EECE 695E: Spectrally-Correlated Random Processes

Take-Home Exam # 3 (Nov. 2025)

Student #: ________________________________

Name: ________________________________

Upload a pdf file that contains all your solutions and answers to the PLMS.

Problem 1:	____ / 10
Problem 2:	____ / 10
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Total:	____ / 60

Problem 1. (10 points) Suppose that $\hat{X}(t)$ is a real-valued zero-mean WSS random process with autocorrelation function $R_{\hat{X}\hat{X}}(\tau) \triangleq E[X(t)X(t-\tau)]$. This random process is up-converted to a real passband random process as

$$\hat{Y}(t) = \Re\{\hat{X}(t)e^{\mathrm{j}2\pi f_c t}\},$$

BPSK.

where $f_c \gg B$ with being B the bandwidth of $\hat{X}(t)$. Answer the following questions.

$R_{\hat{Y}\hat{Y}}(t,s) = \mathrm{Re}\{\cdots e^{j2\pi f_c(t+s)}\}$

(a) (2 points) Find the mean path of $\hat{Y}(t)$.

(b) (3 points) Find the autocorrelation function $R_{\hat{Y}\hat{Y}}(t,s)$ of $\hat{Y}(t)$ in terms of $R_{\hat{X}\hat{X}}(\tau)$.

$= R_{\hat{Y}\hat{Y}}(t + \frac{1}{2f_c}, s + \frac{1}{2f_c})$

(c) (5 points) Show that $\hat{Y}(t)$ is a WSCS random process with cycle period $1/(2f_c)$.

Problem 2. (10 points) In Problem 1, let $S_{\hat{X}\hat{X}}(f)$ be the Fourier transform of $R_{\hat{X}\hat{X}}(\tau)$ with respect to τ. Answer the following questions.

(a) (4 points) Show that the Loève bi-frequency spectrum $S_{\hat{X}\hat{X}}(f_1, f_2)$ can be written as

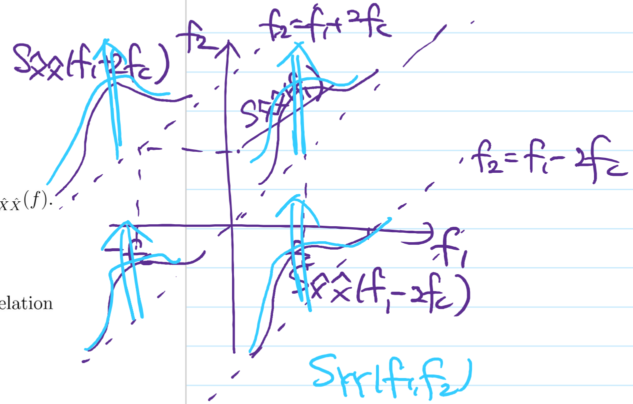
$$S_{\hat{X}\hat{X}}(f_1, f_2) = S_{\hat{X}\hat{X}}(f_1)\delta(f_1 - f_2)$$

and that the complementary bi-frequency spectrum $\tilde{S}_{\hat{X}\hat{X}}(f_1, f_2)$ can be written as

$$\tilde{S}_{\hat{X}\hat{X}}(f_1, f_2) = S_{\hat{X}\hat{X}}(f_1)\delta(f_1 + f_2)$$

or, equivalently,

$$\tilde{S}_{\hat{X}\hat{X}}(f_1, -f_2) = S_{\hat{X}\hat{X}}(f_1)\delta(f_1 - f_2).$$



(b) (5 points) Derive the Loève bi-frequency spectrum $\boxed{S_{\hat{Y}\hat{Y}}(f_1, f_2)}$ in terms of $S_{\hat{X}\hat{X}}(f)$.

(c) (1 point) Sketch $|S_{\hat{Y}\hat{Y}}(f_1, f_2)|$ and confirm that there can be non-zero correlation only on the line $f_1 = f_2$ and the lines parallel to $f_1 = f_2$ with spacing $\pm 2f_c$.

