

VFT Representation of the Generalized Nyquist Criterion

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December 1, 2025

Abstract

This paper begins by deriving the **Unit Energy** ($\|s(t)\|^2 = 1$) condition for the pulse shape function $s(t)$, which is necessary to set the average power of a **random pulse-modulated signal** to a target power P in digital communications. We analyze the simplest example, the Nyquist band-limited Sinc pulse, and verify its energy spectral density $|S(f)|^2$ and total energy in both the time and frequency domains. Crucially, we define the **Vectorized Fourier Transform (VFT)** $\underline{s}(f)$ through the **FRESH(FREquency – SHift)Vectorization** of the pulse function $s(t)$.

Subsequently, we demonstrate that the **Square-Root Nyquist Criterion** for a single-stream system using a matched filter receiver is expressed using the VFT norm-square as: $\frac{1}{T} \underline{s}^H(f) \underline{s}(f) = 1$ (within the Nyquist Zone).

Finally, the discussion is extended to a multi-stream system transmitting K streams, where we derive the **Generalized Nyquist Criterion** that simultaneously satisfies **Zero Inter-Symbol Interference (ISI)** and **Zero Inter-Stream Interference (IZI)**. This condition is concisely expressed as a matrix equation, where the Grammian matrix of $\mathbf{S}(f)$ (the matrix whose columns are the VFTs $\underline{s}_k(f)$ of the K pulses) must be the identity matrix within the Nyquist Zone $|f| \leq \frac{1}{2T}$:

$$\frac{1}{T} \mathbf{S}^H(f) \mathbf{S}(f) = \mathbf{I}_K, \quad \text{for all } |f| \leq \frac{1}{2T}$$

1. Average Power Condition of Random Signal and Unit Energy Pulse Assertion

1.1. Definition of Random Signal $X(t)$ and Nyquist Criterion

The random signal $X(t)$ is defined in the form of a pulse train modulation as:

$$X(t) = A \sum_{n=-\infty}^{\infty} d_n s(t - nT)$$

Here, A is the amplitude constant, and $s(t)$ is the pulse shape function.

To use a Nyquist pulse that satisfies the **Zero Intersymbol Interference (ISI)** condition in a digital communication system, it is generally convenient to normalize the pulse function $s(t)$ to satisfy the **Unit Energy** condition. This means setting the energy of $s(t)$, $\|s(t)\|^2$, to 1 as follows:

$$\|s(t)\|^2 = \int_{-\infty}^{\infty} s^*(t) s(t) dt = 1$$

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$$\|s(t)\|^2 = \int_{-\infty}^{\infty} s^*(t)s(t)dt = 1$$

- T : Symbol Period.
- $\|s(t)\|^2$: Total energy of $s(t)$, assumed to be $\|s(t)\|^2 = 1$.
- $(d_n)_n$: A sequence with proper-complex, zero-mean, unit-variance, i.i.d. elements, satisfying the following statistical properties:

$$\begin{aligned}\mathbb{E}[d_n] &= 0 \\ \mathbb{E}[|d_n|^2] &= 1 \\ \mathbb{E}[d_n d_k^*] &= \delta_{n-k} \\ \mathbb{E}[d_n^2] &= 0 \quad (\text{Proper-complex condition})\end{aligned}$$

1

Q. What is A
that makes $\langle E\{X(t)|^2\} \rangle$
= P

$$\begin{aligned}① & \sqrt{P} \\ ② & \sqrt{P/T} \\ ③ & \sqrt{PT}\end{aligned}$$

1.2. Derivation of Average Power P_X and Determination of Amplitude A

The average power P_X of $X(t)$ is calculated by the time average $\frac{1}{T} \int_0^T \mathbb{E}[|X(t)|^2]dt$. First, we calculate $\mathbb{E}[|X(t)|^2]$.

$$\begin{aligned}\mathbb{E}[|X(t)|^2] &= \mathbb{E}\left[\left(A \sum_n d_n s(t-nT)\right) \left(A \sum_k d_k^* s^*(t-kT)\right)\right] \\ &= A^2 \sum_n \sum_k s(t-nT) s^*(t-kT) \mathbb{E}[d_n d_k^*]\end{aligned}$$

Due to the property of the sequence d_n , $\mathbb{E}[d_n d_k^*] = \delta_{n-k}$, only terms with $n = k$ remain.

$$\mathbb{E}[|X(t)|^2] = A^2 \sum_n |s(t-nT)|^2$$

Now, we take the time average of $\mathbb{E}[|X(t)|^2]$.

$$P_X = \frac{1}{T} \int_0^T \mathbb{E}[|X(t)|^2]dt = \frac{A^2}{T} \int_0^T \sum_n |s(t-nT)|^2 dt$$

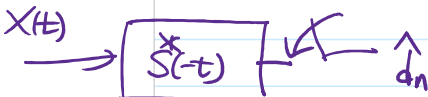
By swapping the order of integration and summation, and using the property that the sum of the integration intervals covers the infinite interval, we get:

$$P_X = \frac{A^2}{T} \int_{-\infty}^{\infty} |s(t)|^2 dt = \frac{A^2}{T} \cdot \|s(t)\|^2$$

1.3. Determination of Amplitude A

Applying the condition that the average power P_X of the signal must satisfy the target power P , $P_X = P$, and substituting the condition $\|s(t)\|^2 = 1$:

sample
@ $t=nT$



$$\hat{d}_n = d_n \int_{-\infty}^{\infty} |s(t)|^2 dt + ISI$$

$$\int_{-\infty}^{\infty} s^*(t) s(t-nT) dt = \delta[n]$$

1.3. Determination of Amplitude A

Applying the condition that the average power P_X of the signal must satisfy the target power P , $P_X = P$, and substituting the condition $\|s(t)\|^2 = 1$:

$$P = \frac{A^2}{T} \cdot 1$$

Since $A^2 = PT$, the amplitude constant is determined as:

$$A = \sqrt{PT}$$

Therefore, when $s(t)$ is a unit energy pulse, the amplitude constant required to make the average power of the random signal $X(t)$ equal to P is $A = \sqrt{PT}$.

2. Derivation of the Unit Energy Pulse $s(t)$

Now, we continue the discussion by setting

$$X(t) = \sqrt{PT} \sum_{n=-\infty}^{\infty} d_n s(t - nT)$$

For the simplest example, we assume $S(f)$ is a constant (c) within the Nyquist band. The Nyquist bandwidth is $\frac{1}{2T}$.

$$S(f) = \begin{cases} c & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases} = c \cdot \text{rect}(fT)$$

2

2.1. Determination of Amplitude Constant c

Since $s(t)$ must be a unit energy pulse ($\|s(t)\|^2 = 1$), the integral of $|S(f)|^2$ in the frequency domain must also be 1, according to Parseval's theorem.

$$\|s(t)\|^2 = \int_{-\infty}^{\infty} |S(f)|^2 df = 1$$

Since $|S(f)|^2 = c^2$:

$$\int_{-1/(2T)}^{1/(2T)} c^2 df = c^2 \left[\frac{1}{2T} - \left(-\frac{1}{2T} \right) \right] = c^2 \cdot \frac{1}{T}$$

This value must be equal to 1:

Since $|S(f)|^* = c^*$:

$$\int_{-1/(2T)}^{1/(2T)} c^2 df = c^2 \left[\frac{1}{2T} - \left(-\frac{1}{2T} \right) \right] = c^2 \cdot \frac{1}{T}$$

This value must be equal to 1:

$$c^2 \cdot \frac{1}{T} = 1 \Rightarrow c^2 = T \Rightarrow c = \sqrt{T}$$

Thus, $S(f)$ satisfying the **Unit Energy condition** is defined as:

$$S(f) = \begin{cases} \sqrt{T} & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

2.2. Derivation of the Time Function $s(t)$

$s(t)$ is derived through the Inverse Continuous-Time Fourier Transform (ICTFT) of $S(f)$.

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df = \int_{-1/(2T)}^{1/(2T)} \sqrt{T} e^{j2\pi ft} df$$

Calculating this integral and applying Euler's formula gives:

$$s(t) = \frac{\sqrt{T}}{2\pi t} \left[e^{j\frac{\pi t}{T}} - e^{-j\frac{\pi t}{T}} \right] = \sqrt{T} \cdot \frac{1}{\pi t} \sin\left(\frac{\pi t}{T}\right)$$

Expressed in terms of the normalized Sinc function $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$:

$$s(t) = \frac{\sqrt{T}}{T} \cdot \text{sinc}\left(\frac{t}{T}\right) = \frac{1}{\sqrt{T}} \text{sinc}\left(\frac{t}{T}\right)$$

3. Energy Spectral Density $|S(f)|^2$

The **Energy Spectral Density (ESD)** $G_s(f)$ for the unit energy pulse $s(t)$ is defined as $|S(f)|^2$.

$$|S(f)|^2 = (\sqrt{T})^2 \cdot \begin{cases} 1 & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

$$|S(f)|^2 = \begin{cases} T & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

4. Verification of Total Energy E_s (Frequency Domain)

The total energy E_s is confirmed to be 1 through integration in the frequency domain.

$$E_s = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_{-1/(2T)}^{1/(2T)} T df$$

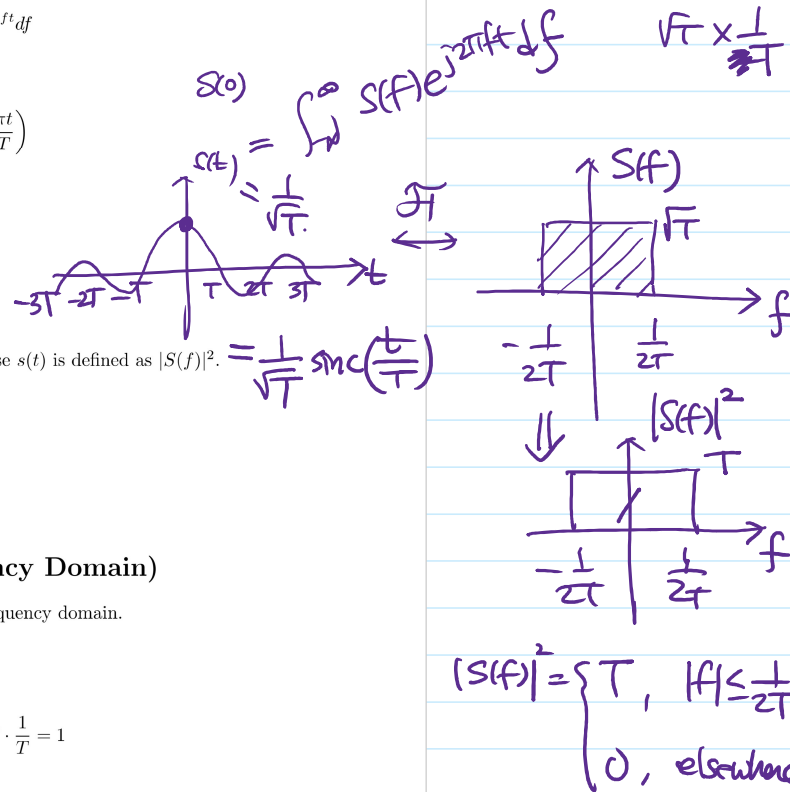
$$E_s = T \cdot \left[f \right]_{-1/(2T)}^{1/(2T)} = T \cdot \left(\frac{1}{2T} - \left(-\frac{1}{2T} \right) \right) = T \cdot \frac{1}{T} = 1$$

$$\text{((S))} \quad \boxed{\frac{1}{T} \int_{-1/(2T)}^{1/(2T)} S^H(f) S(f) df = 1, \forall f}$$

$$S(f) = S(f)$$

$$|S(f)|^2 = \|S(f)\|^2$$

$$= \int_{-1/(2T)}^{1/(2T)} S^H(f) S(f) df = \int_{-1/(2T)}^{1/(2T)} T df = 1$$



5. Verification of Total Energy E_s (Time Domain)

The total energy E_s is confirmed to be 1 through time domain integration $\|s(t)\|^2$.

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{T}} \cdot \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} \right)^2 dt$$

Applying the substitution $x = \frac{\pi t}{T}$ ($dt = \frac{T}{\pi} dx$):

$$E_s = \frac{1}{T} \int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x} \right)^2 \frac{T}{\pi} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx$$

Substituting the special integral $\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx = \pi$:

$$E_s = \frac{1}{\pi} \cdot \pi = 1$$

6. FRESH Vectorization and Vectorized Fourier Transform (VFT)

6.1. Definition of FRESH Vectorization $\underline{s}(t)$

In a communication system with period T , the **FRESH (FREquency-SHift) vectorization** $\underline{s}(t)$ of the pulse shape function $s(t)$ represents the components of $s(t)$ shifted in frequency by $\frac{1}{T}$ intervals, in vector form. While theoretically infinite-dimensional, it becomes finite-dimensional for band-limited pulses.

