Fourier Duality among Wigner-Ville Distributions, Bi-Frequency Spectra, and Spectral Correlation Density Functions: Analysis for Non-Stationary Complex Random Process

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December 11, 2025

Abstract

This document establishes the fundamental Fourier duality among three crucial concepts for analyzing non-stationary complex random processes: the **Statistical Wigner-Ville Distribution (WVD)**, the general **Bi-Frequency Spectra (BFS)**, and the transformed **Spectral Correlation Density Function (SCD)**. We first establish the WVD ($W_X(t,f)$) as a time-frequency representation, and demonstrate that the SCD ($S_X(\alpha,f)$) is derived from the BFS ($S_{XX}(f_1,f_2)$) by transforming individual frequencies (f_1,f_2) to average frequency (f) and cycle frequency (α). The key relationship is shown to be a **two-dimensional Fourier Transform pair** connecting the WVD and the SCD ($\mathcal{F}_{t\to\alpha}\{W_X(t,f)\} = S_X(\alpha,f)$). This duality is essential for characterizing non-stationarity: time-dependence in the WVD translates directly into non-zero correlation components at $\alpha \neq 0$ in the SCD (representing off-diagonal correlation in the BFS). Furthermore, the document emphasizes the necessity of the complementary components, the **Complementary WVD ($\tilde{W}_X$)** and the **Complementary SCD ($\tilde{S}_X$)**, for the complete second-order statistical characterization of improper processes. Finally, the relationship between the WVDs/SCDs of a real bandpass signal $Y(t)$ and its complex baseband counterpart $X(t)$ is derived, demonstrating how the $\tilde{S}_X$ component explicitly causes spectral correlation at $\alpha = \pm 2f_c$, which is critical for analyzing improper processes.

1 Introduction to Non-Stationary Analysis Tools

The analysis of non-stationary random processes often involves two key statistical representations derived from the two-variable correlation functions. We define:

- **Autocorrelation Function:** $R_{XX}(t_1,t_2) \triangleq \mathbb{E}\{X(t_1)X^*(t_2)\}$
- **Complementary Autocorrelation Function:** $\tilde{R}_{XX}(t_1,t_2) \triangleq \mathbb{E}\{X(t_1)X(t_2)\}$

To establish the connection to time-frequency analysis, we transform the time variables into **average time (t)** and **time difference (τ)**:

$$t = \frac{t_1+t_2}{2}$$
$$\tau = t_1 - t_2$$

The transformed correlation functions are:

- **Transformed Autocorrelation Function:** $R_X(t,\tau) \triangleq R_{XX}(t+\tau/2, t-\tau/2)$
- **Transformed Complementary Autocorrelation Function:** $\tilde{R}_X(t,\tau) \triangleq \tilde{R}_{XX}(t+\tau/2, t-\tau/2)$

2 Wigner-Ville Distribution (WVD)

2.1 Deterministic Wigner-Ville Distribution

The WVD was originally introduced in quantum mechanics and first applied to signal analysis for deterministic signals $x(t)$. The deterministic WVD, $W_x(t,f)$, is defined directly from the signal $x(t)$ using the instantaneous autocorrelation kernel.

Recall from the previous discussion that the Energy Spectral Density (ESD), $E_{xx}(f)$, of a finite-energy signal is the Fourier Transform of its *time-average(d)* autocorrelation $r_{xx}(\tau)$. The WVD extends this concept to the time-varying case by taking the Fourier Transform of the **instantaneous autocorrelation kernel**, $x(t+\tau/2)x^*(t-\tau/2)$, with respect to the delay τ:

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-j2\pi f\tau}d\tau \tag{1}$$

This definition provides the instantaneous energy distribution of the signal $x(t)$ across the time-frequency plane (t,f):

$$\int_{-\infty}^{\infty} W_x(t,f)dt = E_{xx}(f) \tag{2}$$

This relationship demonstrates that the Wigner-Ville Distribution, $W_x(t,f)$, serves as the **Instantaneous Spectral Density (ISD)**. When integrated (or averaged) over time t, it perfectly conserves the signal's long-term frequency energy distribution, the **Energy Spectral Density (ESD)**, $E_{xx}(f)$.

Detailed Proof (Time Integration Property) We prove that the time integration of the WVD equals the ESD, $E_{xx}(f) = |X(f)|^2$.