Problem 3. (10 points) Suppose that $X(t)$ is a real-valued WSS random process with mean $\mu_X \neq 0$ and the autocorrelation function $R_{\hat{X}\hat{X}}(\tau) \triangleq E[\hat{X}(t)\hat{X}(t-\tau)]$ of the zero-mean part $\hat{X}(t)$, i.e., $X(t) = \mu_X + \hat{X}(t)$. This random process is up-converted to a real passband random process as

$$Y(t) = \Re\{X(t)e^{j2\pi f_c t}\},$$

where $f_c \gg B$ with being B the bandwidth of $X(t)$. Answer the following questions.

(a) (2 points) Find the mean path $\mu_Y(t)$ of $Y(t)$ and also find the fundamental period of $\mu_Y(t)$.

(b) (3 points) Using Problem 1-(b), find the autocorrelation function $R_{YY}(t,s)$ of $Y(t)$ in terms of μ_X and $R_{\hat{X}\hat{X}}(\tau)$ and also show that

$$R_{YY}(t,s) = R_{YY}\left(t + \frac{1}{2f_c}, s + \frac{1}{2f_c}\right), \forall (t,s).$$

(c) (3 points) Using (a) and (b), show that $Y(t)$ is in general a WSCS random process with fundamental cycle period $1/f_c$.

(d) (2 points) Show that $Y(t)$ is a WSCS random process with fundamental cycle period $1/(2f_c)$ when $\mu_X = 0$.

Problem 4. (10 points) In Problem 3, let $S_{\hat{X}\hat{X}}(f)$ be the Fourier transform of $R_{\hat{X}\hat{X}}(\tau)$ with respect to τ. Answer the following questions.

(a) (2 points) Using Problem 2-(a), show that the Loève bi-frequency spectrum $S_{XX}(f_1, f_2)$ can be written as

$$S_{XX}(f_1,f_2) = \mu_X^2\delta(f_1)\delta(f_2) + S_{\hat{X}\hat{X}}(f_1)\delta(f_1-f_2) = \{\mu_X^2\delta(f_1) + S_{\hat{X}\hat{X}}(f_1)\}\delta(f_1-f_2)$$

and that the complementary bi-frequency spectrum $\tilde{S}_{XX}(f_1,f_2)$ can be written as

$$\tilde{S}_{XX}(f_1,-f_2) = \mu_X^2\delta(f_1)\delta(f_2) + S_{\hat{X}\hat{X}}(f_1)\delta(f_1-f_2) = \{\mu_X^2\delta(f_1) + S_{\hat{X}\hat{X}}(f_1)\}\delta(f_1+f_2).$$

(b) (4 points) Using Problem 2-(b) and Problem 4-(a), derive the Loève bi-frequency spectrum $S_{YY}(f_1,f_2)$ in terms of μ_X and $S_{\hat{X}\hat{X}}(f)$.

(c) (3 points) Sketch $|S_{YY}(f_1,f_2)|$ and confirm that there can be non-zero correlation only on the line $f_1 = f_2$ and the lines parallel to $f_1 = f_2$ with spacing $\pm 2f_c$.

(d) (1 point) Discuss why $Y(t)$ is not WSCS with fundamental cycle period $1/(2f_c)$ even though there are non-zero correlation only on the line $f_1 = f_2$ and the lines parallel to $f_1 = f_2$ with spacing $\pm 2f_c$.

Q. X(t) is SOS RP.
What is the property of Y(t)?

A. Y(t) is WSCS w/ fundamental cycle period $\frac{1}{f_c}$ or $\frac{1}{2f_c}$

non-zero mean X(t)   zero-mean X(t)

(i) $\mu_Y(t) = \mu_Y(t + \frac{1}{f_c}), \forall t$

(ii) $R_{YY}(t,s) = R_{YY}(t+\frac{1}{2f_c}, s+\frac{1}{2f_c}), \forall (t,s)$

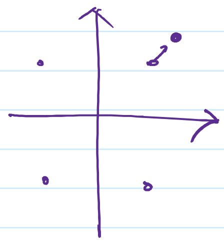
$\frac{1}{2f_c}$ X

$\boxed{\frac{1}{f_c}}$

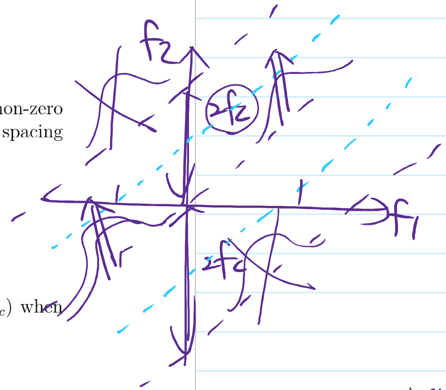
Problem 5. (20 points) Suppose that $X(t)$ is a complex-valued SOS random process with mean μ_X, autocorrelation function $R_{XX}(\tau) \triangleq E[X(t)X^*(t-\tau)]$, and complementary autocorrelation function $\tilde{R}_{XX}(\tau) \triangleq E[X(t)X(t-\tau)]$. This random process is up-converted to a real passband random process as

