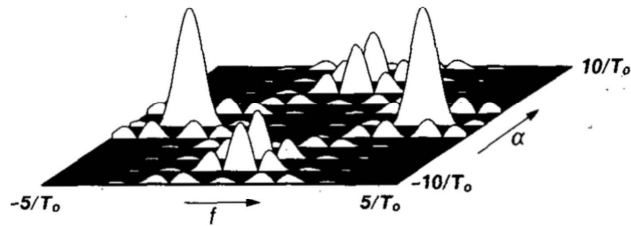


Fig. 5. Spectral correlation magnitude surface for staggered quaternary-phase-shift keying with carrier frequency $= 3.3/T_0$.

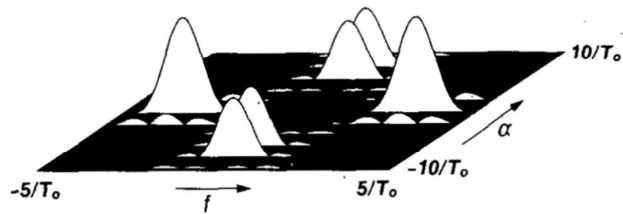


Fig. 6. Spectral correlation magnitude surface for minimum-shift keying with carrier frequency $= 3.3/T_0$.





Fig. 7. Spectral correlation magnitude surface for Manchester-encoded binary phase-shift keying with carrier frequency = $3.3/T_0$.

Figure 5: Fig. 4 in [2]

여기서 l 번째 순환 스펙트럼 밀도 함수 $S_Y^{(l)}(f)$ 는 기저대역 신호의 CPSD $S_X^{(k)}(f)$ 및 CCPSD $\tilde{S}_X^{(k)}(f)$ 의 합으로 구성되며, 순환 주파수 $\alpha = l/T$ 에서 이산적으로 존재한다:

$$S_Y^{(l)}(f) = \frac{1}{4} \left[S_X^{(l)}(f - f_c) + (S_X^{(-l)}(f + f_c))^* + \tilde{S}_X^{(l-m_0)}(f) + (\tilde{S}_X^{(-l-m_0)}(-f))^* \right]$$

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5.3. SCD/CSCD와 CPSD/CCPSD의 관계 및 PSD 도출

$X(t)$ 가 T -주기 SOCS 과정일 경우, SCD와 CSCD는 이산적인 순환 주파수 $\alpha = k/T$ 에서만 존재하며, 각각 CPSD $S_X^{(k)}(f)$ 및 CCPSD $\tilde{S}_X^{(k)}(f)$ 에 의해 결정된다:

$$S_X(\alpha, f) = \sum_{k=-\infty}^{\infty} S_X^{(k)}(f) \delta\left(\alpha - \frac{k}{T}\right)$$

$$\tilde{S}_X(\alpha, f) = \sum_{k=-\infty}^{\infty} \tilde{S}_X^{(k)}(f) \delta\left(\alpha - \frac{k}{T}\right)$$

References

- [1] W. A. Gardner, "Spectral correlation of modulated signals: part I - analog modulation," *IEEE Trans. Commun.*, vol. 35, no. 6, pp. 584-594, June 1987. <https://ieeexplore.ieee.org/document/1096820>
- [2] W. A. Gardner, "Spectral correlation of modulated signals: part II - digital modulation," *IEEE Trans. Commun.*, vol. 35, no. 6, pp. 595-601, June 1987. <https://ieeexplore.ieee.org/document/1096816>

**The Wigner Distribution Function, SCD/CSCD
Analysis, and Spectral Relationship of Up-converted
Passband Signals for Complex Stochastic Processes:**