1. Substitute the WVD definition and change the order of integration: We start by substituting the definition of $W_x(t,f)$ and invoking Fubini's theorem to swap the integration order of t and τ:

$$\int_{-\infty}^{\infty} W_x(t,f)dt = \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\left[\int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)dt\right]d\tau$$

2. Relate the inner time integral to the Autocorrelation Function $r_{xx}(\tau)$: The inner integral, $I(\tau)$, is analyzed using a change of variables.

$$I(\tau) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)dt$$

Let the substitution variable be $u = t-\tau/2$, which implies $t = u+\tau/2$ and $dt = du$. Substituting into $I(\tau)$:

$$\begin{aligned} I(\tau) &= \int_{-\infty}^{\infty} x((u+\tau/2)+\tau/2)x^*(u)du \\ &= \int_{-\infty}^{\infty} x(u+\tau)x^*(u)du \\ &= x(\tau) * x^*(-\tau) = r_{xx}(\tau) \end{aligned}$$

3. Apply the Wiener-Khinchin Theorem: Substituting this result back into the expression for WVD:

$$\int_{-\infty}^{\infty} W_x(t,f)dt = \int_{-\infty}^{\infty} r_{xx}(\tau)e^{-j2\pi f\tau}d\tau$$

The right-hand side is the Fourier Transform of the autocorrelation function, $\mathcal{F}\{r_{xx}(\tau)\}$. By the Parseval's theorem for energy signals, this is equal to the Energy Spectral Density (ESD), $|X(f)|^2$.

$$\mathcal{F}\{r_{xx}(\tau)\} = |X(f)|^2 = E_{xx}(f)$$

4. Conclusion:

$$\therefore \quad \int_{-\infty}^{\infty} W_x(t,f)dt = E_{xx}(f)$$

This confirms that the WVD satisfies the marginal property of energy conservation.

2.1.1 Signal Recovery from WVD

A significant property of the WVD, unlike the conventional Power Spectral Density (PSD) which loses phase information, is that the original signal $x(t)$ can be **recovered** from its WVD $W_x(t,f)$. Since the WVD is a **bi-linear transform**, this recovery is possible **up to a constant phase factor** ($e^{j\theta}$).

- The WVD retains enough information about the signal's phase (embedded in the instantaneous autocorrelation kernel) to allow for reconstruction.
- The core inversion is achieved by taking the Inverse Fourier Transform of $W_x(t,f)$ with respect to f to recover the autocorrelation kernel:

$$x(t+\tau/2)x^*(t-\tau/2) = \int_{-\infty}^{\infty} W_x(t,f)e^{j2\pi f\tau}df$$

- **Recovery of Instantaneous Power at $t = 0$:** Setting the time $t = 0$ and the delay $\tau = 0$ in the kernel recovery process allows us to find the initial energy of the signal:

$$|x(0)|^2 = \int_{-\infty}^{\infty} W_x(0,f)df$$

- **Signal Recovery:** More generally, the signal $x(t)$ can be recovered. For instance, setting $\tau = 2t$ and assuming $x(0) \neq 0$ allows recovery:

$$x(2t)x^*(0) = \int_{-\infty}^{\infty} W_x(t,f)e^{j4\pi ft}df$$

which implies

$$x(t) = \frac{1}{x^*(0)}\int_{-\infty}^{\infty} W_x(t/2,f)e^{j2\pi ft}df \tag{3}$$

This formula is valid only when $\mathbf{x(0) \neq 0}$. The known value of $x(0)$ acts as a **reference point** that fixes the unknown constant phase factor. This demonstrates that $x(t)$ is uniquely determined (up to $e^{j\theta}$) by the knowledge of $W_x(t,f)$.

2.2 Statistical Wigner-Ville Distribution (WVD)

For a random process $X(t)$, the statistical WVD, $W_X(t,f)$, is obtained by taking the ensemble average of the deterministic WVD, or equivalently, by applying the Fourier transform to the transformed autocorrelation function $R_X(t,\tau)$:

$$W_X(t,f) = \mathbb{E}\{W_x(t,f)\} = \mathcal{F}_{\tau\to f}\{R_X(t,\tau)\} = \int_{-\infty}^{\infty} R_X(t,\tau)e^{-j2\pi f\tau}d\tau \tag{4}$$

The statistical WVD provides the **instantaneous power spectral content** of the random process $X(t)$ at time t.

2.2.1 Relation with PSD

The long-term Power Spectral Density (PSD), $S_{XX}(f)$, which was proved to be the Fourier Transform of the time-averaged autocorrelation function $R_{Avg}(\tau)$, is recovered by averaging the statistical WVD over all time t:

$$S_{XX}(f) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} W_X(t,f)dt \tag{5}$$

This fundamental relationship confirms that the statistical WVD serves as the **Instantaneous Power Spectral Density (IPSD)** from which the long-term average spectrum (PSD) is obtained via time averaging.

2.2.2 Recovery of Autocorrelation Function

Unlike the deterministic WVD, the statistical WVD cannot recover the random process $X(t)$ itself, as the ensemble averaging process inherently removes phase information ($\mathbb{E}\{X(t)\}$). However, $W_X(t,f)$ can perfectly recover the statistical information it was derived from.

The **transformed two-variable autocorrelation function**, $R_X(t,\tau)$, is fully recoverable from $W_X(t,f)$ by applying the Inverse Fourier Transform with respect to the frequency variable f:

$$R_X(t,\tau) = \mathcal{F}^{-1}_{f\to\tau}\{W_X(t,f)\} = \int_{-\infty}^{\infty} W_X(t,f)e^{j2\pi f\tau}df \tag{6}$$

This confirms the direct and reversible mathematical duality between the instantaneous spectral content ($W_X(t,f)$) and the statistical temporal correlation ($R_X(t,\tau)$).