$$Y(t) = \Re\{X(t)e^{j2\pi f_c t}\},$$

where $f_c \gg B$ with being B the bandwidth of $X(t)$. Let $S_{XX}(f)$ be the Fourier transform of $R_{XX}(\tau)$ with respect to f and $\tilde{S}_{XX}(f)$ be the Fourier transform of $\tilde{R}_{XX}(\tau)$ with respect to f. Answer the following questions.



(a) (3 points) Show that in general $Y(t)$ is WSCS with fundamental cycle period $1/f_c$.



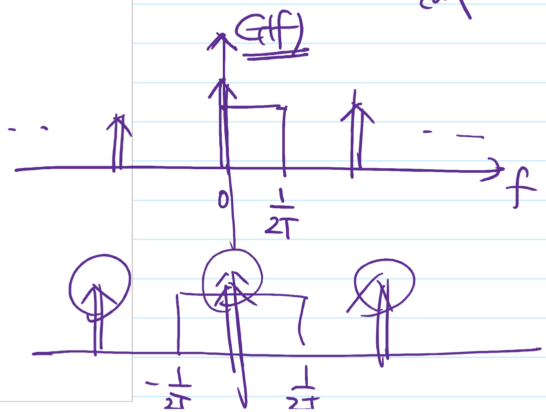
$\frac{1}{2f_c}$

(b) (5 points) Show that the Loève bi-frequency spectrum $S_{YY}(f_1, f_2)$ exhibits non-zero correlation only on the line $f_1 = f_2$ and the lines parallel to $f_1 = f_2$ with spacing $\pm 2f_c$

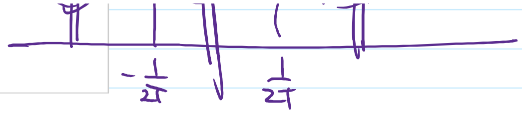
(c) (5 points) Show that $Y(t)$ is WSCS with fundamental cycle period $1/(2f_c)$ when $\mu_X = 0$.

$X(t)$ ↑ complex SOCS

(d) (4 points) Discuss whether $Y(t)$ is WSS or not when $X(t)$ is proper. Hint. Consider both of the cases with $\mu_X \neq 0$ and $\mu = 0$.



(d) (3 points) Discuss whether p-FRESH vectorizer can work or not when $\mu_X \neq 0$.



□Q. Nyquist Criterion. — comm. theory (Sampling, ISI) / control

A. $\begin{cases} X(t) = \sum_{n=-\infty}^{\infty} d_n\, p(t-nT) \\ x(t) = \sum_{n=-\infty}^{\infty} \hat{d}_n\, p(t-nT) \end{cases}$ $d_n \in$ Modulation Constellation . . . $\sum_{n=-\infty}^{\infty} |\hat{d}_n|^2 < \infty$

(i) $X(t) \longrightarrow \boxed{p^*(-t)} \xrightarrow{Y(t)}$ sample @ $t = mT$ $\longrightarrow Y(mT)$

$$\boxed{Y(mT) = d_m, \ \forall m.}$$

Q. what condition does $p(t)$ satisfies for ?

$\boxed{A.}$

$\hat{p}(t) = p(t) * p^*(-t)$

$$Y(mT) = \underbrace{d_m \|p(t)\|^2}_{\hat{p}(0\cdot T)} + \underbrace{\sum_{n \neq m} d_n\, \hat{p}((m-n)T)}_{\text{interference}}$$

