The Complementary Bi-Frequency Spectrum: Analysis of Complex-Valued Random Processes

Gemini AI Assistant and Joon Ho Cho

November 6, 2025

Abstract

This document introduces and defines the **Complementary Bi-frequency Spectrum** ($\tilde{S}_{XX}(f_1, f_2)$), also known as the Pseudo Bi-frequency Spectrum, which is essential for the comprehensive spectral analysis of **complex-valued random processes** $X(t)$. Unlike the standard Bi-frequency Spectrum, which uses the Hermitian autocorrelation function, the complementary spectrum is based on the **pseudo autocorrelation function** $\tilde{R}_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X(t_2)\}$. The definition is derived via a standard double Fourier transform, and its primary significance lies in characterizing the **properness** (or circular symmetry) of the complex process and revealing **frequency coupling** information.

1 Introduction to Complex Correlation Functions

For a complex-valued random process $X(t)$, two second-order correlation functions are generally required to fully describe its statistical behavior:

1. The **Autocorrelation Function (ACF)**: $R_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X^*(t_2)\}$.
2. The **Pseudo Autocorrelation Function (PACF)** or **Complementary Autocorrelation Function**: $\tilde{R}_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X(t_2)\}$.

While the ACF is used to define the standard spectral density, the PACF is crucial for analyzing the complex process's statistical symmetry.

2 Definition of the Complementary Bi-Frequency Spectrum

The **Complementary Bi-frequency Spectrum** $\tilde{S}_{XX}(f_1, f_2)$ is defined as the two-dimensional Fourier Transform of the Pseudo Autocorrelation Function $\tilde{R}_{XX}(t_1, t_2)$.

2.1 Formal Definition

The Complementary Bi-frequency Spectrum is given by the following integral transform:

$$\tilde{S}_{XX}(f_1, f_2) = \iint_{-\infty}^{\infty} \tilde{R}_{XX}(t_1, t_2) e^{-j2\pi(f_1 t_1 + f_2 t_2)} dt_1 dt_2 \tag{1}$$

Note that this is a **standard double Fourier transform** (with a positive sign in the time domain, resulting in a sum in the frequency domain) and differs from the symplectic Fourier transform used for the standard Bi-frequency Spectrum (S_{XX}).

3 Significance and Applications

The $\tilde{S}_{XX}(f_1, f_2)$ spectrum provides unique information not contained in the standard spectrum $S_{XX}(f_1, f_2)$.

3.1 Characterizing Properness (Circular Symmetry)

The primary role of the Complementary Spectrum is to determine if the complex-valued random process $X(t)$ is **proper** .

- A process $X(t)$ is **proper** (i.e., its real and imaginary parts have no statistical correlation with each other) if and only if its pseudo autocorrelation function is zero for all time shifts: $\tilde{R}_{XX}(t_1, t_2) = 0$.
- Consequently, the process is proper if and only if the Complementary Bi-frequency Spectrum is zero for all frequencies:

$$\text{X(t) is proper} \iff \tilde{S}_{XX}(f_1, f_2) = 0 \quad \text{for all } f_1, f_2$$

If $\tilde{S}_{XX}(f_1, f_2)$ is non-zero, the process is deemed **improper**. Improper processes require analysis of both $X(t)$ and its conjugate $X^*(t)$ for optimal signal processing.

3.2 Frequency Coupling and Summation Information

The structure of the transform integral (Eq. 1), featuring the term $e^{-j2\pi(f_1 t_1 + f_2 t_2)}$, highlights the spectrum's sensitivity to **frequency summation** or coupling:

- $\tilde{S}_{XX}(f_1, f_2)$ reveals statistical correlation between the frequency component at f_1 and the frequency component at f_2.
- This information is particularly relevant in analyzing non-linear systems or mixing processes where frequency components are added (e.g., in spectral mixing or harmonic generation).

3.3 Relation to PSD (WSS Case)

For a **Wide-Sense Stationary (WSS)** process, $\tilde{R}_{XX}(t_1, t_2)$ depends only on the time difference $\tau = t_1 - t_2$. In this specialized case, the Complementary Bi-frequency Spectrum is non-zero only along the anti-diagonal where $f_1 = -f_2$:

$$\tilde{S}_{XX}(f_1, f_2) = \tilde{S}_{XX}(f_1) \cdot \delta(f_1 + f_2)$$

where $\tilde{S}_{XX}(f_1)$ is the single-variable **Pseudo Power Spectral Density (PPSD)**, which is the Fourier Transform of $\tilde{R}_{XX}(\tau)$.