An SOCS Example

Gemini AI Assistant and Joon Ho Cho

November 14, 2025

$$S_{xx}(f) = \int_{-\infty}^{\infty} W_x(t, f) dt$$

$$x(t) \in L^2$$

$$\mu_x(t) = \mu_x(t+T), \forall t$$

November 14, 2025

Abstract

This document investigates the spectral analysis of a complex stochastic process $X(t)$ which is a T -periodic **Second-Order Cyclostationary Process (SOCS)**. The analysis begins by defining the process's second-order statistics, namely the **Autocorrelation Function (ACF)** and the **Complementary Autocorrelation Function (CACF)** kernels. These kernels are then used to introduce their time-frequency domain counterparts: the **Wigner Distribution Function (WDF)** and the **Complementary WDF (CWDF)**. By performing a Fourier transform of the WDF and CWDF with respect to the center time t , the document rigorously defines the **Spectral Correlation Density (SCD)** and the **Complementary Spectral Correlation Density (CSCD)**, and establishes their equivalence through coordinate transformation with the Bi-frequency Spectrum representations. The core analysis focuses on the real passband signal $Y(t)$, formed by up-converting $X(t)$ at a carrier frequency f_c . Explicit formulas for the ACF and WDF of $Y(t)$ are derived. Most importantly, the paper presents the relationship where the SCD $S_Y(\alpha, f)$ of the up-converted signal $Y(t)$ is expressed as a linear combination of the baseband SCD $S_X(\alpha, f)$ and CSCD $\tilde{S}_X(\alpha, f)$. Finally, the spectral properties are detailed for the specific case where $X(t)$ is an SOCS process, showing the discrete nature of the SCD and CSCD in terms of Cyclic Power Spectral Density (CPSD) and Complementary CPSD (CCPSD).

1. Definition of Wigner Distribution Function (WDF) and Complementary WDF (CWDF)

The WDF is defined using the ACF kernel as a function of the center time t and time difference τ . The Complementary WDF (CWDF) is defined using the CACF kernel.

1.1. WDF $W_X(t, f)$ of the Complex Baseband Signal $X(t)$

The WDF of $X(t)$ is defined as the Fourier transform of the ACF kernel $R_{XX}(t + \tau/2, t - \tau/2)$ with respect to the time difference τ :

$$W_X(t, f) = \int_{-\infty}^{\infty} R_{XX}(t + \tau/2, t - \tau/2) e^{-j2\pi f \tau} d\tau$$

1.2. Definition of Complementary Wigner Distribution Function (CWDF)

The **CWDF** $\tilde{W}_X(t, f)$ of $X(t)$ is defined as the Fourier transform of the CACF kernel $\tilde{R}_{XX}(t + \tau/2, t - \tau/2)$ with respect to the time difference τ :

$$\tilde{W}_X(t, f) = \int_{-\infty}^{\infty} \tilde{R}_{XX}(t + \tau/2, t - \tau/2) e^{-j2\pi f \tau} d\tau$$

2. Definition of Spectral Correlation Density (SCD) and Com-

2. Definition of Spectral Correlation Density (SCD) and Complementary SCD (CSCD)

The SCD and CSCD are defined by taking the Fourier transform of the WDF $W_X(t, f)$ and CWDF $\tilde{W}_X(t, f)$, respectively, with respect to the center time t .

2.1. Definition of Spectral Correlation Density (SCD)

The **SCD** $S_X(\alpha, f)$ of $X(t)$ is defined as the Fourier transform of the WDF $W_X(t, f)$ with respect to the time t :

$$S_X(\alpha, f) = \mathcal{F}_t \{W_X(t, f)\} = \int_{-\infty}^{\infty} W_X(t, f) e^{-j2\pi\alpha t} dt$$

where α is the **Cyclic Frequency**.

2.2. Definition of Complementary Spectral Correlation Density (CSCD)

The **CSCD** $\tilde{S}_X(\alpha, f)$ of $X(t)$ is defined as the Fourier transform of the CWDF $\tilde{W}_X(t, f)$ with respect to the time t :

$$\tilde{S}_X(\alpha, f) = \mathcal{F}_t \{\tilde{W}_X(t, f)\} = \int_{-\infty}^{\infty} \tilde{W}_X(t, f) e^{-j2\pi\alpha t} dt$$

3. Relationship between Bi-frequency Spectrum and SCD/CSCD

The Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ and the Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ are equivalent spectral representations linked to $S_X(\alpha, f)$ and $\tilde{S}_X(\alpha, f)$ through a frequency variable transformation.

3.1. Relationship between Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ and SCD $S_X(\alpha, f)$

The SCD can be obtained by transforming the Bi-frequency Spectrum to the coordinate system of $\alpha = f_1 - f_2$ and $f = (f_1 + f_2)/2$:

$$S_X(\alpha, f) = S_{XX}(f + \alpha/2, f - \alpha/2) \quad S_{XX}(f_1, f_2)$$

The Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ is defined as the 2D Fourier transform of the ACF $R_{XX}(t, s)$ with respect to t and s .