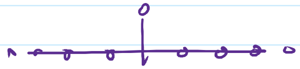
$\hat{p}(0\cdot T)$ interference

intersymbol

$$\Longleftrightarrow \boxed{\hat{p}(nT) = \delta_n} = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$

$$\boxed{\hat{p}(t)} = p(t) * p^*(-t)$$

FD



$$\Longleftrightarrow \quad \hat{p}(f) = P(f)\,P^*(f) = |P(f)|^2$$

$$\frac{1}{T}\sum_{l=-\infty}^{\infty} \left|P\left(f-\frac{l}{T}\right)\right|^2 = 1, \ \forall f$$

cf) $g(t) \leftrightarrow G(f)$

$g(nT) = h_n \leftrightarrow H(e^{j2\pi f})$

$$\underbrace{H(e^{j2\pi fT})}_{\text{period } \frac{1}{T}} = \frac{1}{\underbrace{T}_{\text{sampling rate}}}\underbrace{\sum_{l=-\infty}^{\infty} G\left(f-\frac{l}{T}\right)}_{\text{folded spectrum}}$$

Orthogonality $p(t),\ p^*(t-nT)$

$$\boxed{\int_{-\infty}^{\infty} p(t)\,p^*(t-nT)\,dt = \delta_n}$$

(ii) $x(t) \longrightarrow \boxed{g^*(-t)} \xrightarrow{y(t)}$ sample @ $t=mT$ $\ y(mT)$

$$\boxed{y(mT) = \delta_m}, \ \forall m. \qquad \hat{p}(t) = p(t) * g^*(-t)$$

$$\frac{1}{T}\sum_{l=-\infty}^{\infty} P\left(f-\frac{l}{T}\right) Q^*\left(f-\frac{l}{T}\right) = 1, \ \forall f$$

Bi-Orthogonality $p(t)\ \ g^*(t-nT)$

$\int_{-\infty}^{\infty}$ $*$

Bi-Orthogonality $p(t)$ $\beta^*(t-nT)$

$$\int_{-\infty}^{\infty} p(t)\,\beta^*(t-nT)\,dt = \delta_n.$$

□ DFT of $p(t)$ & $\beta(t)$ $\mathbf{P}(f)$, $\mathbf{B}(f)$

(i) orthogonality

$$\frac{1}{T}\mathbf{P}^H(f)\,\mathbf{P}(f) = 1, \quad \forall f \in \left[-\frac{1}{2T}, \frac{1}{2T}\right)$$

(ii) Bi-orthogonality

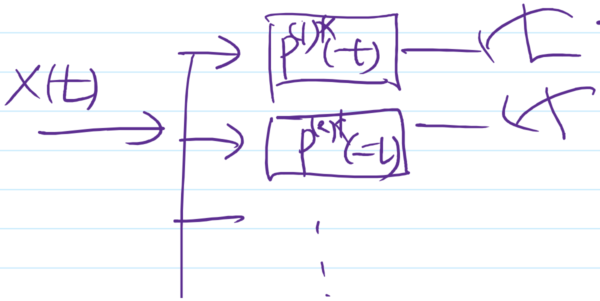
$$\frac{1}{T}\mathbf{P}^H(f)\,\mathbf{B}(f) = 1, \quad \left(\Leftrightarrow \frac{1}{T}\mathbf{B}^H(f)\,\mathbf{P}(f) = 1\right), \quad \forall f \in \left[-\frac{1}{2T}, \frac{1}{2T}\right)$$

□ Multi-Stream Transmission.

$$X(t) = \sum_{l=1}^{L}\sum_{n=-\infty}^{\infty} d_n^{(l)}\, p^{(l)}(t-nT) \qquad d_n^{(l)} \in l\text{th constellation}$$

$$x(t) = \sum_{l=1}^{L}\sum_{n=-\infty}^{\infty} d_m^{(l)}\, p^{(l)}(t-nT) \qquad \|d_m^{(l)}\| < \infty, \ \forall l.$$

(i)



$r^{(1)}(mT) = d_m^{(1)}$ — (*)

$r^{(2)}(mT) = d_m^{(2)}$

No ISI (intersymbol, inter-stream)