The k -th element of $\underline{s}(t)$ is defined as:

$$[\underline{s}(t)]_k = s(t) e^{-j2\pi \frac{k}{T} t}, \quad k \in \mathbb{Z}$$

Thus, the **FRESH vectorization** of $s(t)$ is:

$$\underline{s}(t) = \begin{pmatrix} \dots \\ s(t) e^{j2\pi \frac{1}{T} t} \\ s(t) \\ s(t) e^{-j2\pi \frac{1}{T} t} \\ \dots \end{pmatrix}$$

The order of the elements in this vector may vary depending on the definition of k , but it is generally listed symmetrically around $k = 0$. In this paper, we assume the order of k is $\dots, -1, 0, 1, \dots$

6.2. Definition of Vectorized Fourier Transform (VFT) $\underline{s}(f)$

The **Vectorized Fourier Transform (VFT)** $\underline{s}(f)$ of $\underline{s}(t)$ is the element-wise Fourier transform of each element of $\underline{s}(t)$. (We use $\underline{s}(f)$ for notational consistency.)

$$[\underline{s}(f)]_k = \mathcal{F}\{[\underline{s}(t)]_k\} = \mathcal{F}\{s(t) e^{-j2\pi \frac{k}{T} t}\}$$

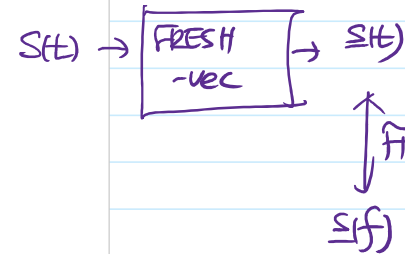
According to the **Frequency Shift Property** of the Fourier transform, $\mathcal{F}\{s(t) e^{-j2\pi f_0 t}\} = S(f + f_0)$, with $f_0 = \frac{k}{T}$:

$$[\underline{s}(f)]_k = S\left(f + \frac{k}{T}\right)$$

The **VFT** of $s(t)$ is:

$$\left(\begin{array}{c} \dots \end{array} \right)$$

$$= \underline{s}^H(f) \underline{s}(f) = \int_{-\infty}^{\infty} \dots$$



$$[\underline{s}(f)]_k = S\left(f + \frac{k}{T}\right)$$

The **VFT** of $s(t)$ is:

$$\underline{s}(f) = \begin{pmatrix} \cdots \\ S(f + \frac{1}{T}) \\ S(f) \\ S(f - \frac{1}{T}) \\ \cdots \end{pmatrix}$$

4

Square-Root Nyquist Criterion for $s(t)$.

$$\frac{1}{T} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} S^H(f) S(f) df = 1, \quad \forall |f| \leq \frac{1}{2T}$$

6.3. Example Revisited: ESD Representation using VFT

The special case derived in Section 3, where the Energy Spectral Density is constant only within the Nyquist bandwidth, is:

$$|S(f)|^2 = \begin{cases} T & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

Dividing this equation by $\frac{1}{T}$ represents the **frequency domain characteristic of a unit energy pulse**:

$$\frac{1}{T} |S(f)|^2 = \begin{cases} 1 & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

Using the VFT $\underline{s}(f)$, the above special case condition is expressed as:

$$\frac{1}{T} \underline{s}^H(f) \underline{s}(f) = 1, \quad \text{for all } |f| \leq \frac{1}{2T}$$

This shows that the VFT's norm square, scaled by $(1/T)$, is uniformly 1 within the Nyquist Zone.

7. VFT Representation of the Square-Root Nyquist Criterion

7.1. Time Domain Definition of the Square-Root Nyquist Criterion

A pulse shape function $s(t)$ satisfies the **Square-Root Nyquist Criterion** in digital communications if the **composite pulse $g(t)$ **, defined by the convolution of $s(t)$ with its complex conjugate time-reversed pulse $s^*(-t)$, satisfies the **Nyquist Criterion**, even if $s(t)$ itself does not satisfy the ISI-free condition.

$$g(t) = s(t) * s^*(-t)$$

Assuming the unit energy condition for $s(t)$, the composite pulse $g(t)$ satisfies the ISI-free condition when sampled at intervals of kT : it is 1 only for $k = 0$ and 0 for other $k \neq 0$.

$$g(kT) = [s(t) * s^*(-t)]|_{t=kT} = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \in \mathbb{Z}, k \neq 0 \end{cases} = \delta_k$$

Here, $g(t)$ is equivalent to the **Autocorrelation Function** of $s(t)$, and $g(0) = \int_{-\infty}^{\infty} |s(t)|^2 dt = \|s(t)\|^2 =$

$$S(f) * S^*(-f) = g(t)$$

$$g(nT) = \delta_n$$

$$g(kT) = [s(t) * s^*(-t)]|_{t=kT} = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \in \mathbb{Z}, k \neq 0 \end{cases} = \delta_k$$

Here, $g(t)$ is equivalent to the **Autocorrelation Function** of $s(t)$, and $g(0) = \int_{-\infty}^{\infty} |s(t)|^2 dt = \|s(t)\|^2 = 1$.

7.2. Frequency Domain Representation of the Square-Root Nyquist Criterion

The Fourier transform $G(f)$ of the composite pulse $g(t)$ relates to the Fourier transform $S(f)$ of $s(t)$ as follows:

$$G(f) = \mathcal{F}\{s(t) * s^*(-t)\} = S(f)S^*(f) = |S(f)|^2$$

The frequency domain condition for $g(t)$ to satisfy the Nyquist Criterion is that the sum of its components shifted by $\frac{1}{T}$ intervals is equal to the constant $\frac{1}{T}$.

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} G\left(f + \frac{k}{T}\right) = \frac{1}{T}, \quad \text{for all } f \in \mathbb{R}$$

Substituting $G(f) = |S(f)|^2$ yields the frequency domain condition for the **Square-Root Nyquist Criterion** of $s(t)$:

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} \left| S\left(f + \frac{k}{T}\right) \right|^2 = 1, \quad \text{for all } |f| \leq \frac{1}{2T}$$

5

7.3. VFT Representation of the Square-Root Nyquist Criterion

Using the **Vectorized Fourier Transform (VFT)** of $s(t)$, the frequency domain summation condition of the Square-Root Nyquist Criterion is expressed in the form of the vector norm-square.

The norm-square of the VFT, $\underline{s}^H(f)\underline{s}(f)$, exactly matches the periodic version of $|S(f)|^2$ of $s(t)$.

$$\underline{s}^H(f)\underline{s}(f) = \sum_{k=-\infty}^{\infty} S^*\left(f + \frac{k}{T}\right) S\left(f + \frac{k}{T}\right) = \sum_{k=-\infty}^{\infty} \left| S\left(f + \frac{k}{T}\right) \right|^2$$

Therefore, the frequency domain condition of the Square-Root Nyquist Criterion is concisely expressed using the VFT as:

$$\frac{1}{T} \underline{s}^H(f)\underline{s}(f) = 1, \quad \text{for all } |f| \leq \frac{1}{2T}$$

This means that the $\frac{1}{T}$ -scaled Periodic Spectral Density of the pulse function $s(t)$ must be uniformly 1 within the Nyquist Zone.

8. Generalized Nyquist Criterion for Multi-Stream Systems

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8.1. Definition of Multi-Stream Signal $X(t)$ and Receiver Structure

In a multi-stream system transmitting K streams, the transmitted signal $X(t)$ is defined as the sum of K orthogonal pulse trains:

$$X(t) = \sum_{k=1}^K \left(\sum_{n=-\infty}^{\infty} d_k[n] s_k(t - nT) \right)$$

Here, $d_k[n]$ is the n -th symbol of the k -th stream, and $s_k(t)$ is the pulse shape function used for the k -th stream.

The receiver is assumed to use **Matched Filter** receivers to demodulate each stream. To demodulate the k' -th stream, the matched filter $s_{k'}^*(-t)$ corresponding to the pulse $s_{k'}(t)$ is used. The output $y_{k'}(t)$ of the matched filter is:

$$y_{k'}(t) = X(t) * s_{k'}^*(-t) = \sum_{k=1}^K \sum_{n=-\infty}^{\infty} d_k[n] [s_k(t - nT) * s_{k'}^*(-t)]$$

The output signal $y_{k'}(t)$ is sampled at intervals of the symbol period T to obtain the symbol $\hat{d}_{k'}[m] = y_{k'}(mT)$.

8.2. Zero ISI and Zero Inter-Stream Interference Conditions

The condition for simultaneously satisfying **Zero Inter-Symbol Interference (ISI)** and **Zero Inter-Stream Interference (IZI)** is that the sampled output $\hat{d}_{k'}[m]$ must contain only the target symbol $d_{k'}[m]$, and no symbols from other streams ($k \neq k'$) or from other times ($n \neq m$). That is:

$$\hat{d}_{k'}[m] = y_{k'}(mT) = d_{k'}[m], \quad \text{for all } k'$$

The sampling condition for the convolution term to satisfy this is:

$$[s_k(t - nT) * s_{k'}^*(-t)]|_{t=mT} = \delta_{k,k'} \delta_{m,n}$$

Substituting $t' = t - nT$ and $t = mT$ yields:

$$[s_k(t') * s_{k'}^*(-t')]|_{t'=(m-n)T} = \delta_{k,k'} \delta_{m,n}$$

Thus, a value must exist only when $m = n$, and it must be 1 only when $k = k'$.

- ISI-free Condition ($k = k'$):** The composite pulse $g_k(t) = s_k(t) * s_k^*(-t)$ must satisfy the Nyquist Criterion.

$$g_k(nT) = \delta_{n,0} \Rightarrow \frac{1}{T} \sum_m |S_k(f + \frac{m}{T})|^2 = 1$$

- IZI-free Condition ($k \neq k'$):** The composite pulse $h_{k,k'}(t) = s_k(t) * s_{k'}^*(-t)$ must be 0 when sampled at T intervals.

$$h_{k,k'}(nT) = 0, \quad \text{for all } n \in \mathbb{Z} \Rightarrow \sum_m S_k(f + \frac{m}{T}) S_{k'}^*(f + \frac{m}{T}) = 0$$

6

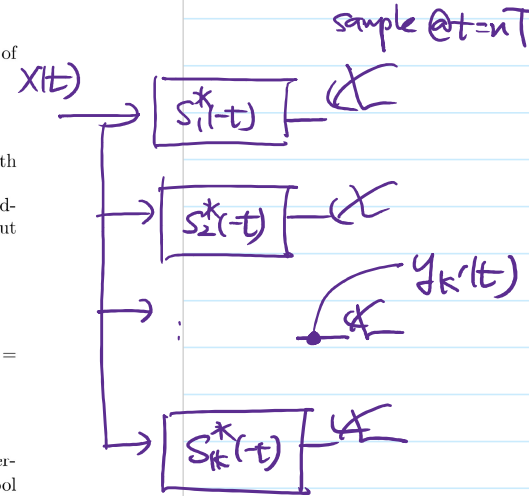
Orthogonality
Bi-orthogonality

$$\frac{1}{T} S^H(f) S(f) = I_K, \quad \forall f \leq \frac{1}{2T}$$

$$\frac{1}{T} R^H(f) S(f) = I_K$$

$$\|s_k(t)\|^2 = 1$$

$$d_k[n]$$



$$(i) \frac{1}{T} S_k^H(f) S_k(f) = 1, \quad \forall f \leq \frac{1}{2T}$$

$$(ii) \frac{1}{T} S_{k'}^H(f) S_k(f) = \delta_{k,k'}, \quad \forall f \leq \frac{1}{2T}$$

$$\frac{1}{T} S_{k'}^H(f) \begin{bmatrix} S_1(f) & \dots & S_K(f) \end{bmatrix} = \underline{e}_k^T \in \mathbb{C}^K$$

$$\frac{1}{T} \begin{bmatrix} S_1(f) & \dots & S_K(f) \end{bmatrix}^H \begin{bmatrix} \underline{e}_1^T \\ \underline{e}_2^T \\ \vdots \\ \underline{e}_K^T \end{bmatrix} = I_K$$

$$\frac{1}{T} S^H(f) S(f) = I_K, \quad \forall f \leq \frac{1}{2T}$$

$T \rightarrow \tau \rightarrow \omega \rightarrow \dots \rightarrow \frac{1}{T}$

$d_k[m]$

8.3. VFT Matrix Representation of the Generalized Nyquist Criterion

These two conditions (ISI-free and IZI-free) can be integrated into a single matrix equation within the Nyquist Zone $|f| \leq \frac{1}{2T}$ using the VFT.

