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Widely-Linear Processing of Complex-Valued Signals: Propriety, Circularity, and Optimal Filtering

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Abstract

This document is a summary and variation of Chapter 1 of the influential book [1] on the statistical processing of complex-valued data. Complex-valued signals are essential tools for the efficient analysis and implementation of communication and signal processing systems. Complex signals enhance processing efficiency by converting passband signals to the equivalent baseband (Section 1), and they concisely represent core functionalities such as quadrature modulation and phase synchronization. This paper focuses on the processing of **Improper Complex Random Variables (ICRVs)**, which exhibit **statistical asymmetry** (Section 2). An ICRV has a non-zero correlation with its complex conjugate (complementary variance $\tilde{C}_{ZZ} \neq 0$), which implies that conventional complex linear processing ($Y = aZ + b$) is insufficient to optimally utilize all the second-order statistical information of the signal (i.e., C_{ZZ} and $\tilde{C}_{ZZ}$). To overcome this limitation and achieve optimal linear performance, **Widely-Linear Processing (WL Processing)**, which uses both the input signal Z and its conjugate Z^*, is required (Section 3). Finally, we clarify the relationship between the second-order symmetry (**Propriety**) and the full distribution symmetry (**Circularity**) of complex random variables (Section 4). WL processing is expressed in the form $Y = aZ + bZ^* + c$ and forms the foundation for designing optimal filtering and detection algorithms for complex random vectors and random processes.

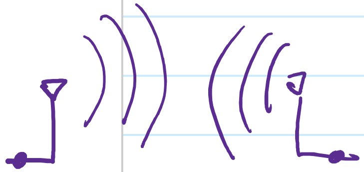
1. The Necessity of Complex-Valued Signals in Communications and Signal Processing

$Z = X + jY = [X$

1. The Necessity of Complex-Valued Signals in Communications and Signal Processing

$Z = X + jY \equiv \begin{bmatrix} X \\ Y \end{bmatrix}$

Complex-valued signals are more than just mathematical tools in communication and signal processing; they are essential core elements for efficiently analyzing and implementing systems. Although mathematically equivalent to handling the real and imaginary parts as a 2×1 real vector, complex signals offer a concise and elegant way to manage phenomena that would be difficult or inefficient to process with real-valued signals alone.



1.1. Convenience of Equivalent Baseband Signal

Instead of processing a real-valued signal $s_p(t)$ with a high carrier frequency (f_c) in the **Passband**, the **Complex Envelope** $s_l(t)$—which shifts the frequency down to the **Baseband** (near DC)—allows all filtering and processing steps to be performed efficiently at a lower sampling rate.

$\boxed{x(t)}$ $y(t)$

$= \mathrm{Re}\{\boxed{x_l(t)} e^{j2\pi f_c t}\}$

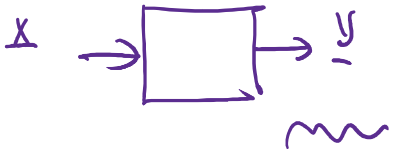
$$s_p(t) = \mathrm{Re}\{s_l(t)e^{j2\pi f_c t}\}$$

Here, $s_l(t)$ is in the form $I(t)+jQ(t)$, containing the In-phase (I) and Quadrature-phase (Q) components.

1.2. Intrinsic Nature of the Fourier Transform (FFT/DFT)

The **Discrete Fourier Transform (DFT)** or **FFT**, which is essential for frequency domain analysis, outputs a generally **complex-valued** spectrum $(X[k])$ even when taking a real-valued signal $(x[n])$ as input. This is because the spectrum of a real signal satisfies a **Conjugate Symmetric** relationship between positive and negative frequencies, and complex values are necessary to express this symmetry.

$$X[k] = X^*[-k \pmod N)]$$



$x(t) = \mathrm{Re}\{\underset{\sim}{x_l(t)}\, e^{j(2\pi f_c t + \theta)}\}$

1.3. Representation of Quadrature Modulation Techniques

$x_l(t) = \sum d_n\, p(t - nT)$

Modern quadrature modulation techniques like **QPSK and QAM** are represented by symbols **s** on the complex plane, which is the natural space for the communication system's **Constellation Diagram**.

$x(t) = \sum (d_n \, p(t-n))$

complex

Modern quadrature modulation techniques like **QPSK and QAM** are represented by symbols $\mathbf{s}$ on the complex plane, which is the natural space for the communication system's **Constellation Diagram**.

$$\mathbf{s} = I + jQ$$

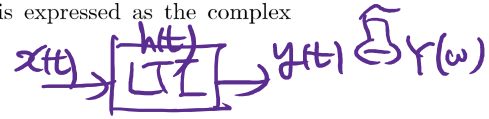
The complex components I and Q represent information carried on two orthogonal carriers (cosine and sine), simplifying the modulation and demodulation process.