2.2.3 Property: The WVD $W_X(t,f)$ is always real.

This fundamental property holds because the transformed autocorrelation function $R_X(t,\tau)$ is **conjugate symmetric** with respect to the time-lag τ. Specifically, $R_X(t,\tau)$ satisfies the property:

$$R_X^*(t,\tau) = R_X(t,-\tau)$$

Since the WVD is the Fourier Transform of a function that is conjugate symmetric, the resulting WVD, $W_X(t,f)$, must be equal to its own complex conjugate ($W_X(t,f) = W_X^*(t,f)$), confirming that $W_X(t,f)$ is **real-valued** for all t and f.

3 Bi-frequency Spectra

The concept of the bi-frequency spectrum is crucial for analyzing the non-stationarity of a process by correlating different frequency components. We define two primary forms derived from the correlation and complementary correlation between $X(t_1)$ and $X(t_2)$.

3.1 Bi-frequency Spectrum (Symmetric Form)

The **Bi-frequency Spectrum**, $S_{XX}(f_1,f_2)$, is a frequency-frequency distribution that expresses the correlation between two frequency components, f_1 and f_2. It is defined as the Symplectic Fourier Transform of $R_{XX}(t_1,t_2)$:

$$S_{XX}(f_1,f_2) = \iint_{-\infty}^{\infty} R_{XX}(t_1,t_2)e^{-j2\pi(f_1t_1-f_2t_2)}dt_1dt_2 \tag{7}$$

3.2 Complementary Bi-frequency Spectrum (Anti-Symmetric Form)

The **Complementary Bi-frequency Spectrum**, $\tilde{S}_{XX}(f_1,f_2)$, is defined as the Fourier Transform of the complementary autocorrelation function $\tilde{R}_{XX}(t_1,t_2)$ with an additive sign in the exponential:

$$\tilde{S}_{XX}(f_1,f_2) = \iint_{-\infty}^{\infty} \tilde{R}_{XX}(t_1,t_2)e^{-j2\pi(f_1t_1+f_2t_2)}dt_1dt_2 \tag{8}$$

This complementary form is often encountered in the analysis of non-stationary processes, particularly in relation to the Ambiguity Function (AF), due to its time-frequency representation duality.

4 Fourier Transform Duality

The Wigner-Ville Distribution (WVD) and the Spectral Correlation Density Function (SCD) constitute a **two-dimensional Fourier Transform pair**. This duality links the joint time-lag representation of the autocorrelation function, $R_X(t,\tau)$, directly to the joint cycle-frequency representation of the spectral correlation, $S_X(\alpha,f)$.

4.1 Duality between WVD (W_X) and Spectral Correlation Density Function (S_X)

The relationship connects the WVD, $W_X(t,f)$, derived from the transformed autocorrelation $R_X(t,\tau)$, with the Spectral Correlation Density Function $S_X(\alpha,f)$. The entire system of transformations is defined by the following two Fourier Transform relationships:

4.1.1 Relationship Formula (Two-Dimensional Duality)

The full 2D Fourier Transform relationship is based on the conjugate variable pairs:

- $t \leftrightarrow \alpha$ (Time $\leftrightarrow$ Cycle Frequency)
- $\tau \leftrightarrow f$ (Time Lag $\leftrightarrow$ Average Frequency)

1. WVD from Autocorrelation (Lag $\leftrightarrow$ Frequency): The WVD $W_X(t,f)$ is the Fourier Transform of the transformed autocorrelation $R_X(t,\tau)$ with respect to the time lag τ:

$$W_X(t,f) = \mathcal{F}_{\tau\to f}\left\{R_X(t,\tau)\right\} = \int_{-\infty}^{\infty} R_X(t,\tau)e^{-j2\pi f\tau}d\tau \tag{9}$$

2. SCD from WVD (Time $\leftrightarrow$ Cycle Frequency): The Spectral Correlation Density Function $S_X(\alpha,f)$ is the Fourier Transform of the WVD $W_X(t,f)$ with respect to the time t, where the cycle frequency α is conjugate to t:

$$S_X(\alpha,f) = \mathcal{F}_{t\to\alpha}\left\{W_X(t,f)\right\} = \int_{-\infty}^{\infty} W_X(t,f)e^{-j2\pi\alpha t}dt \tag{10}$$

Combining these, $S_X(\alpha,f)$ is the 2D Fourier Transform of $R_X(t,\tau)$ with respect to both variables:

$$S_X(\alpha,f) = \mathcal{F}_{t\to\alpha}\mathcal{F}_{\tau\to f}\left\{R_X(t,\tau)\right\}$$

4.1.2 Interpretation of the Duality

- **WVD ($\mathbf{W_X(t,f)}$):** Time-frequency domain representation, showing the **instantaneous power** at frequency f at time t.
- **SCD ($\mathbf{S_X(\alpha,f)}$):** Spectral Correlation Density Function. It shows the spectral content of the WVD's variation over time t. The variable α is the **cycle frequency** conjugate to t.
- **Non-Stationarity Measure:** Non-stationary behavior is defined by $W_X(t,f)$ being time-dependent. If $W_X(t,f)$ is time-dependent, its Fourier Transform with respect to t, $S_X(\alpha,f)$, will be non-zero for $\alpha \neq 0$. Thus, **non-zero correlation components** ($\alpha \neq 0$) directly quantify the **non-stationarity** (e.g., cyclostationarity) of the process.
- **WSS Case:** If the process is WSS, $W_X(t,f)$ is constant with respect to t. Consequently, $S_X(\alpha,f)$ becomes a Dirac delta function $\delta(\alpha)$, confirming that $S_{XX}(f_1,f_2)$ is non-zero only on the main diagonal $f_1 = f_2$.