3.1.1 Geometric Interpretation of the SCD Transformation

The coordinate transformation from the Bi-frequency plane (f_1, f_2) to the Cyclic Spectrum plane (α, f) is described by the linear mapping:

$$\begin{bmatrix} \alpha \\ f \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Geometrically, this transformation represents a combination of rotation and non-uniform scaling:

1. **Rotation:** The (f_1, f_2) plane is effectively rotated by 45° counter-clockwise. The α axis becomes perpendicular to the main diagonal ($f_1 = f_2$), and the f axis becomes parallel to the main diagonal.
2. **Scaling:** When decomposed into a 45° clockwise rotation followed by a scaling and reflection:

$$\begin{bmatrix} f \\ \alpha \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

This implies that after rotation, the graph is scaled by $1/\sqrt{2}$ along the f direction (mean frequency) and by $-\sqrt{2}$ along the α direction (cyclic frequency), which includes a reflection (flip) and a scaling by $\sqrt{2}$.

$$\begin{aligned} S_X(\alpha, f) &\equiv S_{XX}(f_1, f_2) \\ \tilde{S}_X(\alpha, f) &\equiv \tilde{S}_{XX}(f_1, f_2) \end{aligned}$$

3.2. Relationship between Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ and CSCD $\tilde{S}_X(\alpha, f)$

The CSCD can be obtained by transforming the Complementary Bi-frequency Spectrum:

$$\tilde{S}_X(\alpha, f) = \tilde{S}_{XX}(f + \alpha/2, -f + \alpha/2)$$

The Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ is defined as the 2D Fourier transform of the CACF $\tilde{R}_{XX}(t, s)$ with respect to t and s .

3.2.1 Geometric Interpretation of the CSCD Transformation

The transformation from the Bi-frequency plane (f_1, f_2) to the Cyclic Spectrum plane (α, f) is defined by the linear mapping:

$$\begin{bmatrix} \alpha \\ f \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

This matrix can be decomposed as a reflection followed by the standard SCD transformation matrix:

$$\begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Geometrically, the CSCD transformation is achieved by the following sequence of operations on the $\tilde{S}_{XX}(f_1, f_2)$ surface:

1. **Reflection (Flip):** The Complementary Bi-frequency Spectrum is first reflected across the f_1 axis (i.e., $f_2 \rightarrow -f_2$). This is equivalent to applying the reflection matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
2. **SCD Transformation:** The standard SCD geometric transformation (rotation by 45° and non-uniform scaling/flip) is then applied to the reflected spectrum.

4. ACF, WDF, and SCD Representation of the Up-converted Signal $Y(t)$

$Y(t)$ is the real passband signal obtained by up-converting $X(t)$ to the carrier frequency f_c :

$$Y(t) = \text{Re} \{ X(t) e^{j2\pi f_c t} \} = \frac{1}{2} (X(t) e^{j2\pi f_c t} + X^*(t) e^{-j2\pi f_c t})$$

4.1. Autocorrelation Function (ACF) of $Y(t)$

Under the zero-mean assumption, the ACF of $Y(t)$, $R_{YY}(t, s) = E[Y(t)Y(s)]$, is expressed in terms of the baseband ACF R_{XX} and CACF $\tilde{R}_{XX}$:

$$R_{YY}(t + \tau/2, t - \tau/2) = \frac{1}{2} \text{Re} \left\{ R_{XX}(t + \tau/2, t - \tau/2) e^{j2\pi f_c \tau} + \tilde{R}_{XX}(t + \tau/2, t - \tau/2) e^{j4\pi f_c t} \right\}$$

4.2. Relationship between $W_Y(t, f)$ and $W_X, \tilde{W}_X$

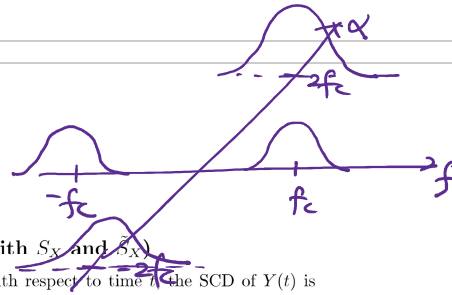
The WDF of $Y(t)$, $W_Y(t, f)$, is expressed in terms of the WDF $W_X(t, f)$ and CWDF $\tilde{W}_X(t, f)$ of $X(t)$ as:

$$W_Y(t, f) = \frac{1}{4} [W_X(t, f - f_c) + W_X^*(t, f + f_c)] + \frac{1}{4} [\tilde{W}_X(t, f) e^{j4\pi f_c t} + \tilde{W}_X^*(t, -f) e^{-j4\pi f_c t}]$$

$$W_Y(t, f) = \frac{1}{4} [W_X(t, f - f_c) + W_X^*(t, f + f_c)] + \frac{1}{4} [\tilde{W}_X(t, f) e^{j4\pi f_c t} + \tilde{W}_X^*(t, -f) e^{-j4\pi f_c t}]$$