1.4. Perfect Inclusion and Processing of Phase Information

A complex signal perfectly includes both **Magnitude** (A) and **Phase** (ϕ) information simultaneously in the form $s = Ae^{j\phi}$. Phase delay caused by filters or channels is a critical factor in communication systems, and phase synchronization and compensation are only possible using the complex representation.

1.5. Simplicity of Linear System Analysis

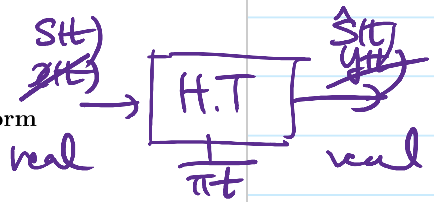
Filtering and channel transmission in communication systems are mostly modeled as **Linear Time-Invariant (LTI)** systems. In the frequency domain, the output $Y(\omega)$ is expressed as the complex product of the input $X(\omega)$ and the system response $H(\omega)$.



$$\underline{Y(\omega) = H(\omega)X(\omega)}$$

Since $H(\omega)$ is a complex function representing both magnitude and phase changes of the system, complex algebra provides the most concise analytical tool.

1.6. Hilbert Transform and Analytic Signal



The **Analytic Signal** $(s_a(t))$ is generated by combining a real signal $s(t)$ and its **Hilbert Transform** $(\hat{s}(t)$**, a** $90°$ **phase shift)**.

$$\underline{s_a(t) = s(t) + \boxed{j}\hat{s}(t)}$$

The analytic signal possesses a **Single-Sided Spectrum** with the negative frequency components perfectly eliminated. This forms the theoretical basis for the equivalent baseband signal and implementation of techniques like **SSB (Single Sideband)** modulation.

1.7. Channel Modeling and Equalization

Wireless channels induce magnitude attenuation and arbitrary phase shift on the signal, which is modeled by a **complex channel gain** (h).

$$h = ae^{j\theta}$$

The **Equalizer** at the receiver, which compensates for channel distortion, is implemented as a complex filter that estimates and multiplies by the inverse of this complex gain, compensating for both magnitude and phase distortion simultaneously.

1.8. Doppler Effect and Frequency Offset Compensation

The **Doppler Offset** (Δf) resulting from the relative movement between the transmitter and receiver introduces a complex rotation component into the received complex baseband signal:

$$e^{j2\pi\Delta ft}$$

The receiver must accurately estimate and counter-rotate this complex rotation component through **Frequency Synchronization**, which is only possible via complex signal processing.

1.9. Representation of High-Dimensional Communication Systems (MIMO)

Systems using multiple antennas, such as **MIMO (Multiple-Input Multiple-Output)**, model the channel with a **complex matrix H**. The input-output relationship of the system is expressed using complex linear algebra.

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

Here, $\mathbf{x}$ and $\mathbf{y}$ are complex signal vectors, and each element of $\mathbf{H}$ contains a complex channel gain. All spatial processing techniques, including **Beamforming**, are based on complex matrix operations.

2. Complex Correlation Coefficient of Maximally Improper Complex Random Variables

This section addresses the statistical analysis and processing of complex-valued random quantities encountered in communications and signal processing. These quantities include **Complex-Valued Scalars**, **Complex-Valued Vectors**, **Complex-Valued Matrices**, as well as the important **Complex-Valued Scalar Signals** and **Complex-Valued Vector Signals** essential for channel modeling and signal analysis.

We focus on the characteristics and processing of these quantities when they appear as **Random Quantities** (i.e., **Random Variables**, **Random Vectors**, or **Random Processes**) to model real-world uncertainty.

Specifically, this section defines and analyzes the **Complex Correlation Coefficient (ρ_Z)**—a metric that quantifies the degree of statistical symmetry of a complex random variable—focusing on the core statistic, the **Complementary Variance** (C_{ZZ}). Furthermore, we detail the geometric and statistical implications of the extreme case where the magnitude of this coefficient is unity, known as the **Maximally**

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2.1. Complex-Valued Random Variables

A **Complex-Valued Random Variable** (Z) is a random variable whose value is a point on the complex plane ($z = x + jy$).