Defining the VFT of the K pulses $s_1(t), \dots, s_K(t)$ as $\underline{s}_k(f)$, the orthogonality and unit energy conditions are determined by the value of $\frac{1}{T} \underline{s}_{k'}^H(f) \underline{s}_k(f)$.

$$\frac{1}{T} \underline{s}_{k'}^H(f) \underline{s}_k(f) = \begin{cases} 1 & \text{for } k = k' \quad (\text{ISI-free}) \\ 0 & \text{for } k \neq k' \quad (\text{IZI-free}) \end{cases}, \quad \text{for all } |f| \leq \frac{1}{2T}$$

We define the K -column matrix $\mathbf{S}(f)$ as $\mathbf{S}(f) = (\underline{s}_1(f) \quad \underline{s}_2(f) \quad \dots \quad \underline{s}_K(f))$, where the K VFT vectors are its column vectors. The (k', k) -th element of the Grammian matrix $\mathbf{S}^H(f) \mathbf{S}(f)$ is exactly $\underline{s}_{k'}^H(f) \underline{s}_k(f)$.

Therefore, the above K^2 identities can be expressed by stating that the Grammian of $\mathbf{S}(f)$ must be the identity matrix I_K within the Nyquist Zone.

$$\star \left[\frac{1}{T} \mathbf{S}^H(f) \mathbf{S}(f) = I_K, \quad \text{for all } |f| \leq \frac{1}{2T} \right] \star \leftarrow K \text{ streams}$$

This is the VFT representation of the **Generalized Nyquist Criterion** for multi-stream systems.

$$\text{((s)) } \frac{1}{T} \underline{s}^H(f) \underline{s}(f) = 1, \quad \forall |f| \leq \frac{1}{2T} \leftarrow K=1$$

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# Continuous-Time Equivalents of Welch Bound Equality Sequences

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**Abstract**—The theory developed on the sequence optimization for equal-power code-division multiple-access (CDMA) systems is generalized in this paper. Unlike previous works, where discrete-time or vector channel models are predominantly employed, the overloaded channels are modeled as strictly band-limited, continuous-time multiple-access channels corrupted by additive white Gaussian noise (AWGN). After being posed as a multiple-input/single-output (MISO) joint transmitter and receiver optimization problem with the total mean-squared error (MSE) as the objective functional to be minimized, the variational problem is solved by using a frequency-domain approach. It is shown that there exist continuous-time equivalents of Welch bound equality (CTE-WBE) sequences as the jointly optimum transmit waveforms and that the matched filters are the jointly optimum receivers. The user capacity of the band-limited channels is characterized by a necessary and sufficient condition for the admissibility of users into the system in terms of the channel load, the received signal-to-noise ratio, and the signal-to-interference-plus-noise ratio (SINR) requirement. It is also shown that CTE-WBE sequences achieve the lower bound on the continuous-time equivalent of total squared correlation (CTE-TSC).

**Index Terms**—Code-division multiple-access (CDMA), frequency-division multiple-access (FDMA), joint transmitter and receiver optimization, time-division multiple-access (TDMA), total squared correlation, user capacity, Welch bound equality (WBE) sequences.

## I. INTRODUCTION

IN code-division multiple-access (CDMA) communications, where multiple-access interference (MAI) is a limiting factor on the system performance, the signature or spreading sequences of users are identified as key design parameters that significantly affect the MAI characteristic [1], [2]. In order to

For symbol-synchronous CDMA systems, since the optimal signature sequences are simply orthogonal ones when the number of users is lower than or equal to the processing gain, it has been of prime interest to find a signature sequence set that minimizes the effect of MAI when the system is overloaded, i.e., when the number of users is greater than the processing gain. It is shown in [7] that Welch bound equality (WBE) sequences [5] that minimize total squared correlation (TSC) are optimum in the sense that the sum capacity is maximized for equal-power overloaded systems. An extension of this problem is considered and the optimum sequences are constructed in [8] for the cases with asymmetric user-power constraints. In [9], WBE sequences are again shown to be optimum for the users employing linear minimum mean-squared error (LMMSE) receivers in the sense that total received power is minimized subject to the equal signal-to-interference-plus-noise ratio (SINR) requirements. Interestingly, it turns out that the LMMSE receivers reduce to matched filters when the sequences are optimized. Unequal SINR requirement cases with LMMSE receivers are also considered in [9], and it is shown that generalized WBE sequences are optimum. For efficient construction of optimal signature sequences, an iterative procedure called minimum mean-squared error (MMSE) iteration is proposed in [10] and the convergence of a modified version of this procedure to the optimum sequences is proved in [11].

Sequence optimizations for symbol asynchronous but chip synchronous CDMA systems have been also investigated. In [12], it is shown that signature sequences for chip-synchronous Gaussian overloaded systems can be optimized by minimizing the total squared asynchronous correlation and that the opti-

Recall.

GNC

$$S(f) = [s_1(f) \cdots s_K(f)]$$

where

$$\rightarrow \left( \frac{1}{T} \right) \int_{-T/2}^{T/2} \|s_k(f)\|^2 df = 1 \quad \text{for } \sum_k |s_k(f)|^2 = 1, \forall f.$$

$$\rightarrow \int_{-\infty}^{\infty} s_k(t) s_k^*(t-nT) dt = \delta_{k,l} \delta_n$$

$$\Leftrightarrow \frac{1}{T} S^H(f) S(f) = I_K, \quad |f| \leq \frac{1}{2T}$$

To show the Gram Matrix of  $S(f)$

$$\sum_{n=-\infty}^{\infty} \left( \int_{-\infty}^{\infty} s_k(t) s_k^*(t-nT) dt \right)^2 \quad \text{Gramian}$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} |s_k^H(f) s_k(f)|^2 df$$

$$\Rightarrow \sum_{k=1}^K \sum_{n=-\infty}^{\infty} \left| \int s_k^*(t) s_k(t-nT) dt \right|^2$$



In code-division multiple-access (CDMA) communications, where multiple-access interference (MAI) is a limiting factor on the system performance, the signature or spreading sequences of users are identified as key design parameters that significantly affect the MAI characteristic [1], [2]. In order to mitigate the detrimental effect of MAI, various sequence sets have been proposed that possess very good cross-correlation properties [3]–[6].

Manuscript received January 13, 2004; revised November 17, 2004. This work was supported in part by the National Science Foundation under Grant CCR-0329794 and in part by the Ministry of Information and Communication, Korea, under grant for ITRC program (BrOMA@POSTECH) supervised by the Institute of Information Technology Assessment. The material in this paper was presented in part at the IEEE Wireless Communications and Networking Conference, Atlanta, GA, March 2004, and in part at the IEEE International Conference on Communications, Paris, France, June 2004.

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Communicated by M. Médard, Associate Editor for Communications. Digital Object Identifier 10.1109/TIT.2005.853323

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sequence optimizations for symbol asynchronous but chip synchronous CDMA systems have been also investigated. In [12], it is shown that signature sequences for chip-synchronous Gaussian overloaded systems can be optimized by minimizing the total squared asynchronous correlation and that the optimization can be performed by using an iterative algorithm. In [13], chip-synchronous multipath channels are considered and iterative algorithms are proposed.

The commonality of the above previous works is in that the observation model is either a vector or a discrete-time model. The users are assumed symbol synchronous or chip synchronous with the chip waveforms satisfying Nyquist's condition for zero interchip interference. Thus, the system design problem is just reduced to the signature *sequence* design problem. This approach could be suboptimum if the users are completely asynchronous, so that the effect of the chip waveform is not ignorable. Moreover, this channel model does not allow straightforward extension of the equivalence of various multiple-access techniques, such as CDMA, frequency-division multiple-access (FDMA), and time-division multiple-access (TDMA), established for nonoverloaded Gaussian multiple-access channels to overloaded Gaussian multiple-access channels.

In this paper, the signature *waveform* optimization problem is investigated for overloaded equal-power CDMA communications over strictly band-limited, continuous-time, multiple-access channels corrupted by additive white Gaussian noise (AWGN). After being posed as a multiple-input/single-output (MISO) joint transmitter and receiver optimization problem with the total mean-squared error (MSE) as the objective functional to be minimized, the variational problem is solved by using a frequency-domain approach [14]–[18]. It is shown that there exist continuous-time equivalents of Welch bound equality (CTE-WBE) sequences as the optimum signature waveforms and that CTE-WBE sequences achieve the lower bound on the continuous-time equivalent of total squared correlation (CTE-TSC). Thus, the theory of the sequence optimization for discrete-time or vector channel CDMA systems is successfully generalized in this paper to continuous-time systems. It is also shown that there exist an FDMA and a TDMA-like optimum systems that achieve the lower bound on the CTE-TSC.

The rest of this paper is organized as follows. In Section II, the signal and system models are described. In Section III, the jointly optimum transmit and receive waveforms are derived via a frequency-domain approach. Examples and discussions are provided in Section IV, and concluding remarks are offered in

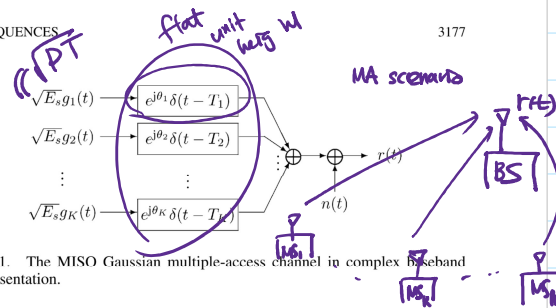


Fig. 1. The MISO Gaussian multiple-access channel in complex baseband representation.

where  $j$  denotes  $\sqrt{-1}$ ,  $\theta_k$  is the phase of the  $k$ th received signal, and  $n(t)$  is the proper-complex AWGN with  $\mathbb{E}\{n(t)n(s)^H\} = N_0\delta(t-s)$ . The superscript  $H$  denotes conjugation for scalars and conjugate transposition for vectors, and the function  $\delta(t)$  denotes the Dirac delta function. Fig. 1 depicts this simple MISO Gaussian multiple-access channel.

The received signal is processed by  $K$  linear receivers generating the outputs given by

$$y_k[m] = \int_{-\infty}^{\infty} w_k(t - mT)^H r(t) dt \quad (3)$$

for  $k = 1, 2, \dots, K$ , where  $w_k(t)$  is the receive waveform to de-

$$\begin{aligned} &\Rightarrow \sum_{k=1}^K \sum_{n=-\infty}^{\infty} \left| \int s_k^*(t) s_k(t - nT) \right|^2 \\ &= \frac{1}{T} \int_{-\infty}^{\infty} \left\| \sum_{k=1}^K s_k^H(f) S(f) \right\|^2 df \\ &= \frac{1}{T} \int_{-\infty}^{\infty} \sum_{k=1}^K s_k^H(f) S(f) S^H(f) s_k(f) df \\ &\Rightarrow \sum_{k=1}^K \sum_{n=-\infty}^{\infty} \left| \int s_k^*(t) s_k(t - nT) \right|^2 \\ &= \text{tr} \left( \frac{1}{T} \int_{-\infty}^{\infty} S^H(f) S(f) S^H(f) S(f) df \right) \end{aligned}$$

Def.

$$X(f) = \sum d[m] s(t - mT) \text{ is a}$$

The rest of this paper is organized as follows. In Section II, the signal and system models are described. In Section III, the jointly optimum transmit and receive waveforms are derived via a frequency-domain approach. Examples and discussions are provided in Section IV, and concluding remarks are offered in Section V.

## II. SIGNAL AND SYSTEM MODEL

In this section, we describe the signal and system model. There are  $K$  active users. Each user transmits an independent sequence of uncorrelated data symbols by using a linear modulation. The  $k$ th user's complex baseband equivalent of the transmitted signal is modeled as

$$g_k(t) \triangleq \sum_{m=-\infty}^{\infty} d_k[m] s_k(t - mT) \quad (1)$$

where the data symbols  $(d_k[m])_{m \in \mathbb{Z}}$  are assumed to be uncorrelated proper-complex random variables with mean zero and variance one, and  $1/T$  [symbols/s] is the symbol transmission rate of the linear modulation scheme. The set  $\mathbb{Z}$  is the set of all integers. The  $k$ th user's transmit waveform  $s_k(t)$  is assumed strictly band-limited with bandwidth  $W/2$  [Hz] in baseband or  $W$  [Hz] in passband. Since we consider communications over overloaded channels, it is assumed that  $WT < K$ , i.e.,  $1 + \beta < K$ , where the excess bandwidth  $\beta (> -1)$  is defined by the relation  $WT = 1 + \beta$ .