α cycle frequency

$$S_Y(\alpha, f) = \frac{1}{4} [S_X(\alpha, f - f_c) + \dots]$$



4.3. SCD Expression of $Y(t)$ (Relationship with S_X and $\tilde{S}_X$)

Taking the Fourier transform of the $W_Y(t, f)$ relationship with respect to time t , the SCD of $Y(t)$ is expressed using the SCD and CSD of $X(t)$:

$$S_Y(\alpha, f) = \frac{1}{4} [S_X(\alpha, f - f_c) + S_X^*(-\alpha, f + f_c)] + \frac{1}{4} [\tilde{S}_X(\alpha - 2f_c, f) + \tilde{S}_X^*(-\alpha + 2f_c, -f)]$$

$S_X(\alpha, f) = \tilde{S}_X(\alpha, f)$

Due to impropriety of $X(t)$

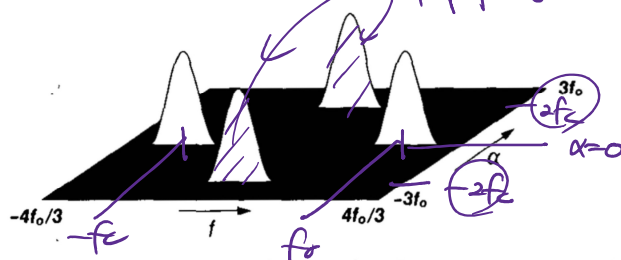


Fig. 1. Spectral correlation magnitude surface for an amplitude-modulated sine wave with carrier frequency f_0 .

Figure 1: Fig.1 in [1]

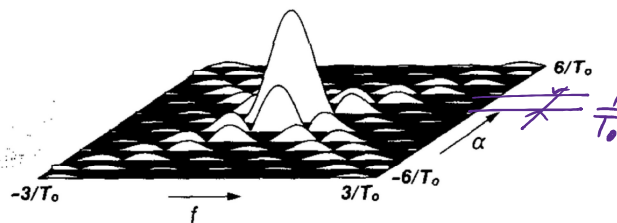


Fig. 2. Spectral correlation magnitude surface for a pulse-amplitude

$X(t)$ real baseband.
 $f_c = 0$



Fig. 2. Spectral correlation magnitude surface for a pulse-amplitude modulated pulse train with pulse rate $1/T_0$.

Figure 2: Fig.2 in [1]

5. Characteristics of the T -periodic SOCS Complex Stochastic Process $X(t)$ and Characteristics of $Y(t)$

$X(t)$ is defined as a T -periodic SOCS process, where its second-order statistics are T -periodic. This characteristic determines the properties of the up-converted signal $Y(t)$.

4

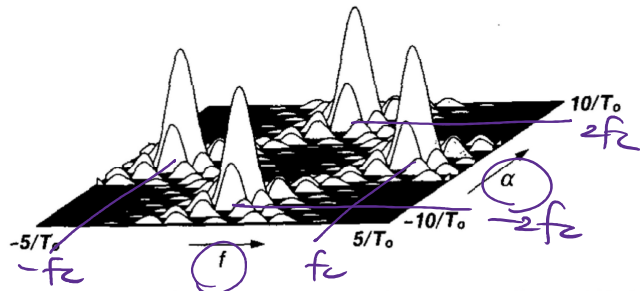
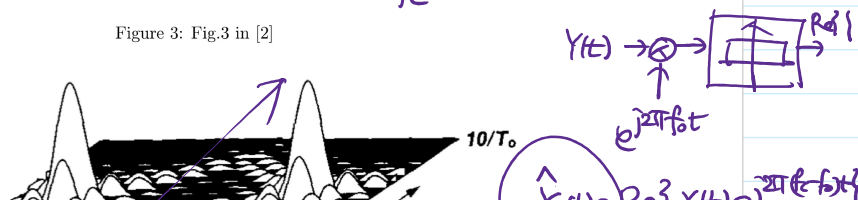


Fig. 3. Spectral correlation magnitude surface for amplitude-shift keying (and binary phase-shift keying) with carrier frequency $f_c = 3.3/T_0$.