4.2 Duality between Complementary WVD ($\tilde{W}_X$) and Complementary Spectral Correlation Density Function ($\tilde{S}_X$)

For processes that exhibit specific forms of non-stationarity, such as improperness or cyclostationarity, the **Complementary WVD** provides a powerful analytic tool, dual to the Complementary SCD.

4.2.1 Relationship Formula (Complementary Duality)

The full Complementary 2D Fourier Transform relationship is defined as:

- $t \leftrightarrow \alpha$ (Time $\leftrightarrow$ Cycle Frequency)
- $\tau \leftrightarrow f$ (Time Lag $\leftrightarrow$ Negative Average Frequency)

1. Complementary WVD from Complementary Autocorrelation:

$$\tilde{W}_X(t,f) = \mathcal{F}_{\tau\to f}\left\{\tilde{R}_X(t,\tau)\right\} = \int_{-\infty}^{\infty} \tilde{R}_X(t,\tau)e^{-j2\pi f\tau}d\tau \tag{11}$$

2. Complementary SCD from Complementary WVD: The Complementary Spectral Correlation Density Function $\tilde{S}_X(\alpha, f)$ is the Fourier Transform of the Complementary WVD $\tilde{W}_X(t,f)$ with respect to t, incorporating the negative average frequency alignment:

$$\tilde{S}_X(\alpha,f) = \mathcal{F}_{t\to\alpha}\left\{\tilde{W}_X(t,-f)\right\} = \int_{-\infty}^{\infty} \tilde{W}_X(t,-f)e^{-j2\pi\alpha t}dt \tag{12}$$

This duality relates the non-conjugate correlation $\tilde{R}_{XX}(t_1,t_2)$ to the spectral correlation. The complementary spectrum is primarily used for analyzing **second-order non-stationary processes** where the non-conjugate correlation is non-zero (e.g., improper complex signals).

4.3 Relationship between Bi-frequency Spectra and SCDs

The transformation from the general Bi-frequency Spectra ($S_{XX}(f_1,f_2)$ and $\tilde{S}_{XX}(f_1,f_2)$) to the Spectral Correlation Density Functions ($S_X(\alpha,f)$ and $\tilde{S}_X(\alpha,f)$) is a mathematical necessity for establishing the direct two-dimensional Fourier Duality with the Wigner-Ville Distributions (WVDs) and for analyzing cyclostationary signals.

The transformation is based on mapping the individual frequency variables (f_1, f_2) to the average frequency (f) and the cycle frequency (α):

$$\begin{aligned} f &= \frac{f_1+f_2}{2} && \text{(Average Frequency: Conjugate to Time Lag } \tau\text{)} \\ \alpha &= f_1 - f_2 && \text{(Cycle Frequency: Conjugate to Time } t\text{)} \end{aligned}$$

4.3.1 Standard Spectral Correlation Density (SCD) Derivation

The SCD $S_X(\alpha,f)$ is the transformed version of the standard Bi-frequency Spectrum $S_{XX}(f_1,f_2)$:

$$S_X(\alpha,f) \triangleq S_{XX}(f+\alpha/2, f-\alpha/2) \tag{13}$$

The SCD quantifies the correlation between frequency components f_1 and f_2 that are separated by a constant difference α. A non-zero value at $\alpha \neq 0$ implies spectral correlation, characteristic of non-stationary processes like cyclostationary signals.

Derivation from WVD Duality We demonstrate this relationship by starting from the established Fourier duality between the WVD and the SCD. $S_X(\alpha,f)$ is the Fourier Transform of $W_X(t,f)$ with respect to t, and $W_X(t,f)$ is the Fourier Transform of $R_X(t,\tau)$ with respect to τ:

$$S_X(\alpha,f) = \int_{-\infty}^{\infty} W_X(t,f)e^{-j2\pi\alpha t}dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_X(t,\tau)e^{-j2\pi f\tau}d\tau\right]e^{-j2\pi\alpha t}dt$$

Changing the order of integration and substituting $R_X(t,\tau) = R_{XX}(t+\tau/2, t-\tau/2)$ (with unit Jacobian $dtd\tau = dt_1dt_2$):

$$S_X(\alpha,f) = \iint_{-\infty}^{\infty} R_{XX}(t_1,t_2)e^{-j2\pi(\alpha t + f\tau)}dt_1dt_2$$