The  $k$ th bandpass signal is received with signal energy per symbol  $E_k (> 0)$ , propagation delay  $T_k$ , and phase angle  $\theta_k$  in the presence of AWGN with two-sided power spectral density  $N_0/2$ . Since equal power cases are only considered in this paper, we define the common received signal energy per symbol as  $E_s (= E_k, \forall k)$ . Thus, if the energy of  $s_k(t)$  is normalized as

$$\|s_k(t)\| \triangleq \sqrt{\int_{-\infty}^{\infty} |s_k(t)|^2 dt} = 1, \quad \forall k$$

the complex baseband equivalent of the received signal can be written as

$$r(t) = \sum_{k=1}^K \sqrt{E_s} g_k(t - T_k) e^{j\theta_k} + n(t) \quad (2)$$

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$$y_k[m] = \int_{-\infty}^{\infty} w_k(t - mT)^H r(t) dt \quad (3)$$

for  $k = 1, 2, \dots, K$ , where  $w_k(t)$  is the receive waveform to demodulate the  $k$ th user signal. We assume without loss of generality that the receive waveforms have the same bandwidth  $W/2$  as the transmit waveforms. This is well justified because the out-of-band received signal, which only contains the AWGN component, is irrelevant for the symbol detection [19].

The mean-squared error  $\text{MSE}_k$  of the  $k$ th receiver output is defined as

$$\text{MSE}_k \triangleq \mathbb{E}\{|y_k[m] - d_k[m]|^2\} \quad (4)$$

which is clearly a quantity independent of  $m$  when  $w_k(t)$  is fixed. Another performance metric is the signal-to-interference-plus-noise ratio  $\text{SINR}_k$  of the  $k$ th user defined as

$$\text{SINR}_k \triangleq \frac{\mathbb{E}\{|\mathbb{E}\{y_k[m] d_k^*[m]\}|^2\}}{\text{Var}\{y_k[m] d_k^*[m]\}} \quad (5)$$

It follows, the transmit and receive waveforms  $s_k(t)$  and  $w_k(t)$ , for  $k = 1, 2, \dots, K$ , are to be optimized, where all the waveforms are restricted to have piecewise smooth<sup>1</sup> frequency responses. Piecewise smoothness is not only a standard assumption in the calculus of variations [21] to exclude practically meaningless pathological functions but also a sufficient condition for the applicability of the Poisson summation formula,<sup>2</sup> which plays an important role in handling the problem in the frequency domain.

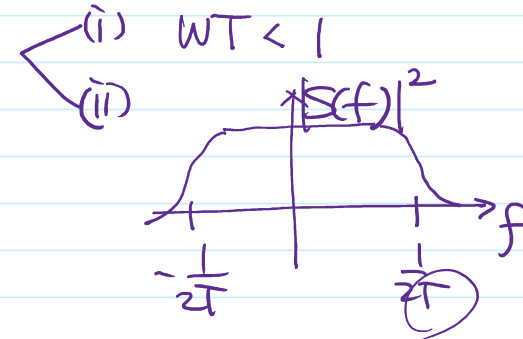
## III. CTE-WBE SEQUENCES

In this section, we perform a joint transmitter and receiver optimization to minimize the total mean-squared error defined as  $\text{TMSE} \triangleq \sum_{k=1}^K \text{MSE}_k$ . Thus, the optimization problem to be solved is formulated as follows.

<sup>1</sup>A function  $f(t)$  is called piecewise smooth if  $f'(t)$  is piecewise continuous [20].

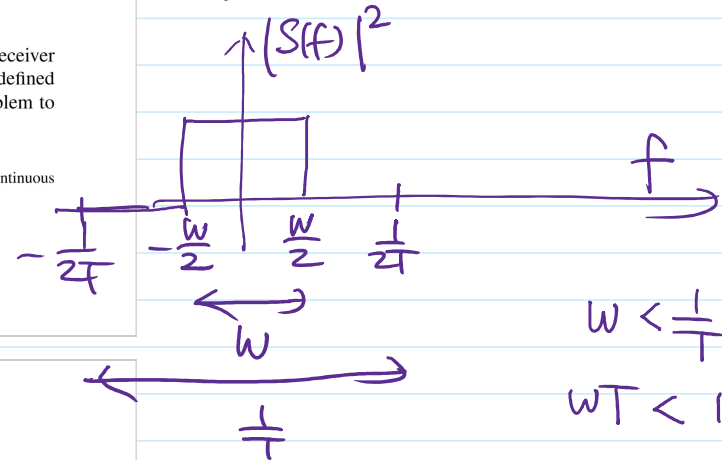
<sup>2</sup>For details, see [22, Proposition 1].

$\tilde{X}(f) = \sum d[m] s(t - mT)$  is a FTN (Faster-Than-Nyquist) transmission.



$$\tilde{X}(t) = \sum d_n s(t - m \frac{T}{\alpha})$$

$\alpha > 1$



Problem 1:

$$\begin{aligned} & \underset{\{(s_k(t), w_k(t))\}_{k=1}^K}{\text{minimize}} && \sum_{k=1}^K \mathbb{E}\{|y_k[m] - d_k[m]|^2\} \\ & \text{subject to} && \|s_k(t)\| = 1, \quad \forall k. \end{aligned} \quad (6)$$

where  $\Re\{\cdot\}$  denotes the real part and  $\phi_{k,k'}$  is defined as

$$\phi_{k,k'} \triangleq \sum_{l=-\infty}^{\infty} |c_{k,k'}(lT)|^2.$$

Now, the Poisson summation formula leads to



$$\begin{aligned} & \underset{\{(s_k(t), w_k(t))\}_{k=1}^K}{\text{minimize}} \quad \sum_{k=1}^K \mathbb{E}\{|y_k[m] - d_k[m]|^2\} \\ & \text{subject to} \quad \|s_k(t)\| = 1, \quad \forall k. \end{aligned} \quad (6)$$

To solve this variational problem, a frequency-domain approach is taken. In the first subsection, the time-domain expression of the objective functional is converted into a frequency-domain expression. In the subsequent subsections, the optimum receive waveforms are found, the optimum transmit waveforms are identified as continuous-time equivalents of WBE sequences, and the user capacity of the continuous-time, strictly band-limited Gaussian multiple-access channel is characterized. In the last subsection, we provide an alternate definition of the optimum solutions and show that they achieve the lower bound on the continuous-time equivalent of total squared correlation.

#### A. Frequency-Domain Expression of Total MSE

By definition, the decision statistic  $y_k[m]$  for the  $k$ th user can be written as

$$\begin{aligned} y_k[m] & \triangleq \int_{-\infty}^{\infty} w_k(t - mT)^H r(t) dt \\ & = \sqrt{E_s} \sum_{k'=1}^K e^{j\theta_{k'}} \\ & \quad \times \sum_{m'=-\infty}^{\infty} d_{k'}[m'] c_{k,k'}((m - m')T - T_{k'}) + n_k[m] \end{aligned} \quad (7)$$

where

$$c_{k,k'}(t) \triangleq \int_{-\infty}^{\infty} w_k(\tau - t)^H s_{k'}(\tau) d\tau$$

is the cross correlation between  $w_k(t)$  and  $s_{k'}(t)$ , and

$$n_k[m] \triangleq \int_{-\infty}^{\infty} w_k(t - mT)^H n(t) dt$$

is the Gaussian noise component with mean 0 and variance  $N_0 \|w_k\|^2$ .

To make the notations simple, we let  $s_k(t)$  absorb the delay  $T_k$  and the effect of carrier phase  $\theta_k$ , so that we set  $T_k = \theta_k = 0, \forall k$ , without loss of generality. This is well justified because, if the  $k$ th optimum transmit waveform is  $s_k^*(t)$  with all-zero delays and phases, the solution with delay  $T_k$  and phase  $\theta_k$  is simply  $s_k^*(t + T_k)e^{-j\theta_k}$ .

By substituting (7) into (4) and (5), we obtain

$$\text{MSE}_k = 1 - 2\Re\{\sqrt{E_s} c_{k,k}(0)\} + N_0 \|w_k\|^2 + E_s \sum_{k'=1}^K \phi_{k,k'} \quad (8)$$

and

$$\text{SINR}_k = \frac{|c_{k,k}(0)|^2}{\frac{N_0}{E_s} \|w_k\|^2 + \sum_{k'=1}^K \phi_{k,k'} - |c_{k,k}(0)|^2} \quad (9)$$

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$$\phi_{k,k'} \triangleq \sum_{l=-\infty}^{\infty} |c_{k,k'}(lT)|^2.$$

Now, the Poisson summation formula leads to

$$\phi_{k,k'} = \sum_{l=-\infty}^{\infty} q_{k,k'}(lT) = \frac{1}{T} \sum_{l=-\infty}^{\infty} Q_{k,k'}\left(\frac{l}{T}\right)$$

where  $Q_{k,k'}(\xi)$  is the Fourier transform of  $q_{k,k'}(t) \triangleq |c_{k,k'}(t)|^2$ . If we define  $W_k(\xi)$  and  $S_k(\xi)$ , respectively, as the Fourier transforms of  $w_k(t)$  and  $s_k(t)$  for each  $k$ ,  $Q_{k,k'}(l/T)$  can be expressed in the frequency domain as

$$\begin{aligned} Q_{k,k'}\left(\frac{l}{T}\right) & = \int_{-\infty}^{\infty} W_k(\xi)^H S_{k'}(\xi) \\ & \quad S_{k'}\left(\xi - \frac{l}{T}\right)^H W_k\left(\xi - \frac{l}{T}\right) d\xi. \end{aligned}$$

Dividing the integration interval into length- $1/T$  subintervals and rearranging the terms, we obtain

$$\phi_{k,k'} = \frac{1}{T} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \mathbf{w}_k(f)^H \mathbf{s}_{k'}(f) \mathbf{s}_{k'}(f)^H \mathbf{w}_k(f) df$$

where  $\mathbf{w}_k(f)$  and  $\mathbf{s}_{k'}(f)$  are, respectively, the vectorized Fourier transforms [18] of  $w_k(t)$  and  $s_{k'}(t)$  with sampling period  $1/T$ , defined as

$$\mathbf{w}_k(f) \triangleq \begin{bmatrix} W_k\left(-\frac{L}{T} + f\right) \\ W_k\left(-\frac{L-1}{T} + f\right) \\ \vdots \\ W_k\left(\frac{L}{T} + f\right) \end{bmatrix} \quad (10)$$

and

$$\mathbf{s}_{k'}(f) \triangleq \begin{bmatrix} S_{k'}\left(-\frac{L}{T} + f\right) \\ S_{k'}\left(-\frac{L-1}{T} + f\right) \\ \vdots \\ S_{k'}\left(\frac{L}{T} + f\right) \end{bmatrix} \quad (11)$$

where  $L \triangleq \lceil \beta/2 \rceil$ . Similarly, the other two terms in the MSE expression can be written as

$$c_{k,k}(0) = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \mathbf{w}_k(f)^H \mathbf{s}_k(f) df$$

and

$$\|w_k\|^2 = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \mathbf{w}_k(f)^H \mathbf{w}_k(f) df.$$

Combining all these expressions, we obtain

$$\begin{aligned} \text{MSE}_k & = 1 + \int_{-\frac{1}{2T}}^{\frac{1}{2T}} [\mathbf{w}_k(f)^H \mathbf{R}(f) \mathbf{w}_k(f) \\ & \quad - 2\Re\{\sqrt{E_s} \mathbf{w}_k(f)^H \mathbf{s}_k(f)\}] df \end{aligned}$$

which yields the frequency-domain expression of the objective functional, the total MSE, as

$$\begin{aligned} \text{TMSE} & = K + \sum_{k=1}^K \int_{-\frac{1}{2T}}^{\frac{1}{2T}} [\mathbf{w}_k(f)^H \mathbf{R}(f) \mathbf{w}_k(f) \\ & \quad - 2\Re\{\sqrt{E_s} \mathbf{w}_k(f)^H \mathbf{s}_k(f)\}] df \end{aligned} \quad (12)$$

where the frequency-domain correlation matrix  $\mathbf{R}(f)$  of the received signal is defined as

$$\mathbf{R}(f) \triangleq N_0 \mathbf{I} + \frac{E_s}{T} \sum_{k'=1}^K \mathbf{s}_{k'}(f) \mathbf{s}_{k'}^H(f) \quad (13)$$

for  $f \in [-1/(2T), 1/(2T))$ , where  $\mathbf{I}$  is the identity matrix.

Although the nominal dimension of the above vector-valued functions and the matrix-valued functions are  $2L + 1$  and  $(2L + 1)$ -by- $(2L + 1)$ , some of the entries must take zero values on some intervals of  $f$  because the signals are assumed strictly band-limited. We adopt the notions of effective vectorized Fourier transform and effective matrix-valued frequency function as defined in [18, Definition 7], so that all the vector-valued and matrix-valued functions are effective ones in the remainder of this paper. We also introduce the following two definitions related to the dimension of effective vector-valued functions.

**Definition 1:** The degrees of freedom  $N(f)$  for  $f \in [-1/(2T), 1/(2T))$  is defined as the dimension of the effective vectorized Fourier transform  $\mathbf{s}_k(f)$ .

