Duality in Quadratic Time-Frequency Analysis: A Comparison of the Wigner-Ville Distribution and the Ambiguity Function

Gemini AI Assistant and Joon Ho Cho

November 6, 2025

Abstract

The Wigner-Ville Distribution (WVD) and the Ambiguity Function (AF) are two foundational tools in quadratic signal analysis, both derived from the same instantaneous autocorrelation kernel. This paper outlines their profound mathematical connection as a **two-dimensional Fourier Transform pair**, highlighting their shared properties, such as bi-linearity and the retention of signal phase information. Crucially, we distinguish between their respective domains and primary applications: the WVD operates in the **Time-Frequency** domain for instantaneous spectral analysis, while the AF operates in the **Time Delay-Doppler Frequency** domain, serving as the essential tool for analyzing resolution and uncertainty in radar and sonar systems. This duality underscores a universal principle in signal processing, mapping internal signal characteristics (t, f) to external propagation characteristics (τ, ν).

1 Introduction

The Wigner-Ville Distribution ($W_x(t, f)$) and the Ambiguity Function ($\chi(\tau, \nu)$) are two cornerstones of advanced signal processing, particularly in the analysis of non-stationary signals. Both functions belong to Cohen's class of time-frequency representations and share a direct mathematical relationship, often used interchangeably depending on the application domain: spectral analysis or radar/communication system design.

2 Mathematical Definitions

Both the WVD and the AF are constructed from the instantaneous autocorrelation kernel $\tilde{r}_x(t, \tau)$, defined as:

$$\tilde{r}_x(t, \tau) = x(t + \tau/2)x^*(t - \tau/2)$$

2.1 Wigner-Ville Distribution (WVD)

The WVD is a Fourier Transform of the kernel with respect to the **time lag τ**, resulting in a distribution across the **Time (t)-Frequency (f)** plane.

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi f\tau}d\tau \tag{1}$$

The Wigner-Ville Distribution and the Ambiguity Function are referred to as **quadratic transforms** because their definition involves the signal $x(t)$ being multiplied by its complex conjugate $x^*(t)$, resulting in an operation that is of the second power (order) with respect to the signal amplitude. This bi-linear (or quadratic) nature is the source of their most significant analytical feature—the **cross-terms**—which arise when analyzing signals composed of multiple components.

2.2 Ambiguity Function (AF)

The Ambiguity Function is a Fourier Transform of the kernel with respect to the **average time t**, resulting in a distribution across the **Time Delay (τ)-Doppler Frequency (ν)** plane.

$$\chi(\tau, \nu) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi\nu t}dt \tag{2}$$

3 Commonalities: Mathematical Duality and Shared Properties

3.1 Two-Dimensional Fourier Transform Pair

The most fundamental commonality is their **Fourier Duality**. The WVD and the AF form a 2D Fourier Transform pair, meaning one can be derived from the other via a two-dimensional transform:

$$\mathbf{W_x(t, f)} \leftrightarrow \mathcal{F}_{\mathbf{t}\to\nu}, \mathcal{F}_{\mathbf{f}\to-\tau}\chi(\tau, \nu) \tag{3}$$

Specifically:

- The WVD is the 2D Inverse Fourier Transform of the AF.
- The AF is the 2D Fourier Transform of the WVD (with a sign adjustment in the τ or f integral kernel).

3.2 Shared Quadratic Properties

- **Bi-linearity and Cross-Terms:** Both functions are bi-linear transforms of $x(t)$, leading to the phenomenon of **cross-terms** for multicomponent signals. These terms represent interference patterns that can complicate visual interpretation.
- **Signal Recovery:** Both preserve the complete phase information of the signal. The original signal $x(t)$ can be **recovered** from either the WVD or the AF, uniquely determined up to a constant phase factor ($e^{j\theta}$).
- **Origin Value:** The total energy (E) of the signal is found at the origin of the AF, and the total energy is conserved when the WVD is integrated over both time and frequency:

$$\chi(0, 0) = E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t, f)dtdf$$

4 Differences: Domain, Interpretation, and Application

Despite their mathematical connection, the WVD and AF serve distinct analytical purposes dictated by their respective domains.

4.1 Domain and Interpretation

- **WVD $(\mathbf{t}, \mathbf{f})$:** This is the **internal representation** of the signal's energy distribution. It shows **when** the signal has energy at **what frequency**. It is ideal for analyzing signals with time-varying frequency content, such as frequency-modulated (chirp) or cyclostationary signals.
- **AF (τ, ν):** This is the **external representation** related to propagation and measurement. It shows how the signal behaves under **time delay (τ)** and **Doppler shift (ν)**. The AF's peak structure describes the system's ability to distinguish targets.

4.2 Primary Applications

- **WVD Applications (Spectral Analysis):**
  1. Non-stationary signal analysis (e.g., speech, biological signals).
  2. Tracking instantaneous frequency and group delay.
  3. Analyzing the energy concentration of complex modulation schemes.
- **AF Applications (Radar/Sonar):**

1. **Resolution and Ambiguity:** Analyzing the ambiguity (lack of separability) between two targets that differ in both range (related to τ) and velocity (related to ν).
2. **Waveform Design:** Designing radar pulse waveforms that minimize sidelobes in the AF plane to reduce unwanted clutter and maximize target resolution.
3. **Channel Modeling:** Describing communication channels that are simultaneously time-dispersive (τ) and frequency-dispersive (ν).

In summary, the WVD and AF are two sides of the same quadratic coin, each providing unique, critical insights into signal properties when viewed through the appropriate lens of their respective domains.