Figure 3: Fig.3 in [2]



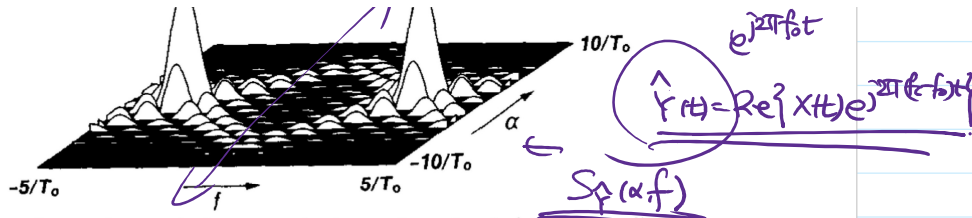


Fig. 4. Spectral correlation magnitude surface for balanced amplitude-phase-shift keying (and quaternary phase-shift keying) with carrier frequency $\approx 3.3/T_0$.

Figure 4: Fig. 4 in [2]

5.1. $X(t)$'s Second-Order Statistics and Spectral Characteristics

5.1.1 Autocorrelation Function (ACF) and Complementary Autocorrelation Function (CACF)

The ACF $R_{XX}(t, s) = E[X(t)X^*(s)]$ and CACF $\tilde{R}_{XX}(t, s) = E[X(t)X(s)]$ of $X(t)$ are both T -periodic:

$$R_{XX}(t, s) = R_{XX}(t + T, s + T), \quad \tilde{R}_{XX}(t, s) = \tilde{R}_{XX}(t + T, s + T)$$

The ACF and CACF kernels, $R_{XX}(t + \tau/2, t - \tau/2)$ and $\tilde{R}_{XX}(t + \tau/2, t - \tau/2)$, exhibit a **Jointly Periodic** characteristic with respect to the center time t .

5.1.2 Fourier Series Expansion of ACF and CACF Kernels

Since the kernel of the T -periodic SOCS process has a period T with respect to the center time t , it can be expanded into a Fourier series as follows.

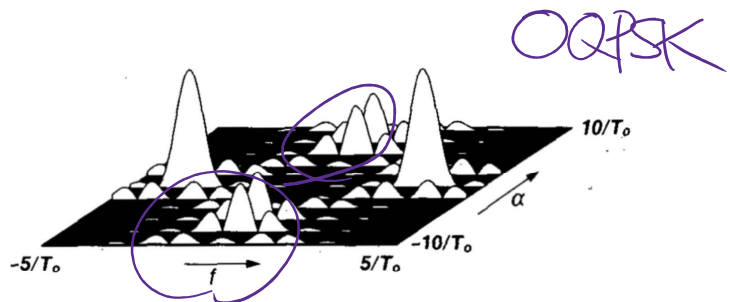


Fig. 5. Spectral correlation magnitude surface for staggered quaternary-phase-shift keying with carrier frequency $\approx 3.3/T_0$.

Fig. 5. Spectral correlation magnitude surface for staggered quaternary-phase-shift keying with carrier frequency = $3.3/T_0$.

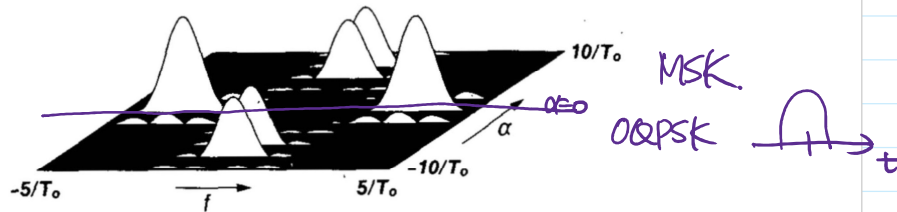


Fig. 6. Spectral correlation magnitude surface for minimum-shift keying with carrier frequency = $3.3/T_0$.

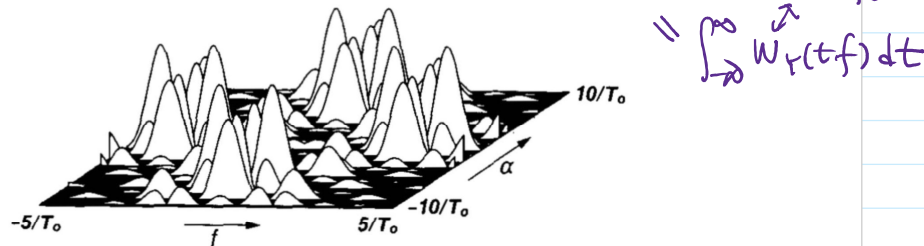


Fig. 7. Spectral correlation magnitude surface for Manchester-encoded binary phase-shift keying with carrier frequency = $3.3/T_0$.