By [18, Section IV-A], the degrees of freedom  $N(f)$  as a function of  $f$  is obtained as

$$N(f) = \begin{cases} 1 + \lceil \beta \rceil, & \text{for } |f| < \frac{1+\beta-\lceil \beta \rceil}{2T} \\ \lceil \beta \rceil, & \text{otherwise} \end{cases} \quad (14a)$$

for even  $\lceil \beta \rceil$ , and

$$N(f) = \begin{cases} \lceil \beta \rceil, & \text{for } |f| < \frac{\lceil \beta \rceil - \beta}{2T} \\ 1 + \lceil \beta \rceil, & \text{otherwise} \end{cases} \quad (14b)$$

for odd  $\lceil \beta \rceil$ .

**Definition 2:** The average degrees of freedom  $N$  is defined as

$$N \triangleq T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} N(f) df. = \frac{\int_{-\frac{1}{2T}}^{\frac{1}{2T}} N(f) df}{\frac{1}{T}} \quad (15)$$

From (14), we can easily compute the average degrees of freedom as

$$\begin{aligned} N &= (1 + \lceil \beta \rceil)(1 + \beta - \lceil \beta \rceil) + \lceil \beta \rceil(\lceil \beta \rceil - \beta) \\ &= 1 + \beta \\ &= WT \end{aligned}$$

whether  $\lceil \beta \rceil$  is odd or even. Note that, unlike the processing gain of a CDMA system, this number is not restricted to a positive integer. It can be any positive real number.

From (12), note that the objective functional is minimized by minimizing

$$\sum_{k=1}^K (\mathbf{w}_k(f)^H \mathbf{R}(f) \mathbf{w}_k(f) - 2\Re\{\sqrt{E_s} \mathbf{w}_k(f)^H \mathbf{s}_k(f)\})$$

for each  $f$ . Note also that this minimization can be performed by minimizing the summand for each  $k$ . Hence, although the problem is posed as a joint transmitter and receiver optimization, the problem can be regarded as an optimal signal design problem with linear MMSE receivers being employed. If we apply the well-known solution of an unconstrained quadratic minimization [23, Ch. 6] for each  $k$  and  $f$ , we have

$$\mathbf{w}_k^*(f) = \sqrt{E_s} \mathbf{R}(f)^{-1} \mathbf{s}_k(f) \quad (16)$$

for  $f \in [-1/(2T), 1/(2T))$ , which is the cyclic Wiener filtering solution [24] for the mean-square estimation of  $d_k[m]$ . With this solution, the MMSE is obtained as

$$\text{MMSE}_k = 1 - E_s \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \mathbf{s}_k(f)^H \mathbf{R}(f)^{-1} \mathbf{s}_k(f) df. \quad (17)$$

Consequently, the total MSE after receive waveform optimization reduces to

$$\text{TMSE} = K - E_s \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left[ \sum_{k=1}^K \mathbf{s}_k(f)^H \mathbf{R}(f)^{-1} \mathbf{s}_k(f) \right] df. \quad (18)$$

Since the minimization of  $\text{MSE}_k$  is equivalent to maximizing  $\text{SINR}_k$ , the receiver-only maximized SINR becomes

$$\text{MSINR}_k = \left( \frac{1}{E_s \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \mathbf{s}_k(f)^H \mathbf{R}(f)^{-1} \mathbf{s}_k(f) df} - 1 \right)^{-1}. \quad (19)$$

We can see from (17) and (19) that, as it is well known [25], the MMSE and the MSINR are related by

$$\text{MSINR}_k = \frac{1}{\text{MMSE}_k} - 1 \quad (20a)$$

or by

$$\text{MMSE}_k = \frac{1}{\text{MSINR}_k + 1}. \quad (20b)$$

### C. Transmitter Optimization

In this subsection, we solve our original joint optimization problem *Problem 1*, using the simplified total MSE (18) obtained after the receiver-only optimization performed in the previous subsection. More specifically, we solve the problem given by the following.

*Problem 2:*

$$r^{\frac{1}{2T}} \left\lceil \frac{K}{\dots} \right\rceil$$

whether  $\lceil \beta \rceil$  is odd or even. Note that, unlike the processing gain of a CDMA system, this number is not restricted to a positive integer. It can be any positive real number.

### B. Receiver-Only Optimization

The joint optimization is tackled by first finding the receive waveforms that minimize the total MSE given the transmit waveforms and then finding the transmit waveforms that minimize the total MSE expressed as a functional only of transmit waveforms. In this subsection, we find the vectorized Fourier transform  $\mathbf{w}_k^*(f)$  of the optimum receive waveform given transmit waveforms, and derive the receiver-only minimized total MSE as a functional of  $\{\mathbf{s}_k(f)\}_{k=1}^K$ .

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vious subsection. More specifically, we solve the problem given by the following.

*Problem 2:*

$$\begin{aligned} & \underset{\{\mathbf{s}_k(f)\}_{k=1}^K}{\text{minimize}} && K - E_s \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left[ \sum_{k=1}^K \mathbf{s}_k(f)^H \mathbf{R}(f)^{-1} \mathbf{s}_k(f) \right] df \\ & \text{subject to} && \int_{-\frac{1}{2T}}^{\frac{1}{2T}} a(f) df = K \end{aligned} \quad (21)$$

where the total energy density  $a(f)$  of the transmit waveforms  $\{\mathbf{s}_k(t)\}_{k=1}^K$  is defined as

$$a(f) \triangleq \sum_{k=1}^K a_k(f) \quad (22)$$

3180

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 51, NO. 9, SEPTEMBER 2005

with the energy density  $a_k(f)$  of the vectorized Fourier transform  $\mathbf{s}_k(f)$  defined as

$$a_k(f) \triangleq \mathbf{s}_k(f)^H \mathbf{s}_k(f), \quad \text{for } k = 1, 2, \dots, K.$$

Notice in (21) that the original subsidiary condition  $\|\mathbf{s}_k(t)\| = 1, \forall k$ , is relaxed to  $\sum_{k=1}^K \|\mathbf{s}_k(t)\|^2 = K$ . After finding the solution to Problem 2, we will put the original condition back.

If we define the matrix  $\mathbf{S}(f)$  as

$$\mathbf{S}(f) \triangleq [\mathbf{s}_1(f), \mathbf{s}_2(f), \dots, \mathbf{s}_K(f)] \quad (23)$$

where the columns are the vectorized Fourier transforms of the transmit waveforms, the second term of the objective functional in Problem 2 can be written as<sup>3</sup>

$$\begin{aligned} & E_s \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left[ \sum_{k=1}^K \mathbf{s}_k(f)^H \mathbf{R}(f)^{-1} \mathbf{s}_k(f) \right] df \\ &= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \text{tr} \{ \mathbf{S}(f) \mathbf{S}(f)^H (\sigma^2 \mathbf{I} + \mathbf{S}(f) \mathbf{S}(f)^H)^{-1} \} df \\ &= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sum_{n=1}^{N(f)} \frac{\lambda_n(f)}{\lambda_n(f) + \sigma^2} df \end{aligned} \quad (24)$$

where  $\text{tr}\{\cdot\}$  denotes the trace,  $N(f)$  is the degrees of freedom,  $\{\lambda_n(f)\}_{n=1}^{N(f)}$  are the eigenvalues corresponding to the orthonormal eigenvectors of  $\mathbf{S}(f) \mathbf{S}(f)^H$ , and  $\sigma^2 \triangleq \frac{N_0 T}{E_s}$ .

Now, we are ready to tackle the transmit waveform optimization. The optimization is performed in three steps: 1) The minimum value is obtained given the distribution of the total energy density. 2) The optimum distribution of the total energy is found.

*Lemma 1:* If  $A, B$ , and  $C$  are all positive, the function  $f(x)$  of  $x \in [0, C]$ , defined as

$$f(x) \triangleq Ax/(A+x) + B(C-x)/(B+(C-x))$$

has the unique maximum  $(1/(A+B) + 1/C)^{-1}$  at  $x = x^* \triangleq AC/(A+B)$ .

*Proof:* Straightforward by showing  $f'(x) = 0$  iff  $x = x^*$  and  $f''(x) < 0$  for  $0 \leq x \leq C$ .  $\square$

*Proposition 2:* The solution to the variational problem

$$\begin{aligned} & \underset{a(f)}{\text{minimize}} && K - T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{N(f)a(f)}{a(f) + \sigma^2 N(f)} df \\ & \text{subject to} && a(f) \geq 0, \forall f, \quad \text{and} \quad \int_{-\frac{1}{2T}}^{\frac{1}{2T}} a(f) df = K \end{aligned}$$

is given by

$$a^*(f) = \left( \frac{K}{N} \right) TN(f) \quad (27)$$

for  $f \in [-1/(2T), 1/(2T))$ .

*Proof:* Suppose that the integration of  $a(f)$  over

$$f \in \{f : N(f) = 1 + \lceil \beta \rceil\}$$

equals  $x$ , for some  $x \in [0, K]$ . Note that the integration interval corresponds to  $\{f : |f| < (1 + \beta - \lceil \beta \rceil)/(2T)\}$  for an even number of  $\lceil \beta \rceil$ , while  $\{f : |f| > (\lceil \beta \rceil - \beta)/(2T)\}$  for an odd number of  $\lceil \beta \rceil$ . By dividing the intervals  $f \in \{f : N(f) = 1 + \lceil \beta \rceil\}$  and  $f \in \{f : N(f) = \lceil \beta \rceil\}$ , respectively, into  $M_1$  and  $M_2$  equal-length subintervals, and then letting  $M_1$  and  $M_2$

Now, we are ready to tackle the transmit waveform optimization. The optimization is performed in three steps: 1) The minimum value is obtained given the distribution of the total energy density. 2) The optimum distribution of the total energy is found. 3) Finally, a necessary and sufficient condition for the optimality is found.

*Proposition 1:* Given  $a(f) \geq 0$  for all  $f$ , the solution to the optimization problem

$$\begin{aligned} & \underset{\{\lambda_n(f)\}_{n=1}^{N(f)}}{\text{minimize}} \quad K - T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sum_{n=1}^{N(f)} \frac{\lambda_n(f)}{\lambda_n(f) + \sigma^2} df \\ & \text{subject to } \lambda_n(f) \geq 0, \quad \forall n \text{ and } \sum_{n=1}^{N(f)} \lambda_n(f) = a(f) \end{aligned} \quad (25)$$

is given by

$$\lambda_n^*(f) = \frac{a(f)}{N(f)} \quad (26)$$

for  $n \in \{1, 2, \dots, N(f)\}$  and  $f \in [-1/(2T), 1/(2T)]$ .

*Proof:* By using, for each  $f$  [9, Example 3.1] based on the majorization theory [27], we straightforwardly obtain the conclusion.  $\square$

Now, with the minimized objective functional found by substituting the solution (26) to (25), we take the second step.

<sup>3</sup>The approach is similar to those taken in [9] and [26], where the observations are in a finite-dimensional Euclidean space.