2.1.1 Equivalent Real Vector Representation and Complex Statistics

A complex-valued random variable $Z = X + jY$ can be represented by a 2×1 **Real Random Vector** $\mathbf{Z}_r$ using its real part (X) and imaginary part (Y). (We assume $E[Z] = 0$ for simplicity, neglecting the mean.)

$$\boxed{Z} = X + jY \iff \mathbf{Z}_r = \boxed{\begin{pmatrix} X \\ Y \end{pmatrix}} \rightarrow \begin{bmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y \\ \rho_{XY}\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

X E[X] Var(X)

The key elements defining the statistical characteristics of Z are the complex variance (C_{ZZ}) and the **Complementary Variance ($\tilde{C}_{ZZ}$)**. If σ_X^2 is the variance of X, σ_Y^2 is the variance of Y, and ρ_{XY} is the correlation coefficient between X and Y, they are expressed as follows:

$$\begin{cases} \sigma_X^2 \\ \sigma_Y^2 \\ \rho_{XY} \end{cases}$$

- **1. Complex Variance (C_{ZZ}):** (Conjugate Variance)

$$C_{ZZ} = E[ZZ^*] = \sigma_X^2 + \sigma_Y^2$$

- **2. Complementary Variance ($\tilde{C}_{ZZ}$):** (Non-conjugate Variance)

$$\tilde{C}_{ZZ} = E[Z^2] = \sigma_X^2 - \sigma_Y^2 + j(2\rho_{XY}\sigma_X\sigma_Y)$$

$$\mathrm{Cov}(X, Y) = \rho_{XY}\sigma_X\sigma_Y$$

Definition of Properness: A complex random variable Z is defined as a **Proper Complex Random Variable** when its **Complementary Variance (C_{ZZ}) is zero**. If $C_{ZZ} \neq 0$, Z is defined as an **Improper Complex Random Variable**.

DSB-WC

2.1.2 Proper Complex Gaussian Random Variable

The necessary and sufficient conditions for a complex random variable Z to be proper ($C_{ZZ} = 0$) imply two conditions:

1. **Equal Variances:** $\sigma_{\mathbf{X}}^2 = \sigma_{\mathbf{Y}}^2$ 2. **Uncorrelation:** $\rho_{\mathbf{XY}} = \mathbf{0}$

When Z is Proper Complex Gaussian, denoting the complex variance C_{ZZ} as σ^2, we have $\sigma_X^2 = \sigma_Y^2 = \sigma^2/2$, and $\mathbf{X}$ and $\mathbf{Y}$ are independent and identically distributed.

* **Complex Variance Relationship:**

$$\boxed{\sigma_{\mathbf{Z}}^2} = \boxed{\sigma_{\mathbf{X}}^2} + \boxed{\sigma_{\mathbf{Y}}^2} = \frac{\sigma^2}{2} + \frac{\sigma^2}{2} = \sigma^2$$

Probability Density Function (PDF) of a Proper Complex Gaussian Random Variable
$Z \sim \mathcal{CN}(\mu, \sigma^2)$ is denoted, and its PDF $f_Z(z)$ is as follows: ($z = x + jy$):

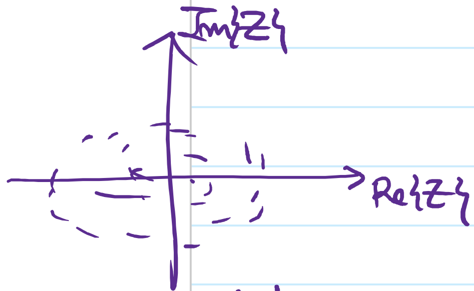
$$f_Z(z) = \frac{1}{\pi\sigma^2} \exp\left(-\frac{|z-\mu|^2}{\sigma^2}\right)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

2.1.3 Improper Complex Gaussian Random Variable

If Z follows a Gaussian distribution and $C_{ZZ} \neq 0$, it is an **Improper Complex Gaussian** random variable. This condition implies that $\mathrm{Var}(X) \neq \mathrm{Var}(Y)$ or $\rho_{XY} \neq 0$.

Statistical Properties of Improper Complex Gaussian: When Z is improper, the covariance matrix $\mathbf{C}_r$ of the real vector $\mathbf{Z}_r$ is not proportional to the identity matrix, and the complementary covariance matrix is additionally required for efficient analysis using complex operations.

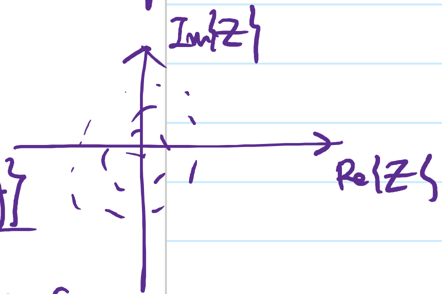


$$\mathbf{C}_r = \begin{pmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y \\ \rho_{XY}\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

2.2. Complex Correlation Coefficient

The **Complex Correlation Coefficient (ρ_Z)** is a measure of how improper Z is.