Transformation of the Exponential Term We express the exponent argument in terms of the original time variables t_1 and t_2:

$$\alpha t + f\tau = \alpha\left(\frac{t_1+t_2}{2}\right) + f(t_1-t_2)$$

Grouping the terms with t_1 and t_2:

$$\alpha t + f\tau = \left(f+\frac{\alpha}{2}\right)t_1 + \left(\frac{\alpha}{2}-f\right)t_2$$

To match the Bi-frequency Spectrum definition, we rearrange the t_2 term to have a negative sign in front of the average frequency component:

$$\alpha t + f\tau = \left(f+\frac{\alpha}{2}\right)t_1 - \left(f-\frac{\alpha}{2}\right)t_2$$

Matching with Bi-frequency Spectrum Substituting this back into the SCD integral:

$$S_X(\alpha, f) = \iint_{-\infty}^{\infty} R_{XX}(t_1, t_2) e^{-j2\pi\left[\left(f+\frac{\alpha}{2}\right)t_1 - \left(f-\frac{\alpha}{2}\right)t_2\right]} dt_1 dt_2$$

By comparing this result to the Bi-frequency Spectrum definition (Equation 7), $S_{XX}(f_1, f_2)$, which uses the Symplectic Fourier Transform $\mathcal{F}_{t_1 \to f_1} \mathcal{F}_{t_2 \to -f_2}$, we see the direct coordinate transformation:

$$f_1 = f + \alpha/2 \quad \text{and} \quad f_2 = f - \alpha/2$$

This confirms the relationship: $S_X(\alpha, f) \triangleq S_{XX}(f + \alpha/2, f - \alpha/2)$.

4.3.2 Complementary Spectral Correlation Density (C-SCD) Derivation

The Complementary SCD $\tilde{S}_X(\alpha, f)$ is the transformed version of the Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$. Based on the duality definition chosen for consistency with the overall spectral correlation analysis (Section 4.2.2), the relationship is derived as:

$$\tilde{S}_X(\alpha, f) \triangleq \tilde{S}_{XX}\left(\frac{\alpha}{2} - f, \frac{\alpha}{2} + f\right) \tag{14}$$

The C-SCD quantifies the non-conjugate spectral correlation. While the SCD relates $X(f_1)$ to $X^*(f_2)$, the C-SCD relates $X(f_1)$ to $X(f_2)$. Non-zero values in $\tilde{S}_X(\alpha, f)$ indicate signal improperness, which is essential for analyzing processes like real-valued bandpass signals or single sideband modulated signals.

Derivation from Complementary WVD Duality We start from the Complementary Duality definition (Eq. 12), which uses the $\tilde{W}_X(t, -f)$ term to ensure consistency with the subsequent $S_Y(\alpha, f)$ derivation:

$$\tilde{S}_X(\alpha, f) = \int_{-\infty}^{\infty} \tilde{W}_X(t, -f) e^{-j2\pi\alpha t} dt$$

Substituting the $\tilde{W}_X$ definition and rearranging the terms:

$$\tilde{S}_X(\alpha, f) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \tilde{R}_X(t, \tau) e^{-j2\pi(-f)\tau} d\tau\right] e^{-j2\pi\alpha t} dt$$

Substituting $\tilde{R}_X(t, \tau) = \tilde{R}_{XX}(t + \tau/2, t - \tau/2)$ and converting the integration limits to t_1, t_2:

$$\tilde{S}_X(\alpha, f) = \iint_{-\infty}^{\infty} \tilde{R}_{XX}(t_1, t_2) e^{-j2\pi(\alpha t - f\tau)} dt_1 dt_2$$

Transformation of the Exponential Term Using the inverse coordinate relations $t = (t_1 + t_2)/2$ and $\tau = t_1 - t_2$, the exponent argument is transformed:

$$\alpha t - f\tau = \alpha\left(\frac{t_1 + t_2}{2}\right) - f(t_1 - t_2)$$

Grouping the terms based on t_1 and t_2:

$$\alpha t - f\tau = \left(\frac{\alpha}{2} - f\right) t_1 + \left(\frac{\alpha}{2} + f\right) t_2$$

Matching with Complementary Bi-frequency Spectrum Substituting this back into the $\tilde{S}_X(\alpha, f)$ integral:

$$\tilde{S}_X(\alpha, f) = \iint_{-\infty}^{\infty} \tilde{R}_{XX}(t_1, t_2) e^{-j2\pi\left[\left(\frac{\alpha}{2}-f\right)t_1 + \left(\frac{\alpha}{2}+f\right)t_2\right]} dt_1 dt_2$$

Comparing this to the Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ definition (Eq. 8), which uses the Ordinary Double Fourier Transform $\mathcal{F}_{t_1 \to f_1} \mathcal{F}_{t_2 \to f_2}$, the transformation is established by:

$$f_1 = \frac{\alpha}{2} - f \quad \text{and} \quad f_2 = \frac{\alpha}{2} + f$$

This rigorously confirms the established relationship: $\tilde{S}_X(\alpha, f) \triangleq \tilde{S}_{XX}(\frac{\alpha}{2} - f, \frac{\alpha}{2} + f)$.