~~ISI~~

$$\boxed{p^{(L)*}(-t)}$$

$$y^{(L)}(mT) = d_m^{(L)}$$

(m)ulti(s)tream

~~ISI~~ I

$$(*) \iff \hat{p}^{(1)}(mT) = \delta_n \iff \frac{1}{T}\sum |P^{(1)}(f-\tfrac{\ell}{T})| = 1,\ \forall f$$

$$p^{(1)}(t) * p^{(2)*}(-t)\Big|_{t=nT} = 0,\ \forall n \qquad \frac{1}{T}\sum P^{(1)}(f-\tfrac{\ell}{T})\, P^{(2)*}(f-\tfrac{\ell}{T}) = 0,\ \forall f$$

$$\vdots$$

$$p^{(1)}(t) * p^{(L)*}(-t)\Big|_{t=nT} = 0,\ \forall n$$

$$\iff \frac{1}{T}\,\underline{P}^{(1)H}(f)\,\underline{P}^{(1)}(f) = 1,\ \forall f \iff \frac{1}{T}\,\underline{P}^{(1)H}(f)\begin{bmatrix} \underline{P}^{(1)}(f) \\ \underline{P}^{(2)}(f) \\ \vdots \\ \underline{P}^{(L)}(f) \end{bmatrix} = \underline{e}_{1},\ \forall f$$

$$\frac{1}{T}\,\underline{P}^{(1)H}(f)\,\underline{P}^{(2)}(f) = 0,\ \forall f$$

$$\vdots$$

$$\frac{1}{T}\,\underline{P}^{(1)H}(f)\,\underline{P}^{(L)}(f) = 0,\ \forall f$$

$$\vdots$$

$$\frac{1}{T}\left[\underline{P}^{(1)}(f)\ \underline{P}^{(2)}(f)\ \cdots\ \underline{P}^{(L)}(f)\right]^H \left[\underline{P}^{(1)} \cdots\ \underline{P}^{(L)}(f)\right] = I_L \qquad \forall f$$

$$\frac{1}{T}\left[\mathbf{p}^{(1)}(f)\ \mathbf{p}^{(2)}(f) \cdots \mathbf{p}^{(L)}(f)\right]^H \left[\mathbf{p}^{(1)} \cdots \mathbf{p}^{(L)}(f)\right] = I_L, \quad \forall f$$

$$\Leftrightarrow \boxed{\frac{1}{T}\mathbb{P}^H(f)\,\mathbb{P}(f) = I_L, \ \forall f}$$

(ii)

...

$$\Leftrightarrow \frac{1}{T}\mathbb{P}^H(f)\,\mathbb{Q}(f) = I_L, \ \forall f.$$

□ Q. Why TSR?

A.

KL expansion

$X(t) \qquad 0 \le t < T.$

$$\int_0^T R_{xx}(t,s)\, v_i(s)\, ds = \lambda_i v_i(t), \quad 0 \le t < T.$$

$$(\lambda_i, v_i(t))_{i=1}^{\infty}$$

$$R_{xx}(t,s) = R_{xx}(s,t), \quad \forall (t,s)$$

$$\int_0^T\int_0^T \omega(t)\, R_{xx}(t,s)\, w(s)\, dt\, ds \ge 0,$$

Mercer's theorem

$$R_{xx}(t,s) = \sum_{i=1}^{\infty} \lambda_i\, v_i(t)\, v_i(s)$$

$$X(t) = \sum \sqrt{\lambda_i}\, X_i\, v_i(t).$$

where $\quad X_i = \frac{1}{\sqrt{\lambda_i}}\int_0^T x(t)\, v_i(t)\, dt.$

where $X_i = \frac{1}{\sqrt{\lambda_i}} \int_0^{T} x(t)\, v_i^*(t)\, dt$.

$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix}$ X_i & X_j are uncorrelated.

$\mathrm{cov}(X_i) = \lambda_i$

□ proper-complex WSCS w/ T

$x(t) \longrightarrow \boxed{\text{FRESH vec.}} \longrightarrow \underline{X}(t)$

WSS proper-complex R.V.P

$R_{\underline{xx}}(t, t-\tau) \xrightarrow{\mathcal{F}} S_{\underline{xx}}(f) = \sum \cancel{X_i}\, \underline{v_i}(f)\, \underline{v_i}^H(f)$

$\mathcal{F}^{-1}$

$X_i(f)$

$x(t) = \sum_{\ell} \sum_{n} d_n^{(\ell)} p^{(\ell)}(t - nT)$,

$(d_n^{(\ell)})_n$ $(d_n^{(\ell')})_n$

uncorrelated sequences.