$Z = X + jY$



$$\rho_Z = \frac{\tilde{C}_{ZZ}}{C_{ZZ}} = \frac{E[(Z-\mu)^2]}{E[|Z-\mu|^2]}$$

$$\rho_{XY} \triangleq E\left\{\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right\}$$

2.2.1 Proof that $|\rho_Z| \leq 1$

We will prove that the magnitude of the complex correlation coefficient is always less than or equal to 1 ($|\rho_Z| \leq 1$).

Def.

1. Preparation for Proof: Basic Definitions and Real Vector We assume the mean of Z, $\mu = E[Z]$, is 0 for simplicity (by considering $Z - \mu$).

$Z_\theta = Z e^{j\theta}$

1. Preparation for Proof: Basic Definitions and Real Vector We assume the mean of Z, $\mu = E[Z]$, is 0 for simplicity (by considering $Z - \mu$).

- **Complex Variance (Denominator):** $C_{ZZ} = E[|Z|^2] = \sigma_X^2 + \sigma_Y^2$
- **Complementary Variance (Numerator):** $\tilde{C}_{ZZ} = E[Z^2] = (\sigma_X^2 - \sigma_Y^2) + j(2\rho_{XY}\sigma_X\sigma_Y)$

The **Covariance Matrix $\mathbf{C}_r$** of the equivalent real vector $\mathbf{Z}_r = (X, Y)^T$ is:

$$\mathbf{C}_r = \begin{pmatrix} \sigma_X^2 & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \sigma_Y^2 \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y \\ \rho_{XY}\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

2. Using the Property of the Covariance Matrix The covariance matrix $\mathbf{C}_r$ of a random variable must physically be **Positive Semi-definite**. From this condition, the determinant ($\det(\mathbf{C}_r)$) must always be greater than or equal to 0:

$$\det(\mathbf{C}_r) = \sigma_X^2\sigma_Y^2 - (\rho_{XY}\sigma_X\sigma_Y)^2 = \sigma_X^2\sigma_Y^2(1 - \rho_{XY}^2) \geq 0$$

This inequality is equivalent to the fundamental property of the real correlation coefficient, $|\rho_{XY}| \leq 1$.

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$Z_\theta = Z e^{j\theta}$

$\begin{cases} Z \\ Z_\theta \end{cases}$ $\forall\theta$.

□ Lemma.

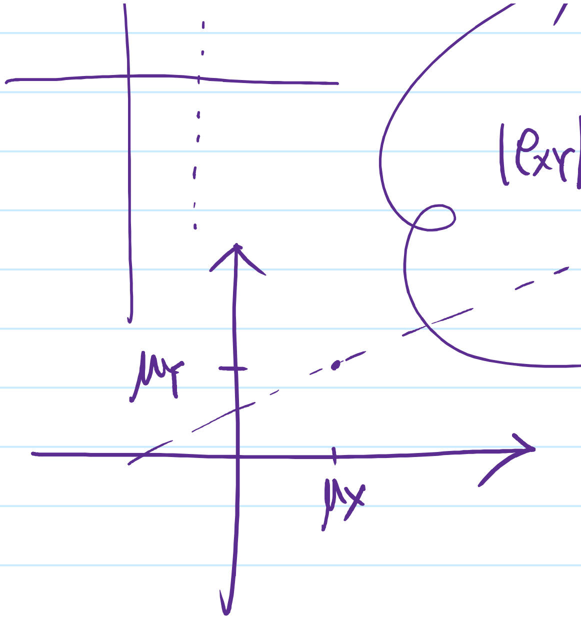
$-1 \leq \rho_{XY} \leq 1$

equalities hold iff - - - - - - → Pearson's

(proof) (i) $\boxed{\rho_{XY}} \triangleq E\left\{\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right\} =$

$|a^T b|^2 \leq (a^T a)(b^T b)$

equality hold iff $a \propto b$



$|a'b| \le (a'a)(b'b)$ equality hold iff $a \propto b$

$|\rho_{XY}|^2 \le E\left\{\left(\frac{X-\mu_X}{\sigma_X}\right)^2\right\} E\left\{\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right\} = 1 \cdot 1$ " " iff

$$\frac{X-\mu_X}{\sigma_X} \overset{a.s.}{=} c\,\frac{Y-\mu_Y}{\sigma_Y}$$

$$Y = \boxed{a}X + b$$

$$Y = \frac{\rho_{XY}\sigma_X\sigma_Y}{\sigma_X^2}(X-\mu_X)+\mu_Y$$
$$= \boxed{\rho_{XY}}\frac{\sigma_Y}{\sigma_X}(X-\mu_X)+\mu_Y.$$

← assumed $\sigma_X \neq 0$.