4.4 Difference between Deterministic WVD and statistical WVDs

While the deterministic WVD serves as a powerful analytic tool for individual signal analysis, the extension to random processes via the statistical WVD introduces limitations concerning the completeness of recoverable statistical information due to the inherent operation of ensemble averaging.

4.4.1 Recovery of Autocorrelation Function

Unlike the deterministic WVD, the statistical WVD cannot recover the random process $X(t)$ itself, as the ensemble averaging process inherently removes phase information ($\mathbb{E}\{X(t)\}$). However, $W_X(t,f)$ can perfectly recover the statistical information it was derived from.

The **transformed two-variable autocorrelation function**, $R_X(t,\tau)$, is fully recoverable from $W_X(t,f)$ by applying the Inverse Fourier Transform with respect to the frequency variable f:

$$R_X(t,\tau) = \mathcal{F}^{-1}_{f\to\tau}\{W_X(t,f)\} = \int_{-\infty}^{\infty} W_X(t,f)e^{j2\pi f\tau}df \tag{15}$$

This confirms the direct and reversible mathematical duality between the instantaneous spectral content ($W_X(t,f)$) and the statistical temporal correlation ($R_X(t,\tau)$).

4.4.2 Completeness of Second-Order Correlation Information

While the deterministic WVD, $W_x(t,f)$, determines the signal $x(t)$ up to a phase ambiguity, the statistical WVD, $W_X(t,f)$, only provides the non-conjugate correlation information, $R_X(t,\tau)$. To fully characterize the **second-order correlation statistics** of the random process $X(t)$, both the standard autocorrelation $R_{XX}(t_1,t_2)$ and the **complementary autocorrelation** $\tilde{R}_{XX}(t_1,t_2)$ are required.

Since $\tilde{R}_X(t,\tau)$ is not derivable from $W_X(t,f)$, the knowledge of the **Complementary WVD,** $\tilde{W}_X(t,f)$, which is the Fourier Transform of $\tilde{R}_X(t,\tau)$, is **necessarily required** to retrieve the 'other half' of the second-order correlation information. This distinction is crucial for analyzing non-stationary processes where the non-conjugate term $\tilde{R}_{XX}$ is significant (e.g., improper complex signals).

5 Relation between Real Signal WVD and Complex Signal WVDs

In practical signal processing, we often deal with a real-valued random process $Y(t)$ derived from a complex-valued random process $X(t)$, typically in the form of a bandpass signal centered at frequency f_c. This section establishes the relationship between the statistical Wigner-Ville Distribution of the real signal, $W_Y(t,f)$, and the two Wigner-Ville Distributions of its complex baseband counterpart, $W_X(t,f)$ and $\tilde{W}_X(t,f)$.

Let $X(t)$ be a zero-mean complex random process, and let the real-valued random process $Y(t)$ be defined by modulating $X(t)$ to a carrier frequency f_c:

$$Y(t) = \mathrm{Re}\{X(t)e^{\mathrm{j}2\pi f_c t}\}$$

The relationship is derived by first expressing the transformed autocorrelation function $R_Y(t,\tau)$ in terms of $R_X(t,\tau)$ and $\tilde{R}_X(t,\tau)$. Based on the provided expansion, $R_Y(t,\tau)$ is:

$$R_Y(t,\tau) = \frac{1}{4}\left[R_X(t,\tau)e^{j2\pi f_c\tau} + R_X^*(t,\tau)e^{-j2\pi f_c\tau} + \tilde{R}_X(t,\tau)e^{j4\pi f_c t} + \tilde{R}_X^*(t,\tau)e^{-j4\pi f_c t}\right]$$

The relationship between $W_Y(t,f)$ and $W_X(t,f)$, $\tilde{W}_X(t,f)$ is given by taking the Fourier Transform $\mathcal{F}_{\tau\to f}$ of the $R_Y(t,\tau)$ expression. This results in the sum of four distinct terms:

$$W_Y(t,f) = \frac{1}{4}W_X(t,f-f_c) + \frac{1}{4}W_X^*(t,-f-f_c) + \frac{1}{4}\tilde{W}_X(t,f)e^{j4\pi f_c t} + \frac{1}{4}\tilde{W}_X^*(t,-f)e^{-j4\pi f_c t}$$

The four terms represent:

1. The spectral energy of $X(t)$ shifted to the positive frequency band (f_c).
2. The spectral energy of $X(t)$ shifted to the negative frequency band ($-f_c$). (Since W_X is real, $W_X^*(t,-f-f_c) = W_X(t,-f-f_c)$ can be used.)
3. The spectral interference term involving $\tilde{W}_X(t,f)$, oscillating at $2f_c$.
4. The conjugate spectral interference term involving $\tilde{W}_X^*(t,-f)$, oscillating at $-2f_c$.

5.1 Special Case: Proper-Complex Process

A complex process $X(t)$ is defined as **proper** (or circularly symmetric) if its complementary autocorrelation function is zero: $\tilde{R}_{XX}(t_1, t_2) = 0$ for all t_1, t_2. This is equivalent to having its complementary WVD equal to zero: $\tilde{W}_X(t, f) = 0$.