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number of  $\lceil \beta \rceil$ , while  $\{f : |f| > (\lceil \beta \rceil - \beta)/(2T)\}$  for an odd number of  $\lceil \beta \rceil$ . By dividing the intervals  $f \in \{f : N(f) = 1 + \lceil \beta \rceil\}$  and  $f \in \{f : N(f) = \lceil \beta \rceil\}$ , respectively, into  $M_1$  and  $M_2$  equal-length subintervals, and then letting  $M_1$  and  $M_2$  tend to infinity, we have the equalities

$$\begin{aligned} & \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{N(f)a(f)}{a(f) + \sigma^2 N(f)} df \\ &= \int_{f \in \{f : N(f)=1+\lceil \beta \rceil\}} \frac{(1 + \lceil \beta \rceil)a(f)}{a(f) + \sigma^2(1 + \lceil \beta \rceil)} df + \int_{f \in \{f : N(f)=\lceil \beta \rceil\}} \frac{\lceil \beta \rceil a(f)}{a(f) + \sigma^2 \lceil \beta \rceil} df \\ &= \lim_{M_1 \rightarrow \infty} \sum_{i=1}^{M_1} \frac{a(f_i)(1 + \lceil \beta \rceil)}{a(f_i) + \sigma^2(1 + \lceil \beta \rceil)} \cdot \frac{1 + \beta - \lceil \beta \rceil}{TM_1} \\ &+ \lim_{M_2 \rightarrow \infty} \sum_{j=1}^{M_2} \frac{a(f_j)\lceil \beta \rceil}{a(f_j) + \sigma^2 \lceil \beta \rceil} \cdot \frac{\lceil \beta \rceil - \beta}{TM_2} \end{aligned} \quad (28)$$

where we used the piecewise continuity of  $a(f)$ . By applying Example 3.1 of [9] to each summation, we obtain an upper bound on (28) given by  $f(x)/\sigma^2$  with  $A$ ,  $B$ , and  $C$  defined as

$$\begin{aligned} A &\triangleq (1 + \beta - \lceil \beta \rceil)(1 + \lceil \beta \rceil)\sigma^2/T \\ B &\triangleq (\lceil \beta \rceil - \beta)(\lceil \beta \rceil)\sigma^2/T \end{aligned}$$

and

$$C \triangleq K$$

respectively, where the bound is achieved if and only if

$$a(f) = xT/(1 + \beta - \lceil \beta \rceil), \quad \text{for } f \in \{f : N(f) = 1 + \lceil \beta \rceil\}$$

and

$$a(f) = (K - x)T/(\lceil \beta \rceil - \beta), \quad \text{for } f \in \{f : N(f) = \lceil \beta \rceil\}.$$

Now we apply Lemma 1 to maximize  $f(x)/\sigma^2$ . Since the optimum value of  $x$  is  $x^* = \frac{AC}{A+B}$ , the optimal total energy allotment is

$$a^*(f) = \begin{cases} (1 + \lceil \beta \rceil) \frac{KT}{1 + \beta}, & \text{for } f \in \{f : N(f) = 1 + \lceil \beta \rceil\} \\ \lceil \beta \rceil \frac{KT}{1 + \beta}, & \text{for } f \in \{f : N(f) = \lceil \beta \rceil\} \end{cases}$$

which simplifies to (27).  $\square$

Note that with the above optimum solution, the minimized total MSE becomes

$$\sum_{k=1}^K \text{MMSE}_k^* = K \left( 1 - \frac{1}{\frac{K}{N} + \frac{N_0}{E_s}} \right). \quad (29)$$

than one subinterval and assigning different sets of WBE sequences to different subintervals, we can use even more than two WBE sequence sets in the construction of a set of CTE-WBE sequences. The lengths of these subintervals can be chosen in infinitely many different ways, which also leads to different CTE-WBE sequences. Therefore, the conclusion follows.  $\square$

In terms of a set of CTE-WBE sequences, the optimum transmit waveforms can be described as follows.

*Theorem 1:* The set of the optimum transmit waveforms is a CTE-WBE sequence set with parameter  $(K, W, T)$ , and the optimum receivers are matched filters matched to each transmit waveforms.

*Proof:* From (26) and (27), we can see that the necessary and sufficient condition for  $\{\mathbf{s}_k^*(f)\}_{k=1}^K$  to be an optimum solution is given by

total MSE becomes

$$\sum_{k=1}^K \text{MMSE}_k^* = K \left( 1 - \frac{1}{\frac{K}{N} + \frac{N_0}{E_s}} \right). \quad (29)$$

Before the final solution is presented, we define the set of the CTE-WBE sequences as follows.

**Definition 3:** A set of CTE-WBE sequences with parameter  $(K, W, T)$  is defined, when  $K \geq WT$ , as a set of  $K$  unit-energy complex-valued waveforms with bandwidth  $W/2$ , satisfying

$$\mathbf{S}(f)\mathbf{S}(f)^H = \left(\frac{K}{N}\right) T \mathbf{I}_{N(f)} \quad (30)$$

for  $f \in [-1/(2T), 1/(2T))$ .

The existence and the uniqueness of a set of CTE-WBE sequences are examined as follows.

**Remark 1:** A set of CTE-WBE sequences with parameter  $(K, W, T)$  exists.

**Proof:** We prove by construction. Thanks to the existence of a WBE sequence set [5] for any  $K$  and  $N(f) (\leq K)$ , we can find two WBE sequence sets  $\{\mathbf{u}_k\}_{k=1}^K$  and  $\{\mathbf{v}_k\}_{k=1}^K$  with dimensions  $1 + \lceil \beta \rceil$  and  $\lceil \beta \rceil$ , respectively. Note that, by definition

$$\sum_{k=1}^K \mathbf{u}_k \mathbf{u}_k^H = \frac{K}{1 + \lceil \beta \rceil} \mathbf{I}_{1 + \lceil \beta \rceil}$$

and

$$\sum_{k=1}^K \mathbf{v}_k \mathbf{v}_k^H = \frac{K}{\lceil \beta \rceil} \mathbf{I}_{\lceil \beta \rceil}.$$

Thus, if we set for all  $k$

$$\mathbf{s}_k(f) = \sqrt{(1 + \lceil \beta \rceil)/(1 + \beta)} \mathbf{u}_k, \quad \text{for } f \in \{f : N(f) = 1 + \lceil \beta \rceil\}$$

and

$$\mathbf{s}_k(f) = \sqrt{(\lceil \beta \rceil)/(1 + \beta)} \mathbf{v}_k, \quad \text{for } f \in \{f : N(f) = \lceil \beta \rceil\}$$

then  $\{\mathbf{s}_k(f)\}_{k=1}^K$  becomes a set of CTE-WBE sequences satisfying (30).  $\square$

**Remark 2:** A set of CTE-WBE sequences with parameter  $(K, W, T)$  is not unique.

**Proof:** During the construction of a set of CTE-WBE sequences described in Remark 1, we have more than one choices of WBE sequence sets because, given  $K$  and  $N(f)$ , a WBE sequence set is not uniquely determined. By dividing the intervals  $\{f : N(f) = 1 + \lceil \beta \rceil\}$  and  $\{f : N(f) = \lceil \beta \rceil\}$  into more

waveforms.

**Proof:** From (26) and (27), we can see that the necessary and sufficient condition for  $\{\mathbf{s}_k^*(f)\}_{k=1}^K$  to be an optimum solution is given by

$$\lambda_n^*(f) = \left(\frac{K}{N}\right) T \triangleq \lambda^* \quad (31)$$

for  $n = 1, 2, \dots, N(f)$  and  $f \in [-1/(2T), 1/(2T))$ . If  $\mathbf{S}^*(f)$  satisfies

$$\mathbf{H}(f) \triangleq \mathbf{S}^*(f) \mathbf{S}^*(f)^H = \left(\frac{K}{N}\right) T \mathbf{I}_{N(f)} \quad (32)$$

then, obviously, (31) is true. Let  $h_{i,j}(f)$  be the  $(i, j)$ th entry of the Hermitian matrix  $\mathbf{H}(f)$  and  $\mathbf{v}_k(f)$  be the  $k$ th orthonormal eigenvector of  $\mathbf{H}(f)$ . Then, there exists  $\{a_{i,n}(f)\}_{n=1}^{N(f)}$  such that

$$\mathbf{e}_i = \sum_{n=1}^{N(f)} a_{i,n}(f) \mathbf{v}_n(f)$$

where  $\mathbf{e}_i$  is the  $i$ th standard basis vector of the  $N(f)$ -dimensional Euclidean space. Thus, if the equality in (31) holds, then (32) is true because

$$h_{i,j}(f) = \mathbf{e}_i^H \mathbf{H}(f) \mathbf{e}_j = \mathbf{e}_i^H \left( \sum_{n=1}^{N(f)} a_{j,n}(f) \mathbf{H}(f) \mathbf{v}_n(f) \right) = \sum_{n=1}^{N(f)} a_{j,n}(f) \mathbf{e}_i^H \mathbf{H}(f) \mathbf{v}_n(f)$$

In addition, from (16) and (32), we observe that the optimum receivers are the matched filters. Therefore, the conclusion follows.  $\square$

Note that Remark 1, the existence of equal energy waveforms satisfying (32), justifies the use of the relaxed subsidiary condition in (21). Note also that the carrier phases do not affect the optimality of the waveforms because

$$\mathbf{S}^*(f) \mathbf{S}^*(f)^H = \sum_{k=1}^K (\mathbf{s}_k^*(f) e^{j\theta_k}) (\mathbf{s}_k^*(f) e^{j\theta_k})^H$$

for any choice of  $\{\theta_k\}_{k=1}^K$ . We may be tempted to search for an alternate expression, of the above optimality condition, that is simply in terms of the cross-correlation values  $\int_{-\infty}^{\infty} s_k^*(t) s_{k'}^*(t) dt, \forall k, k'$ . Unfortunately, this is not possible, which is verified in the next section.

#### D. User Capacity Characterization

In this subsection, we characterize the user capacity of a strictly band-limited, proper-complex Gaussian multiple-access



channel in terms of the channel load, the SINR, and the received signal-to-noise ratio.

**Definition 4:** The channel load  $\rho$  of a strictly band-limited multiple-access channel with bandwidth  $W/2$ , an additive proper-complex white Gaussian noise, and  $K$  equal-power users having the common symbol transmission rate  $1/T$  is defined as

$$\rho \triangleq \frac{K}{N} \quad (33)$$

where  $N = WT = 1 + \beta$  is the average degrees of freedom.

Now, that the channel is overloaded can be alternately defined as  $\rho > 1$ . Similarly to [9], we introduce the notion of admissibility and find a necessary and sufficient condition for the user admissibility.

**Definition 5:**  $K$  equal-power users with the same symbol transmission rate  $1/T$  and the same SINR requirement  $\gamma$  are admissible to the strictly band-limited proper-complex Gaussian multiple-access channel with bandwidth  $W/2$  if there exist pairs of transmit and receive waveforms with a feasible received symbol energy  $E_s$  such that  $\text{SINR}_k \geq \gamma, \forall k \in \{1, 2, \dots, K\}$ .

**Theorem 2:** Users are admissible if and only if

$$\gamma < \frac{1}{\rho - 1}. \quad (34)$$

*Proof:* By using the relation (20) and the result (29), the jointly maximized MSINR and its upper bound can be easily obtained as

$$\text{MSINR}_k^* = \frac{1}{(\rho - 1) + \frac{N_0}{E_s}} < \frac{1}{\rho - 1} \quad (35)$$

for all  $k$ . If  $K$  users are admissible, then by Definition 5,  $\gamma \leq \text{SINR}_k, \forall k$ . Since  $\text{SINR}_k$  is upper-bounded by  $\text{MSINR}_k^*$  and  $\text{MSINR}_k^*$  is upper-bounded by  $1/(\rho - 1)$ , we have (34). Conversely, if (34) is satisfied, then, by the intermediate-value theorem, there exists  $x > 0$  such that

$$\gamma \leq 1/(\rho - 1 + x) < 1/(\rho - 1).$$

Thus, by choosing this  $x$  as  $N_0/E_s$ , CTE-WBE sequences with parameter  $(K, W, T)$  as the transmit waveforms, and matched filters as the receive waveforms, we achieve

$$\text{SINR}_k = 1/(\rho - 1 + N_0/E_s), \quad \forall k.$$

Hence, the users are admissible.  $\square$

A major difference from the results in [9] and [12] is that the channel load  $\rho$  defined in this paper can be any real number, not alone a rational number, that is larger than one. Another major difference is that the SINR in [9] and [12] is defined as twice the SINR defined in this paper because a real baseband signaling is concerned in the references.<sup>4</sup>

#### E. Alternate Definition of CTE-WBE Sequences

defined. The basic CDMA model [28] with equal-power users is given by

$$\mathbf{r} = \sum_{k=1}^K \sqrt{E_s} d_k \mathbf{s}_k + \mathbf{n} \quad (36)$$

where  $K$  is the number of users,  $E_s$  is the symbol energy,  $d_k$  is the data symbol of the  $k$ th user,  $\mathbf{s}_k$  is the  $k$ th user's normalized spreading sequence, and  $\mathbf{n}$  is an additive proper-complex Gaussian noise with mean zero and covariance matrix  $N_0 \mathbf{I}$ , where all the vectors are  $N$ -dimensional. The data symbols are modeled as zero-mean, independent, proper-complex random variables with unit variance.

If each user employs a matched-filter receiver matched to the desired sequence, then the  $k$ th user's decision statistic  $y_k$  is given by

$$y_k = \mathbf{s}_k^H \mathbf{r} = \sqrt{E_s} d_k + \sum_{k' \neq k} \sqrt{E_s} d_{k'} \mathbf{s}_k^H \mathbf{s}_{k'} + n_k \quad (37)$$

where  $n_k \triangleq \mathbf{s}_k^H \mathbf{n}$ . If the SINR of the  $k$ th user  $\gamma_k$  is defined as

$$\gamma_k \triangleq \frac{\mathbb{E}(|\mathbb{E}\{y_k|d_k\}|^2)}{\text{Var}\{y_k|d_k\}} \quad (38)$$

then the well-known expression

$$\gamma_k = \frac{E_s}{N_0 + \sum_{k' \neq k} E_s \mathbf{s}_k^H \mathbf{s}_{k'}} \quad (39)$$

is obtained. The TSC

$$\text{TSC} \triangleq \sum_{k=1}^K \sum_{k'=1}^K |\mathbf{s}_k^H \mathbf{s}_{k'}|^2 \quad (40)$$

thus can be viewed as the summation of the signal power and the interference power for all matched-filter outputs, divided by the symbol energy.