3. Deriving the Relationship between $|\tilde{C}_{ZZ}|^2$ and C_{ZZ}^2 To simplify $|\tilde{C}_{ZZ}|^2$ using C_{ZZ}^2, we utilize the identity $(\sigma_X^2+\sigma_Y^2)^2 = (\sigma_X^2-\sigma_Y^2)^2 + 4\sigma_X^2\sigma_Y^2$.

$$|\tilde{C}_{ZZ}|^2 = (\sigma_X^2-\sigma_Y^2)^2 + (2\rho_{XY}\sigma_X\sigma_Y)^2$$

We add and subtract $4\sigma_X^2\sigma_Y^2$ to the above equation to separate the $\det(\mathbf{C}_r)$ term:

$$\begin{aligned} |\tilde{C}_{ZZ}|^2 &= \underbrace{(\sigma_X^2-\sigma_Y^2)^2 + 4\sigma_X^2\sigma_Y^2}_{=C_{ZZ}^2} - 4\sigma_X^2\sigma_Y^2 + 4\sigma_X^2\sigma_Y^2\rho_{XY}^2 \\ &= C_{ZZ}^2 - 4\sigma_X^2\sigma_Y^2(1-\rho_{XY}^2) \\ &= C_{ZZ}^2 - 4\det(\mathbf{C}_r) \end{aligned}$$

4. Final Conclusion Since $\det(\mathbf{C}_r) \ge 0$, the following inequality holds:

$$|C_{ZZ}|^2 = C_{ZZ}^2 - 4\det(\mathbf{C}_r) \le C_{ZZ}^2$$

Taking the square root and dividing by C_{ZZ}, we prove that the **magnitude of the complex correlation coefficient** is always less than or equal to 1.

↑ Im{Z}

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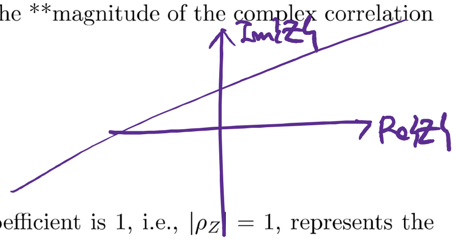

$$|\rho_{\mathbf{Z}}| = \frac{|\mathbf{C_{ZZ}}|}{\mathbf{C_{ZZ}}} \leq \mathbf{1}$$

2.2.2 Meaning of $|\rho_Z| = 1$: Maximally Improper

The case where the magnitude of the complex correlation coefficient is 1, i.e., $|\rho_Z| = 1$, represents the state of **Maximally Improperness** that a complex random variable Z can possess.

As derived in 2.2.1, the necessary and sufficient condition for $|\rho_Z| = 1$ is that the determinant of the equivalent real covariance matrix $\mathbf{C}_r$ is zero:

$$\det(\mathbf{C}_r) = \sigma_X^2 \sigma_Y^2 (1 - \rho_{XY}^2) = 0$$

1. Geometric Meaning of $\det(\mathbf{C}_r) = 0$ $\det(\mathbf{C}_r) = 0$ signifies the existence of **Perfect Linear Dependence** between the components X and Y of the real vector $\mathbf{Z}_r = (X, Y)^T$. This means that the probability mass of the random variable Z is concentrated on a **single straight line (1-dimensional space)** rather than the complex plane ($\mathbb{R}^2$).

- **Distribution Shape:** Z consists of X and Y that satisfy a linear equation of the form $Y = aX + b$. This line is tilted at a specific angle in the complex plane.

2. Relationship between ρ_Z and ρ_{XY} and the Limitation of ρ_{XY} The condition $\det(\mathbf{C}_r) = 0$ is satisfied in two broad categories, and $|\rho_Z| = 1$ holds in both:

- **Category A: Perfect Correlation ($|\rho_{\mathbf{XY}}| = \mathbf{1}$)**
  In this case, $\sigma_X^2 > 0$ and $\sigma_Y^2 > 0$, and X and Y have the relationship $Y = aX + b$ ($a \neq 0$). The distribution of Z is **diagonal** rather than parallel to the X or Y axes.
- **Category B: Zero Variance ($\sigma_{\mathbf{X}}^2 = \mathbf{0}$ or $\sigma_{\mathbf{Y}}^2 = \mathbf{0}$)**
  Since either X or Y is a constant, $\det(\mathbf{C}_r) = 0$ is satisfied.
  - When $\sigma_{\mathbf{Y}}^2 = \mathbf{0}$ ($Z = X + j\mu_Y$): Z is distributed along a line parallel to the **X**-axis. (Canonical example: a real random variable $Z = X$).
  - When $\sigma_{\mathbf{X}}^2 = \mathbf{0}$ ($Z = \mu_X + jY$): Z is distributed along a line parallel to the **Y**-axis.