In this case, the relationship simplifies significantly:

$$W_Y(t,f) = \frac{1}{4}\left[W_X(t, f-f_c) + W_X^*(t, -f-f_c)\right] = \frac{1}{4}\left[W_X(t, f-f_c) + W_X(t, -f-f_c)\right]$$

This result shows that the WVD of the real bandpass signal $Y(t)$ is simply the sum of the WVD of the complex baseband signal $X(t)$ shifted symmetrically to $\pm f_c$, and scaled by 1/4 (due to the $\mathrm{Re}\{\cdot\}$ and power distribution factors).

5.2 Relationship between Spectral Correlation Density Functions

Applying the duality relationship $\mathcal{F}_{t\to\alpha}\{W_X(t,f)\} = S_X(\alpha, f)$ to the $W_Y(t,f)$ equation (from Section 5.1) allows us to express the Spectral Correlation Density Function of the real process, $S_Y(\alpha, f)$, in terms of S_X and $\tilde{S}_X$.

The WVD of the real signal $Y(t)$ is:

$$W_Y(t,f) = \frac{1}{4}W_X(t, f-f_c) + \frac{1}{4}W_X^*(t, -f-f_c) + \frac{1}{4}\tilde{W}_X(t,f)e^{j4\pi f_c t} + \frac{1}{4}\tilde{W}_X^*(t,-f)e^{-j4\pi f_c t}$$

Taking the Fourier Transform $\mathcal{F}_{t\to\alpha}$ on both sides of $W_Y(t,f)$:

1. **First Term (Standard SCD):**

   $$\mathcal{F}_{t\to\alpha}\{W_X(t, f-f_c)\} = S_X(\alpha, f-f_c)$$

2. **Second Term (Conjugate SCD):** Utilizing the property $\mathcal{F}\{g^*(t)\} = G^*(-\alpha)$:

   $$\mathcal{F}_{t\to\alpha}\{W_X^*(t, -f-f_c)\} = S_X^*(-\alpha, -f-f_c)$$

   Since $W_X(t,f)$ is real, $W_X(t,f) = W_X^*(t,f)$, which implies $S_X^*(-\alpha, f) = S_X(\alpha, f)$. Therefore, the second term simplifies to $S_X(\alpha, -f-f_c)$.

3. **Third Term (Complementary SCD):** Utilizing the frequency shift property and the Complementary Duality definition $\tilde{S}_X(\alpha, f) = \mathcal{F}_{t\to\alpha}\left\{\tilde{W}_X(t,-f)\right\}$:

   $$\mathcal{F}_{t\to\alpha}\left\{\tilde{W}_X(t,f)e^{j4\pi f_c t}\right\} = \left[\mathcal{F}_{t\to\alpha}\left\{\tilde{W}_X(t,f)\right\}\right]\Big|_{\alpha\to\alpha-2f_c}$$

   Using the identity $\int \tilde{W}_X(t,f)e^{-j2\pi\alpha t}dt = \tilde{S}_X(\alpha, -f)$:

   $$\Rightarrow \tilde{S}_X(\alpha - 2f_c, -f)$$

4. **Fourth Term (Conjugate Complementary SCD):** Utilizing the frequency shift property and the conjugate property $\mathcal{F}\{g^*(t)\} = G^*(-\alpha)$:

   $$\mathcal{F}_{t\to\alpha}\left\{\tilde{W}_X^*(t,-f)e^{-j4\pi f_c t}\right\} = \left[\mathcal{F}_{t\to\alpha}\left\{\tilde{W}_X^*(t,-f)\right\}\right]\Big|_{\alpha\to\alpha+2f_c}$$

   Using the identity $\mathcal{F}_{t\to\alpha}\left\{\tilde{W}_X^*(t,-f)\right\} = \tilde{S}_X^*(-\alpha, f)$:

   $$\Rightarrow \tilde{S}_X^*(-(\alpha+2f_c), f) = \tilde{S}_X^*(-\alpha-2f_c, f)$$

Combining these terms, the Spectral Correlation Density function $S_Y(\alpha, f)$ is:

$$S_Y(\alpha,f) = \frac{1}{4}S_X(\alpha, f-f_c) + \frac{1}{4}S_X^*(-\alpha, -f-f_c) + \frac{1}{4}\tilde{S}_X(\alpha-2f_c, -f) + \frac{1}{4}\tilde{S}_X^*(-\alpha-2f_c, f)$$

To present the relationship with visual symmetry and explicitly using the real-valued nature of $W_X(t,f)$:

$$= \frac{1}{4}S_X(\alpha, f-f_c) + \frac{1}{4}S_X(\alpha, -f-f_c) + \frac{1}{4}\tilde{S}_X(\alpha-2f_c, -f) + \frac{1}{4}\tilde{S}_X^*(-\alpha-2f_c, f)$$

$$= \frac{1}{4}S_X(\alpha, f-f_c) + \frac{1}{4}S_X^*(-\alpha, -f-f_c) + \frac{1}{4}\tilde{S}_X(\alpha-2f_c, -f) + \frac{1}{4}\tilde{S}_X^*(-\alpha-2f_c, f)$$

where we use the property of conjugate symmetry $S_X^*(-\alpha, -f) = S_X(\alpha, -f)$ to simplify the second term, which is mathematically justified because W_X is real.