Welch derived a tight lower bound on the TSC as [5]

$$\sum_{k=1}^K \sum_{k'=1}^K |\mathbf{s}_k^H \mathbf{s}_{k'}|^2 \geq \frac{K^2}{N} \quad (41)$$

for the cases with  $K > N$ . A WBE sequence set is defined as a set of sequences that achieves this lower bound on the TSC.

Now, let us consider a continuous-time CDMA model given by (2). If each user employs a matched-filter receiver matched to the desired signature waveform, then the  $k$ th user's decision statistic  $\hat{y}_k[l]$  for the  $l$ th symbol is given by

$$\begin{aligned} \hat{y}_k[l] &= \int_{-\infty}^{\infty} s_k(t - lT - T_k)^* e^{-j\theta_k} r(t) dt \\ &= \sqrt{E_s} d_k[m] \end{aligned} \quad (42a)$$

$$+ \sum_{k' \neq k} \sum_{m=-\infty}^{\infty} \sqrt{E_s} d_{k'}[m] \tilde{c}_{k,k'}((l-m)T) \quad (42b)$$

$$+ \sum_{m \neq l} \sqrt{E_s} d_k[m] \tilde{c}_{k,k}((l-m)T) \quad (42c)$$

$$+ \tilde{n}_k[m] \quad (42d)$$

where the correlation function  $\tilde{c}_{k,k'}(\tau)$  between two signature

In this subsection, we show that a CTE-WBE sequence set can be alternately defined similarly as a WBE sequence set is

<sup>4</sup>Hence, if the channel is occupied by a single user, the SINR in the references reduces to  $2E_s/N_0$  while that in this paper reduces to  $E_s/N_0$ .

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where the correlation function  $c_{k,k'}(t)$  between two signature waveforms  $s_k(t)$  and  $s_{k'}(t)$  is defined as

$$\tilde{c}_{k,k'}(t) \triangleq \int_{-\infty}^{\infty} s_k(\tau - t)^* s_{k'}(\tau) d\tau \quad (43)$$

CHO AND GAO: CONTINUOUS-TIME EQUIVALENTS OF WELCH BOUND EQUALITY SEQUENCES

and the matched-filter output of the AWGN sampled at  $t = mT$  is given by

$$\tilde{n}_k[m] \triangleq \int_{-\infty}^{\infty} s_k(\tau - mT - T_k)^* e^{-j\theta_k} n(\tau) d\tau. \quad (44)$$

Unlike (37), this decision statistic includes self-induced inter-symbol interference (ISI) term (42c). Thus, if the SINR of the  $k$ th user  $\tilde{\gamma}_k$  is similarly defined as

$$\tilde{\gamma}_k \triangleq \frac{E(|\tilde{y}_k[m]|^2 | d_k[m])}{\text{Var}\{\tilde{y}_k[m] | d_k[m]\}} \quad (45)$$

then expression (46) at the bottom of the page is obtained.

Since the TSC (40) for the basic CDMA model is the summation of the signal power and the interference power for all matched-filter outputs, divided by the symbol energy, we extend this idea to obtain the following definition.

**Definition 6:** The CTE-TSC is defined as

$$\begin{aligned} \text{CTE-TSC} &\triangleq \sum_{k=1}^K \sum_{k'=1}^K \left( \sum_{l=-\infty}^{\infty} |s_k(-t)^H \otimes s_{k'}(t)|_{t=lT}^2 \right) \\ &= \sum_{k=1}^K \sum_{k'=1}^K \sum_{l=-\infty}^{\infty} |\tilde{c}_{k,k'}(lT)|^2 \end{aligned} \quad (47)$$

where  $\otimes$  denotes convolution.

It is straightforward to show that this CTE-TSC reduces to TSC if a symbol synchronous CDMA system without interchip interference is considered.

The following theorem shows that the CTE-WBE sequence set we defined and derived in the previous subsections can be alternately defined as a set of waveforms that minimizes the CTE-TSC.

**Theorem 3:** A set of  $K(> WT)$  unit-energy, complex-valued waveforms with bandwidth  $W/2$  is a CTE-WBE sequence set with parameter  $(K, W, T)$  if and only if the sequence set achieves the lower bound on the CTE-TSC given by

$$\sum_{k=1}^K \sum_{k'=1}^K \left( \sum_{l=-\infty}^{\infty} |s_k(-t)^H \otimes s_{k'}(t)|_{t=lT}^2 \right) \geq \frac{K^2}{N}. \quad (48)$$

**Proof:** The left side of (48) equals  $\sum_{k=1}^K \sum_{k'=1}^K \phi_{k,k'}$  with  $w_k(t) := s_k(t)$ . Hence, using the same technique as (24), it can be rewritten as

Adopting the relaxed subsidiary condition  $\int_{\frac{T}{2T}}^{\frac{T}{2T}} a(f) df = K$  as in Section III-C, we perform the minimization of (49) for each  $f$  similarly to Proposition 1, except for the use of the solution  $x_i = c/M, \forall i$  to the minimization of  $\sum_{i=1}^M x_i^2$  subject to  $\sum_{i=1}^M x_i = c > 0$  for  $x_i \geq 0$ . This optimization leads to the same solution (26) as Proposition 1. The next step is the minimization of  $(1/T) \int_{\frac{T}{2T}}^{\frac{T}{2T}} a(f)^2/N(f) df$  subject to the subsidiary condition. We again divide the integration interval into  $\{f : N(f) = 1 + \lceil \beta \rceil\}$  and  $\{f : N(f) = \lceil \beta \rceil\}$ , and set  $x$  as the proof for Proposition 2. Then, applying the solution to the minimization of  $\sum_{i=1}^M x_i^2$  again, we obtain

$$a(f) = xT/(1 + \beta - \lceil \beta \rceil), \quad \text{for } \{f : N(f) = 1 + \beta\}$$

and

$$a(f) = (K - x)T/(\lceil \beta \rceil - \beta), \quad \text{for } \{f : N(f) = \lceil \beta \rceil\}.$$

Using the fact that the solution to the minimization  $x^2/A + (C - x)^2/B$  is  $x^* = AC/(A + B)$ , we obtain the same solution  $a^*(f) = (K/N)TN(f)$  as (27), which leads to the lower bound  $K^2/N$  as (48). Finally, the conclusion follows by Theorem 1.  $\square$

This theorem completes the generalization of the theory of the sequence optimization for discrete-time equal-power CDMA systems to continuous-time case.

#### IV. EXAMPLES AND DISCUSSIONS

Given the system bandwidth and the number of active users, the optimum solution for the joint transmitter and receiver optimization is not unique. In this section, we present two representative optimal multiple-access systems and provide discussions on CTE-WBE sequences.

**Example 1:** The FDMA system that has the matched-filter receivers and the energy spectral density of the transmit waveform  $s_k(t)$  given flat as

$$|S_k(\xi)|^2 = \begin{cases} \frac{K}{W}, & \text{for } \xi \in [(\frac{W}{K}) \frac{2k-2-K}{2}, (\frac{W}{K}) \frac{2k-K}{2}] \\ 0, & \text{elsewhere} \end{cases} \quad (50)$$

for  $k = 1, 2, \dots, K$ , is an optimum system.

**Proof:** For every  $f$ , the  $k$ th column of the  $N(f)$ -by- $K$  matrix  $\mathbf{S}(f)$  has at most one nonzero entry  $\sqrt{K/W} e^{j\theta_k(f)}$  with  $\theta_k(f)$  being the phase response of the  $k$ th-user signal,

TSC-K = TISI for MF-RF.

Def.

$$\text{TSC} = \mathbf{S}^H \mathbf{S}$$

$$= \text{tr}\{\mathbf{S} \mathbf{S}^H\}$$

where

$$\mathbf{S} = [\mathbf{s}_1 \ \mathbf{s}_2 \ \dots \ \mathbf{s}_K]$$

$$\mathbf{s}_k \in \mathbb{C}^N$$

$$N \leq K$$

$$\|\mathbf{s}_k\| = 1, \forall k$$

Theorem

$$\min_{\mathbf{S}} \text{TSC}$$

$$\Rightarrow \mathbf{S} \mathbf{S}^H = \frac{K}{N} \mathbf{I}_N$$

$$\Rightarrow \min \text{TSC} = \text{tr}(\mathbf{S} \mathbf{S}^H) = \frac{K^2}{N}$$

$$\Leftrightarrow \min \text{TISI} = (\frac{K}{N} - 1)K$$



Proof: The left side of (48) equals  $\sum_{k=1}^K \sum_{k'=1}^K \sum_{l=-\infty}^{\infty} \phi_{k,k'}(lT)$  with  $w_k(t) := s_k(t)$ . Hence, using the same technique as (24), it can be rewritten as

$$\begin{aligned} & \sum_{k=1}^K \sum_{k'=1}^K \phi_{k,k'} \\ &= \sum_{k=1}^K \frac{1}{T} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left( \mathbf{s}_k(f)^H \mathbf{S}(f)^H \mathbf{S}(f) \mathbf{s}_k(f) \right) df \\ &= \frac{1}{T} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \text{tr} \{ \mathbf{S}(f) \mathbf{S}(f)^H \mathbf{S}(f) \mathbf{S}(f)^H \} df \\ &= \frac{1}{T} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sum_{i=1}^{N(f)} |\lambda_i(f)|^2 df. \end{aligned} \quad (49)$$

for  $k = 1, 2, \dots, K$ , is an optimum system.

Proof: For every  $f$ , the  $k$ th column of the  $N(f)$ -by- $K$  matrix  $\mathbf{S}(f)$  has at most one nonzero entry  $\sqrt{K/W} e^{j\theta_k(f)}$  with  $\theta_k(f)$  being the phase response of the  $k$ th-user signal, because the bandwidth  $W/K = (1 + \beta)/(KT)$  is less than the inverse  $1/T$  of the sampling rate. In addition, for every  $f$ , each row of  $\mathbf{S}(f)$  has only one nonzero entry because the user frequency bands are not overlapping. Consequently,  $\mathbf{S}(f) \mathbf{S}(f)^H = (K/N) T \mathbf{I}_{N(f)}$ . Therefore, by Theorem 1, this system is optimum.  $\square$

As shown in Fig. 2, each user's signal takes up an equally divided frequency band with bandwidth  $W/K = (1 + \beta)/(KT)$ . All the magnitude responses are flat, so that we call such signals with flat energy spectral densities the flat-spectrum pulses. It is easy to see that, in general, the phase responses of the transmit

$$\Rightarrow \text{min } |ISI| = \left( \frac{K}{N} - 1 \right) K$$

$$\frac{1}{T} \int_{-T}^T |\mathbf{S}_k^H(f) \mathbf{S}_k(f)|^2 df = ISI_{k,k}$$

$$\tilde{\gamma}_k = \frac{E_s}{N_0 + \sum_{k' \neq k} \sum_{l=-\infty}^{\infty} E_s |\tilde{c}_{k,k'}(lT)|^2 + \sum_{l \neq 0} E_s |\tilde{c}_{k,k}(lT)|^2} \quad (46)$$

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$$[\mathbf{S}_1 \dots \mathbf{S}_K] \quad TSC = \sum_{k,k'} \sum_{l=1}^N |\mathbf{S}_k^H \mathbf{S}_{k'}|^2$$

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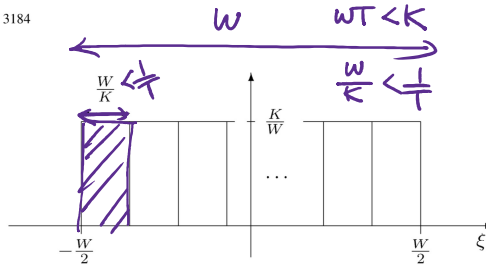


Fig. 2. Energy spectral densities of optimum FDMA signals.

waveforms are very important to maintain the optimality of the system. Note, however, that any phase responses can be chosen for each user's signal in this optimum solution. Even completely asynchronous users with arbitrary delays can achieve the optimum performance by adopting this FDMA scheme.