Limitation of $|\rho_{\mathbf{XY}}| = \mathbf{1}$: The condition $|\rho_{\mathbf{XY}}| = \mathbf{1}$ only covers Category A, and the distribution parallel to the Y or X axes (Category B) makes ρ_{XY} undefined or meaningless. Therefore, the condition $\det(\mathbf{C_r}) = \mathbf{0}$ is the most comprehensive and accurate criterion defining the **Maximally Improperness** where $|\rho_{\mathbf{Z}}| = \mathbf{1}$.

3. Representation as a Phase-Rotated Real Random Variable The 1-dimensional distribution characteristic of $|\rho_Z| = 1$ means that the complex random variable Z is equivalent to **some real random variable W subjected to a constant phase rotation $e^{j\theta}$ (around its mean μ_Z).

$$\boxed{\mathbf{Z} = (\mathbf{W} - \mu_{\mathbf{Z}})\mathbf{e}^{\mathbf{j}\theta} + \mu_{\mathbf{Z}}}$$

Here, W is a **real random variable** satisfying $\mathrm{Im}(W - \mu_Z) = 0$, and θ is a constant value that determines the **angle (Phase Rotation)** of the line Z lies on in the complex plane.

- $\mathbf{Z}$ is expressed as the distribution of the real variable $W - \mu_Z$ on the real axis rotated by θ around μ_Z.
- Since the statistical properties of the complex variable Z are entirely explained by a single real random variable $\mathbf{W}$, such signals offer **the same efficiency as real-valued processing** instead of complex-valued processing. This signifies that the statistical degrees of freedom of $\mathbf{Z}$ have been reduced from 2 (X, Y) to 1 (W).

3. Limitation of Linear Processing of Complex Random Variables and Widely-Linear Processing

When dealing with a complex random variable Z, the most commonly used processing method is the **Complex Linear Transformation**:

$$Y = \boxed{aZ + b}$$

where a and b are complex constants. The aZ term represents a **Linear Transformation**, and the constant b added represents a **Translation**. Strictly speaking, a process that includes a constant translation (addition) is accurately called an **Affine Transformation** or **Affine Processing**. However, in the fields of signal processing and communications, this is conventionally referred to as **Linear Processing** even when the constant term (b) is included. This document adheres to this convention.

3.1. Insufficiency of Complex Linear Processing

Such complex linear processing is perfectly sufficient for analyzing and processing the statistical properties when $\mathbf{Z}$ is a **Proper Complex Random Variable**. When $\mathbf{Z}$ is proper, all its second-order statistical characteristics are completely defined solely by the complex variance (C_{ZZ}), and the complementary variance (C_{ZZ}) is zero.

However, if $\mathbf{Z}$ is an **Improper Complex Random Variable**, i.e., $C_{ZZ} \neq 0$, then complex linear processing becomes **insufficient**.

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However, if $\mathbf{Z}$ is an **Improper Complex Random Variable**, i.e., $C_{ZZ} \neq 0$, then complex linear processing becomes **insufficient**.

- **Information Loss:** An improper random variable Z possesses additional second-order statistical information in the form of C_{ZZ} besides C_{ZZ}. The conventional complex linear filter $Y = aZ$ does not utilize the **conjugate component (Z^*)** of Z. Consequently, it fails to fully exploit the C_{ZZ} information during the processing, leading to the inability to achieve **optimal performance**, such as the Minimum Mean Square Error (MMSE).

To achieve optimal linear estimation or filtering, all second-order statistical information of Z (i.e., C_{ZZ} and C_{ZZ}) must be utilized.

LCL

3.2. Definition of Widely-Linear Processing

Widely-Linear Processing (WL Processing) is a concept designed to utilize all second-order statistical information of an improper complex random variable to achieve optimal linear processing performance.

3.2.1 Broad Definition of Widely-Linear Processing

In the **Broad Definition**, Widely-Linear Processing (WL Processing) encompasses all processing that is **Linear** with respect to the complex random variable Z. This means treating the real and imaginary parts of Z as independent real variables and applying a linear transformation to the resulting real vector. This definition includes the following two equivalent approaches:

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$$Z_r = \boxed{\begin{bmatrix} X \\ Y \end{bmatrix}} = \begin{bmatrix} \frac{Z+Z^*}{2} \\ \frac{Z-Z^*}{2j} \end{bmatrix}$$

1. **Real Composite Vector Approach:**
   Processing that applies a **real-valued affine transformation** to the 2-dimensional real vector $\mathbf{Z}_r = (X, Y)^T$, which is formed by the real part X and imaginary part Y of Z.

   $$\mathbf{Y}_r = \boxed{\mathbf{A}}\mathbf{Z}_r + \mathbf{b}_r$$

   (Where $\mathbf{A}_r$ is a 2×2 real matrix.)