5.2.1 Derivation of the S_X Terms utilizing Real W_X

The simplification of the second term ($W_X^* \to S_X$) is based on the fundamental property that $W_X(t, f)$ is real-valued.

The definition of $S_X(\alpha, f)$ is:

$$\text{(i) } S_X(\alpha, f) = \int_{-\infty}^{\infty} W_X(t, f) e^{-j2\pi\alpha t} dt$$

And with a frequency shift:

$$S_X(\alpha, f - f_c) = \int_{-\infty}^{\infty} W_X(t, f - f_c) e^{-j2\pi\alpha t} dt$$

For the second term in $W_Y(t, f)$, we use the fact that W_X is real ($W_X^* = W_X$) to simplify the integral:

$$\text{(ii) } \int_{-\infty}^{\infty} W_X^*(t, -f - f_c) e^{-j2\pi\alpha t} dt = \left(\int_{-\infty}^{\infty} W_X(t, -f - f_c) e^{j2\pi\alpha t} dt \right)^* = S_X^*(-\alpha, -f - f_c)$$

Since $W_X(t, f)$ is real, $W_X(t, f) = W_X^*(t, f)$, which implies $W_X(t, -f - f_c) = W_X^*(t, -f - f_c)$. Using the standard SCD definition on the real-valued W_X:

$$= \int_{-\infty}^{\infty} W_X(t, -f - f_c) e^{-j2\pi\alpha t} dt = S_X(\alpha, -f - f_c)$$

5.2.2 Interpretation of SCD Relationship

- **WSS/Proper Case ($\alpha = 0$):** If the complex process $X(t)$ is Wide-Sense Stationary (WSS) and Proper ($\tilde{S}_X = 0$), $S_Y(\alpha, f)$ is non-zero only on the main diagonal ($\alpha = 0$). It is defined by the baseband spectrum S_X shifted to $\pm f_c$.
- **Spectral Correlation Components ($\alpha \neq 0$):** The third and fourth terms, which involve the complementary spectrum $\tilde{S}_X$, introduce non-zero components at the cycle frequencies $\alpha = \pm \mathbf{2f_c}$. These components represent the spectral correlation between frequency components separated by $\pm 2f_c$ (i.e., $f_1 - f_2 = \pm 2f_c$). This explicitly confirms that the $\tilde{S}_X$ spectrum is responsible for the inherent **improperness** of the real signal $Y(t)$ when analyzed relative to its complex baseband $X(t)$.

5.3 Implication for Signal Analysis using WVD

- **Proper-Complex Signals:** If $X(t)$ is proper ($\tilde{W}_X(t, f) = 0$), $W_Y(t, f)$ only contains the spectral energy centered around the carrier frequency $\pm f_c$. All relevant instantaneous power information is contained within $W_X(t, f)$.
- **Improper-Complex Signals:** If $X(t)$ is improper (i.e., $\tilde{W}_X(t, f) \neq 0$), the WVD $W_Y(t, f)$ contains the two additional **interference terms** that oscillate at $\pm 2f_c$. These terms explicitly depend on the complementary WVD $\tilde{W}_X(t, f)$ and are critical for accurate analysis.
- **Complete Statistical Characterization:** The dependence of $W_Y(t, f)$ on both $W_X(t, f)$ and $\tilde{W}_X(t, f)$ reinforces the necessity of using both standard and complementary WVDs to fully characterize the second-order statistics of the underlying complex random process $X(t)$.

5.4 Implication for Spectral Correlation Density (SCD) Analysis

The analysis in the SCD domain provides a stationary measure of non-stationarity.

- **Proper Signals (SCD):** For proper signals ($\tilde{S}_X(\alpha, f) = 0$), the SCD of the real signal, $S_Y(\alpha, f)$, exhibits non-zero correlation components only at α values where $S_X(\alpha, f)$ is non-zero (typically near $\alpha = 0$ for WSS signals). The spectral correlation density is essentially confined to the baseband SCD shifted up and down.
- **Improper Signals (SCD):** If $X(t)$ is improper ($\tilde{S}_X(\alpha, f) \neq 0$), the SCD $S_Y(\alpha, f)$ will show distinct spectral correlation components concentrated at the **cycle frequencies $\alpha = \pm 2f_c$**.

- **Spectral Redundancy:** The existence of non-zero correlation at $\alpha = \pm 2f_c$ in $S_Y(\alpha, f)$ is a direct consequence of the spectral redundancy introduced when transforming an improper complex baseband signal $X(t)$ to the real bandpass signal $Y(t)$. Analyzing $S_Y(\pm 2f_c, f)$ allows for the precise estimation and utilization of the baseband improperness ($\tilde{S}_X$) that would otherwise be obscured in a standard power spectral density (PSD) analysis ($\alpha = 0$).

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