Figure 5: Fig. 4 in [2]

ACF Kernel:

$$R_{XX}(t + \tau/2, t - \tau/2) = \sum_{k=-\infty}^{\infty} R_X^{(k)}(\tau) e^{j2\pi k f t}$$

where $R_X^{(k)}(\tau)$ are the Fourier coefficients for the k -th **Cyclic Autocorrelation Function (CAF)**, and

$$\text{CAF} \rightarrow R_X^{(k)}(\tau) \triangleq \frac{1}{\tau} \int_{-\infty}^{\infty} R_{XX}(t + \frac{\tau}{2}, t - \frac{\tau}{2}) e^{j2\pi k f t} dt$$

$$\text{CPSD} \rightarrow S_X^{(k)}(f) \xleftrightarrow{FT} R_X^{(k)}(\tau)$$

their Fourier transform is the **Cyclic Power Spectral Density (CPSD)** $S_X^{(k)}(f)$.
CAF Kernel:

$$W_X(t, f) = \sum_k S_X^{(k)}(f) e^{j2\pi k f t}$$

their Fourier transform is the **Cyclic Power Spectral Density (CPSD)** $S_X^{(k)}(f)$.

CACF Kernel:

$$\tilde{R}_{XX}(t + \tau/2, t - \tau/2) = \sum_{k=-\infty}^{\infty} \tilde{R}_X^{(k)}(\tau) e^{j2\pi \frac{k}{T} t}$$

where $\tilde{R}_X^{(k)}(\tau)$ are the Fourier coefficients for the k -th **Complementary Cyclic Autocorrelation Function (CCACF)**, and their Fourier transform is the **Complementary Cyclic Power Spectral Density (CCPSD)** $\tilde{S}_X^{(k)}(f)$.

5.2. Derivation of $W_Y(t, f)$ based on SOCS and Cyclic Spectrum Expression

If the condition $f_c = m_0/(2T)$ is satisfied, $Y(t)$ satisfies the T -periodic WSCS process, and its WDF $W_Y(t, f)$ is expressed as a Fourier series with frequencies $\alpha_k = k/T$.

$$W_Y(t, f) = \sum_{l=-\infty}^{\infty} S_Y^{(l)}(f) e^{j2\pi \frac{l}{T} t}$$

Here, the l -th cyclic spectral density function $S_Y^{(l)}(f)$ is composed of the sum of the baseband signal's CPSD $S_X^{(k)}(f)$ and CCPSD $\tilde{S}_X^{(k)}(f)$, existing discretely at the cyclic frequency $\alpha = k/T$:

$$S_Y^{(l)}(f) = \frac{1}{4} \left[S_X^{(l)}(f - f_c) + (S_X^{(-l)}(f + f_c))^* + \tilde{S}_X^{(l-m_0)}(f) + \tilde{S}_X^{(-l-m_0)}(-f) \right]$$

5.3. Relationship between SCD/CSCD and CPSD/CCPSD

When $X(t)$ is a T -periodic SOCS process, the SCD and CSCD exist only at discrete cyclic frequencies $\alpha = k/T$, and are determined by the CPSD $S_X^{(k)}(f)$ and CCPSD $\tilde{S}_X^{(k)}(f)$, respectively:

$$S_X(\alpha, f) = \sum_{k=-\infty}^{\infty} S_X^{(k)}(f) \delta\left(\alpha - \frac{k}{T}\right)$$

$$\tilde{S}_X(\alpha, f) = \sum_{k=-\infty}^{\infty} \tilde{S}_X^{(k)}(f) \delta\left(\alpha - \frac{k}{T}\right)$$

References

- [1] W. A. Gardner, "Spectral correlation of modulated signals: part I - analog modulation," *IEEE Trans. Commun.*, vol. 35, no. 6, pp. 584-594, June 1987. <https://ieeexplore.ieee.org/document/1096820>
- [2] W. A. Gardner, "Spectral correlation of modulated signals: part II - digital modulation," *IEEE Trans. Commun.*, vol. 35, no. 6, pp. 595-601, June 1987. <https://ieeexplore.ieee.org/document/1096816>

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