Existence of the Complementary Autocorrelation Function for Complex Second-Order Processes

Gemini AI Assistant and Joon Ho Cho

November 12, 2025

1 Proof: Existence of $\tilde{\mathbf{R}}_{\mathbf{XX}}(t_1, t_2)$

This section proves that if a Complex Random Process $X(t)$ is a **Second-Order Process**, its **Complementary Autocorrelation Function** $\tilde{R}_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}$ must exist (i.e., be finite) for all t_1 and t_2.

1.1 Definitions and Premise

The complex random process $X(t)$ is defined by its real part $X_R(t)$ and imaginary part $X_I(t)$:

$$X(t) = X_R(t) + jX_I(t)$$

Second-Order Condition: $X(t)$ is a second-order process if its second moment is finite for all time t:

$$E\{|X(t)|^2\} < \infty, \quad \forall t \tag{1}$$

Complementary Autocorrelation Function:

$$\tilde{R}_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

1.2 Derivation

1. **Existence of Real Component Second Moments:** From the second-order condition (Equation 1), we expand the squared magnitude:

   $$E\{|X(t)|^2\} = E\{|X_R(t) + jX_I(t)|^2\} = E\{X_R(t)^2\} + E\{X_I(t)^2\}$$

   Since the sum is finite, both components must be individually finite:

   $$E\{X_R(t)^2\} < \infty \quad \text{and} \quad E\{X_I(t)^2\} < \infty \tag{2}$$

   This confirms that $X_R(t)$ and $X_I(t)$ are both second-order processes.

2. **Existence of Real Cross-Correlation Functions (using Cauchy-Schwarz):** Since $X_R(t)$ and $X_I(t)$ are second-order, the **Cauchy-Schwarz inequality** guarantees the existence and finiteness of all four real-valued cross-correlation functions.

   For the real-part autocorrelation $R_{RR}(t_1, t_2) = E\{X_R(t_1)X_R(t_2)\}$:

   $$|E\{X_R(t_1)X_R(t_2)\}| \leq \sqrt{E\{X_R^2(t_1)\}}\sqrt{E\{X_R^2(t_2)\}} < \infty \quad \text{by (2)} \tag{3}$$

   This confirms the existence of R_{RR}. Similarly, all other real cross-correlation functions exist:

   $$\begin{aligned} R_{RR}(t_1, t_2) &= E\{X_R(t_1)X_R(t_2)\} < \infty \\ R_{RI}(t_1, t_2) &= E\{X_R(t_1)X_I(t_2)\} < \infty \\ R_{IR}(t_1, t_2) &= E\{X_I(t_1)X_R(t_2)\} < \infty \\ R_{II}(t_1, t_2) &= E\{X_I(t_1)X_I(t_2)\} < \infty \end{aligned}$$

3. **Existence of the Complementary Autocorrelation $\tilde{\mathbf{R}}_{\mathbf{XX}}$:** We expand $\tilde{R}_{XX}(t_1, t_2)$ using the component processes:

$$\begin{aligned}
\tilde{R}_{XX}(t_1, t_2) &= E\{X(t_1)X(t_2)\} \\
&= E\{(X_R(t_1) + jX_I(t_1))(X_R(t_2) + jX_I(t_2))\} \\
&= E\{X_R(t_1)X_R(t_2) - X_I(t_1)X_I(t_2) + j[X_R(t_1)X_I(t_2) + X_I(t_1)X_R(t_2)]\} \\
&= [E\{X_R(t_1)X_R(t_2)\} - E\{X_I(t_1)X_I(t_2)\}] \\
&\quad + j[E\{X_R(t_1)X_I(t_2)\} + E\{X_I(t_1)X_R(t_2)\}] \\
&= [R_{RR}(t_1, t_2) - R_{II}(t_1, t_2)] + j[R_{RI}(t_1, t_2) + R_{IR}(t_1, t_2)]
\end{aligned}$$

Since all four component real correlation functions $(R_{RR}, R_{II}, R_{RI}, R_{IR})$ are finite, their linear combination, $\tilde{R}_{XX}(t_1, t_2)$, must also be **finite and therefore exists** for all t_1 and t_2.

Q.E.D.