It is worth noting that this optimum FDMA signals cannot be constructed from sets of WBE sequences in such ways that are described in Remark 2. That means, the optimum FDMA signals are not isometric at all to those generated using WBE sequence sets. If a set of CTE-WBE sequences is constructed in such ways that are described in Remark 2, the condition  $a_k(f) = TN(f)/N$  for  $f \in [-1/(2T), 1/(2T)]$  and  $\forall k$ , must be satisfied because the WBE sequences in a set have the same energy for any offset  $f$ . However, it is straightforward to check the energy densities  $\{a_k^*(f)/K\}$  of this optimum FDMA

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 51, NO. 9, SEPTEMBER 2005

and  $S_k(\xi) = 0$  elsewhere, where  $\theta(\xi)$  is the phase response. Hence, the  $(i, j)$ th entry of  $\mathbf{S}(f) \mathbf{S}(f)^H$  becomes

$$[\mathbf{S}(f) \mathbf{S}(f)^H]_{i,j} = \frac{1}{W} \sum_{k=1}^K e^{-j2\pi i k} e^{j2\pi j k} = \frac{K}{W} \mathbf{d}_i^H \mathbf{d}_j$$

where  $\mathbf{d}_i^H$  is the  $(\text{mod}(i, K) + 1)$ th row of the  $K$ -dimensional discrete Fourier transform matrix. Consequently

$$\mathbf{S}(f) \mathbf{S}(f)^H = (K/N) T \mathbf{I}_{N(f)}.$$

Therefore, by Theorem 1, this system is optimum.  $\square$

The waveforms are flat-spectrum pulses that occupy the entire frequency band of the system. Like TDMA systems, the uniform time delays differentiate the users. However, the transmit waveforms of this TDMA-like system are not mutually orthogonal. Unlike the optimum FDMA, the phase responses of the flat-spectrum transmit waveforms of this optimum TDMA-like system cannot be arbitrarily chosen but must be exactly the same after the effect of the uniform time delays are compensated. A simple example can be found by setting  $\theta(\xi) = 0, \forall \xi$

$$s_k(t) = \sqrt{W} \text{sinc} \left( W \left( t - \frac{kT}{K} \right) \right) \quad (52)$$

for  $k = 1, 2, \dots, K$ .

Note that all the cross-correlation functions are not zero functions in the optimum TDMA-like system, while they are

$$\Leftrightarrow \frac{1}{T} \mathbf{S}(f) \mathbf{S}(f)^H = \frac{K}{N} \mathbf{I}_{N(f)}$$

in such ways that are described in Remark 2, the condition  $a_k(f) = TN(f)/N$  for  $f \in [-1/(2T), 1/(2T))$  and  $\forall k$ , must be satisfied because the WBE sequences in a set have the same energy for any offset  $f$ . However, it is straightforward to check the energy densities  $\{a_k^*(f)\}_{k=1}^K$  of this optimum FDMA signals do not satisfy the condition  $a_k(f) = TN(f)/N$  for  $f \in [-1/(2T), 1/(2T))$  and  $\forall k$ . For some  $f$ , there exists some  $k$  such that  $a_k^*(f) = 0$  because, for every  $f$ ,  $(N(f) - K)$  columns of  $\mathbf{S}^*(f)$  are all-zero vectors, which means that the energy densities of corresponding users are zero for that  $f$ . Therefore, the existence of this optimum FDMA signals proves that CTE-WBE sequences include more than what we can construct from WBE sequence sets.

According to our solution derived in the previous section, all the optimum multiple-access schemes have the same total MSE induced by the MAI, the ISI, the AWGN components. In other words, the total amount of interference is preserved as far as the system is optimum. For the above optimum FDMA, ISI is unavoidable because each user has the symbol transmission rate  $1/T$  that is less than the signal bandwidth  $W/K$ , violating the Nyquist minimum-bandwidth condition for zero ISI, while there is no multiple-access interference in each filter output because of the orthogonality of the transmit waveforms. The optimum FDMA system is thus a very special solution that has all the interference in the form of ISI. Let us discuss another special solution.

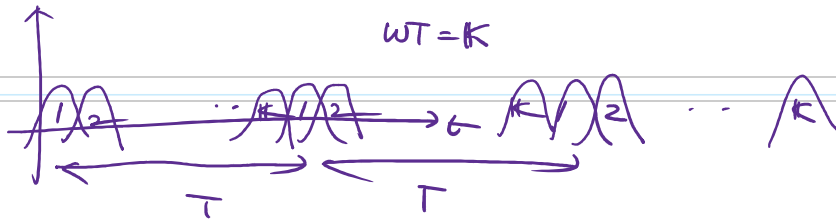
**Example 2:** The TDMA-like system that has the matched-filter receivers and the uniformly time-delayed transmit waveforms  $s_k(t) \triangleq p(t - \frac{kT}{K})$ , for  $k = 1, 2, \dots, K$ , is an optimum system if the energy spectral density of  $p(t)$  is given flat as

$$|P(\xi)|^2 = \begin{cases} \frac{1}{W}, & \text{for } |\xi| \leq \frac{W}{2} \\ 0, & \text{elsewhere.} \end{cases} \quad (51)$$

**Proof:** The Fourier transform of the transmit waveform  $s_k(t)$  can be written as

$$S_k(\xi) = \sqrt{1/W} \exp\left(-j\left(2\pi\xi \frac{kT}{K} - \theta(\xi)\right)\right), \text{ for } |\xi| \leq W/2$$

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for  $k = 1, 2, \dots, K$ .

Note that all the cross-correlation functions are not zero functions in the optimum TDMA-like system, while they are all zero functions in the optimum FDMA. This comparison of the optimum FDMA and optimum TDMA-like systems verifies our claim that the optimality of transmit waveforms cannot be simply described by the cross correlations among the signals.

## V. CONCLUSION

The optimal signaling problem for overloaded channels with the LMMSE receivers is solved in this paper via a frequency-domain approach using vectorized Fourier transforms. It is shown that CTE-WBE sequences are the optimum transmit waveforms and that the matched filters are the optimum linear receivers. The user capacity of the strictly band-limited channel is characterized by a necessary and sufficient condition on the channel load, the signal-to-noise ratio, and the SINR. An alternate definition of the CTE-WBE sequences is also provided as the sequences that achieve the lower bound on the CTE-TSC. It is shown that there exist an FDMA and a TDMA-like optimum systems that achieve the lower bound on the CTE-TSC. Thus, by this work, the theory developed on the sequence optimization for equal-power CDMA systems with the discrete-time or vector channel models [9] is successfully generalized to the continuous-time case. Generalizations for unequal-power systems can be found in [29], [30].

## REFERENCES

- [1] M. B. Pursley, "Performance evaluation for phase-coded spread-spectrum multiple-access communications—Part I: System analysis," *IEEE Trans. Commun.*, vol. COM-25, no. 8, pp. 795–799, Aug. 1977.
- [2] M. B. Pursley and D. V. Sarwate, "Performance evaluation for phase-coded spread-spectrum multiple-access communications—Part II: Code sequence analysis," *IEEE Trans. Commun.*, vol. COM-25, no. 8, pp. 800–803, Aug. 1977.
- [3] R. Gold, "Optimal binary sequences for spread spectrum multiplexing," *IEEE Trans. Inf. Theory*, vol. IT-13, no. 4, pp. 619–621, Oct. 1967.
- [4] —, "Maximal recursive sequences with 3-valued recursive cross-correlation functions," *IEEE Trans. Inf. Theory*, vol. IT-14, no. 1, pp. 154–156, Jan. 1968.
- [5] L. R. Welch, "Lower bounds on the maximum cross correlation of signals," *IEEE Trans. Inf. Theory*, vol. IT-20, no. 3, pp. 397–399, May 1974.
- [6] C.-C. Chen, K. Yao, and E. Biglieri, "Optimal spread spectrum sequences-constructed from Gold codes," in *Proc. IEEE Global Telecommunications Conf.*, 2000, vol. 2, Taipei, Taiwan, Nov. 2000, pp. 867–871.
- [7] M. Rupp and J. L. Massey, "Optimum sequence set for synchronous code-division multiple-access," *IEEE Trans. Inf. Theory*, vol. 40, no. 4, pp. 1261–1266, Jul. 1994.
- [8] G. Golden, J. Mazo, and J. Salz, "Transmitter design for data transmission in the presence of a data-like interferer," *IEEE Trans. Commun.*, vol. 43, no. 2/3/4, pp. 837–850, Feb./Mar./Apr. 1995.
- [9] P. S. Kumar and S. Roy, "Optimization for crosstalk suppression with noncoordinating users," *IEEE Trans. Commun.*, vol. 44, no. 7, pp. 894–905, Jul. 1996.
- [10] J. H. Cho, "Joint transmitter and receiver optimization in additive cyclostationary noise," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3396–3405, Dec. 2004.
- [11] J. M. Wozencraft and I. M. Jacobs, *Principles of Communication Engineering*. New York: Wiley, 1965.
- [12] J. S. Walker, *Fourier Analysis*. New York: Oxford Univ. Press, 1988.

- sequences-constructed from Gold codes," in *Proc. IEEE Global Telecommunications Conf., 2000*, vol. 2, Taipei, Taiwan, Nov. 2000, pp. 867–871.
- [7] M. Rupp and J. L. Massey, "Optimum sequence set for synchronous code-division multiple-access," *IEEE Trans. Inf. Theory*, vol. 40, no. 4, pp. 1261–1266, Jul. 1994.
- [8] P. Viswanath and V. Anantharam, "Optimal sequences and sum capacity of synchronous CDMA systems," *IEEE Trans. Inf. Theory*, vol. 45, no. 5, pp. 1984–1991, Sep. 1999.
- [9] P. Viswanath, V. Anantharam, and D. N. C. Tse, "Optimal sequences, power control, and user capacity of synchronous CDMA systems with linear MMSE multiuser receivers," *IEEE Trans. Inf. Theory*, vol. 45, no. 5, pp. 1968–1983, Sep. 1999.
- [10] S. Ulukus and R. D. Yates, "Iterative construction of optimum signature sequence sets in synchronous CDMA systems," *IEEE Trans. Inf. Theory*, vol. 47, no. 5, pp. 1989–1998, Jul. 2001.
- [11] P. Anigstein and V. Anantharam, "Ensuring convergence of the MMSE iteration for interference avoidance to the global optimum," *IEEE Trans. Inf. Theory*, vol. 49, no. 4, pp. 873–885, Apr. 2003.
- [12] S. Ulukus and R. D. Yates, "User capacity of asynchronous CDMA systems with optimum signature sequences," *IEEE Trans. Inf. Theory*, vol. 50, no. 5, pp. 903–909, May 2004.
- [13] G. S. Rajappan and M. L. Honig, "Signature sequence adaptation for DS-CDMA with multipath," *IEEE J. Sel. Areas Commun.*, vol. 20, no. 2, pp. 384–395, Feb. 2002.
- [14] T. Berger and D. W. Tufts, "Optimum pulse amplitude modulation Part I: Transmitter-receiver design and bounds from information theory," *IEEE Trans. Inf. Theory*, vol. IT-13, no. 2, pp. 196–208, Apr. 1967.
- [15] J. Yang and S. Roy, "On joint transmitter and receiver optimization for multiple-input-multiple-output (MIMO) transmission systems," *IEEE Trans. Commun.*, vol. 42, no. 12, pp. 3221–3231, Dec. 1994.
- [18] J. H. Cho, "Joint transmitter and receiver optimization in additive cyclostationary noise," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3396–3405, Dec. 2004.
- [19] J. M. Wozencraft and I. M. Jacobs, *Principles of Communication Engineering*. New York: Wiley, 1965.
- [20] J. S. Walker, *Fourier Analysis*. New York: Oxford Univ. Press, 1988.
- [21] H. Sagan, *Introduction to the Calculus of Variations*. New York: Dover, 1969.
- [22] J. H. Cho and J. S. Lehnert, "An optimal signal design for band-limited asynchronous DS-CDMA communications," *IEEE Trans. Inf. Theory*, vol. 48, no. 5, pp. 1172–1185, May 2002.
- [23] E. K. P. Chong and S. H. Zak, *An Introduction to Optimization*. New York: Wiley, 1996.
- [24] W. A. Gardner, "Cyclic Wiener filtering: Theory and method," *IEEE Trans. Commun.*, vol. 41, no. 1, pp. 151–163, Jan. 1993.
- [25] U. Madhow and M. L. Honig, "MMSE interference suppression for direct-sequence spread-spectrum CDMA," *IEEE Trans. Commun.*, vol. 42, no. 12, pp. 3178–3188, Dec. 1994.
- [26] T. M. Lok and T. F. Wong, "Transmitter and receiver optimization in multicarrier CDMA systems," *IEEE Trans. Commun.*, vol. 48, no. 7, pp. 1197–1207, Jul. 2000.
- [27] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*. New York: Academic, 1979.
- [28] S. Verdú, *Multuser Detection*. New York: Cambridge Univ. Press, 1998.
- [29] J. H. Cho, "Differential bandwidth and equivalent bandwidth for overloaded multiuser communications," in *Proc. IEEE Int. Symp. Information Theory*, Chicago, IL, Jun./Jul. 2004, p. 511.
- [30] J. H. Cho and L. Gao, "Optimum power and bandwidth allocation for multi-rate FDM and FDMA," *IEEE Trans. Inf. Theory*, submitted for publication.