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2. **Complex Augmented Vector Approach:**
   Processing that applies a **complex-valued affine transformation** to the **Complex Augmented

(Where $\mathbf{A}_r$ is a 2×2 real matrix.)

2. **Complex Augmented Vector Approach:**
Processing that applies a **complex-valued affine transformation** to the **Complex Augmented Vector** $\mathbf{Z}_a = (Z, Z^*)^T$, formed by Z and its complex conjugate Z^*.

$$\begin{pmatrix} Z \\ Z^* \end{pmatrix} \leftarrow \text{redundancy}$$

$$\mathbf{Y} = \mathbf{A}_c \mathbf{Z}_a + c$$

(Where $\mathbf{A}_c$ is a 1×2 or 2×2 complex matrix, and c is a constant vector.)

The broad definition mathematically includes all linear transformations equivalent to utilizing both Z and Z^*.

3.2.2 Narrow Definition of Widely-Linear Processing

In the **Narrow Definition**, Widely-Linear Processing is synonymous with the term **Linear-Conjugate Linear (LCL)** processing and refers exclusively to the following **Complex Augmented Vector Approach**:

- **Narrow Definition (LCL):** A single complex output affine transformation that uses both Z and Z^* to produce the output Y.

  $$Y = aZ + bZ^* + c$$

  Here, a, b, c are complex constants. This expression is equivalent to a 1×2 complex matrix transformation on $\mathbf{Z}_a = (Z, Z^*)^T$.

  $$\boxed{Y = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} Z \\ Z^* \end{pmatrix} + c}$$

This narrow definition (LCL processing) is the most common form referred to as **'Widely-Linear'** in communications and signal processing when dealing with improper signals, as it is the most concise way to utilize all second-order statistics of the improper signal to achieve optimal linear processing performance.

3.3. Extension to Complex Random Vectors and Random Processes

The concept of Widely-Linear Processing extends readily beyond a single complex random variable (Z) to the **Complex Random Vectors (Z)** and **Complex Random Processes** ($Z[n]$ **or** $Z(t)$) encountered in communication systems.

- **Complex Random Vectors (Z):**
  When components of the vector $\mathbf{Z}$ are improper, optimal linear processing uses both the vector $\mathbf{Z}$ and its conjugate vector $\mathbf{Z}^*$ as inputs. That is, the input vector is augmented to $\mathbf{Z}_a = (\mathbf{Z}^T, (\mathbf{Z}^*)^T)^T$, and a matrix-based Widely-Linear transformation $\mathbf{Y} = \mathbf{AZ} + \mathbf{BZ}^* + \mathbf{c}$ is applied. This is crucial for designing optimal linear detectors and equalizers in improper channel environments in MIMO systems.

  $$\begin{bmatrix} Z \\ Z^* \end{bmatrix}$$

- **Complex Random Processes** ($Z[n]$ **or** $Z(t)$)**:**
  When the time-series data $Z[n]$ (a complex random process/signal) is improper, an optimal **Linear Filter** must utilize samples from both the current and past time points. A **Widely-Linear Filter (WLF)** is used, which calculates the output using both the input $Z[n]$ and the conjugate input $Z^*[n]$. This forms the basis for time-varying filtering or Adaptive Filtering algorithms that leverage

both Z and Z^*.

$$\begin{bmatrix} z[n] \\ z^*[n] \end{bmatrix}, \quad \begin{bmatrix} z(t) \\ z^*(t) \end{bmatrix}$$

4. Statistical Symmetries of Complex Random Variables: Circularity and Propriety

This section defines and explores two fundamental statistical symmetries of a complex-valued random variable (CRV) $Z = X + jY$: **Circularity** and **Propriety**. While both describe rotational symmetry in the complex plane, Circularity is a stronger condition related to the full probability distribution, and Propriety relates only to the second-order statistics. The relationship between these two properties is crucial, especially in the context of widely-linear signal processing and estimation theory.

4.1. Definition of Circularity (Rotational Invariance)

Circularity (or strict circularity) is a property related to the full probability distribution of Z.

- **Definition:** A CRV Z is **circular** if its probability density function (PDF) is invariant to an arbitrary phase rotation $e^{j\phi}$.

$$\underline{Z} \sim \underline{e^{j\phi} Z}, \quad \text{for any constant } \phi \in [0, 2\pi) \tag{1}$$

- **Implication:** For Equation (1) to hold, the mean of Z must be zero ($\mu = 0$). Otherwise, $E[e^{j\phi} Z] = e^{j\phi}\mu$ would vary with ϕ, violating the invariance of the distribution.
- **Geometric Interpretation:** The statistical distribution of Z has perfect 360° rotational symmetry in the complex plane.

4.2. Definition of Propriety (Vanishing Complementary Covariance)

Propriety is a property related specifically to the second-order statistics of Z. (As defined in Section 2.1.1.)

Propriety is a property related specifically to the second-order statistics of Z. (As defined in Section 2.1.1.)

- **Definition:** A CRV Z is **proper** if its complementary covariance (or pseudo-covariance) is zero.

$$C_{ZZ} = E[(Z - \mu)^2] = 0 \tag{2}$$

  If $C_{ZZ} \neq 0$, Z is called **improper**.

- **Alternative Conditions:** Propriety is equivalent to the following two conditions holding simultaneously for the real part X and imaginary part Y of Z:
  1. The variances are equal: $\mathrm{Var}(X) = \mathrm{Var}(Y)$.
  2. The real and imaginary parts are uncorrelated: $\mathrm{Cov}(X, Y) = 0$.

- **Geometric Interpretation:** The level curves of the CRV's second-order statistics are circular (or spherical for a CRV vector) rather than elliptical. This means the probability ellipse is a circle.

4.3. Relationship Between the Properties

The relationship between Circularity and Propriety is one-way in the general case, but becomes an equivalence for the specific class of Gaussian variables.

4.3.1 Circularity Implies Propriety (Circularity $\implies$ Propriety)

If Z is circular, it must be proper.

- **Proof Sketch:** If Z is circular, by definition, $Z \sim e^{j\phi}Z$. The mean must be zero ($\mu = 0$).
- The complementary covariance must be invariant to the phase shift:

$$C_{ZZ} = E[Z^2] = E[(e^{j\phi}Z)^2] = E[e^{j2\phi}Z^2] = e^{j2\phi}E[Z^2] = e^{j2\phi}C_{ZZ}$$

- Since this must hold for any ϕ (e.g., $\phi = \pi/4$, where $e^{j2\phi} = e^{j\pi} = -1$), the only way the equality can be satisfied is if:

$$C_{ZZ} = 0$$

- Therefore, any circular CRV is necessarily proper.

4.3.2 Propriety Does Not Imply Circularity (Propriety $\not\Longrightarrow$ Circularity)

A proper CRV is not necessarily circular.

– **Reason:** Propriety ensures the rotational invariance of the **second-order moments** only. Circularity requires invariance of **all** moments (or the entire PDF).

– **Counterexample (Non-Gaussian):** It is possible to construct a non-Gaussian CRV that satisfies $C_{ZZ} = 0$ (Propriety) but whose higher-order moments, such as the fourth-order moment $E[Z^4]$, are not invariant to phase rotation, violating Circularity.

4.4. Special Case: Complex Gaussian Random Variable (CGV)

For a complex Gaussian random variable (CGV), the entire probability distribution is uniquely determined by the mean vector, the covariance matrix, and the complementary covariance matrix. This simplifies the relationship.

4.4.1 Equivalence for CGVs

Assuming a zero-mean CGV ($\mu = 0$), Circularity and Propriety are equivalent conditions.

$$\text{Circularity} \iff \text{Propriety} \quad (\text{For } Z \sim \mathcal{CN}(0, C_{ZZ}, C_{ZZ})) \tag{3}$$

◦ **Proper CGV ($C_{ZZ} = 0$):** This variable is fully described by $\mathcal{CN}(0, C_{ZZ})$. It possesses perfect rotational symmetry, meaning it is also circular.

◦ **Improper CGV ($C_{ZZ} \neq 0$):** This variable is not proper, and hence, it is not circular. Its distribution is elliptical, representing a lack of rotational symmetry. A common example is a purely real Gaussian variable $Z = X \sim \mathcal{N}(0, \sigma^2)$, where $C_{ZZ} = \sigma^2 \neq 0$.

References

[1] P. J. Schreier and L. L. Scharf, *Statistical Signal Processing of Complex-Valued Data: The Theory of Improper and Noncircular Signals.* Cambridge, U.K.: Cambridge University Press, 2010.

FRESH vectorization
p-F " vec
FRESH prop.

9

$\square$

$$\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$

$b_i \in \{1, -1\}$

i.i.d. . . .

$$\underline{z} = H\underline{b} + \underline{n}$$

$$\hat{\underline{b}} = A\underline{z}$$

$$\hat{\underline{b}} = B\underline{z} + C\underline{z}^* = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \underline{z} \\ \underline{z}^* \end{bmatrix} = D\underline{z}_a$$

$$MSE \triangleq E\{|\hat{\underline{b}} - \underline{b}|^2\}$$

$$\hat{\underline{b}} = Re\{\tilde{A}\underline{z}\}$$