The Fourier Duality between Wigner-Ville Distribution (WVD) and Bi-frequency Spectrum: Non-Stationary Analysis

Gemini AI Assistant and Joon Ho Cho

November 6, 2025

Abstract

This document establishes the fundamental Fourier duality between two crucial tools for analyzing non-stationary signals: the **Wigner-Ville Distribution (WVD)**, which provides a time-frequency representation, and the **Bi-frequency Spectrum** ($\mathcal{S}_{XX}(f_1, f_2)$), which provides a frequency-frequency representation. We begin with the deterministic definition of the WVD, extending it to the statistical domain via the autocorrelation function. The key relationship is shown to be a **two-dimensional Fourier Transform pair** connecting the WVD (in t and f) and the Bi-frequency Spectrum (in average frequency f_c and difference frequency f_d). This duality is essential for characterizing non-stationarity: any time-dependence in the WVD translates directly into non-zero off-diagonal terms (where $f_1 \neq f_2$) in the Bi-frequency Spectrum.

1 Introduction to Non-Stationary Analysis Tools

The analysis of non-stationary random processes often involves two key statistical representations derived from the two-variable autocorrelation function $R_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X^*(t_2)\}$. To establish the connection, we first transform the time variables into **average time (t)** and **time difference (τ)**:

$$t = \frac{t_1 + t_2}{2}$$
$$\tau = t_1 - t_2$$

The transformed autocorrelation function is $\tilde{R}_{XX}(t, \tau) = R_{XX}(t + \tau/2, t - \tau/2)$.

2 Wigner-Ville Distribution (WVD)

2.1 Deterministic Wigner-Ville Distribution

The WVD was originally introduced in quantum mechanics and first applied to signal analysis for deterministic signals $x(t)$. The deterministic WVD, $W_x(t, f)$, is defined directly from the signal $x(t)$ using the instantaneous autocorrelation kernel.

Recall from the previous discussion that the Energy Spectral Density (ESD), $E_{xx}(f)$, of a finite-energy signal is the Fourier Transform of its *time-average(d)* autocorrelation $r_{xx}(\tau)$. The WVD extends this concept to the time-varying case by taking the Fourier Transform of the **instantaneous autocorrelation kernel**, $x(t + \tau/2)x^*(t - \tau/2)$, with respect to the delay τ:

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi f\tau} d\tau \tag{1}$$

This definition provides the instantaneous energy distribution of the signal $x(t)$ across the time-frequency plane (t, f).

Crucially, the total energy of the signal in the frequency domain, the ESD, is recovered by integrating the WVD over all time t:

$$\int_{-\infty}^{\infty} W_x(t, f)dt = E_{xx}(f) \tag{2}$$

This relationship confirms that the WVD serves as the **instantaneous spectral density** from which the long-term energy distribution (ESD) is obtained via time integration (averaging).

2.1.1 Signal Recovery from WVD

A significant property of the WVD, unlike the conventional Power Spectral Density (PSD) which loses phase information, is that the original signal $x(t)$ can be **recovered** from its WVD $W_x(t,f)$. Since the WVD is a **bi-linear transform**, this recovery is possible **up to a constant phase factor** ($e^{j\theta}$).

- The WVD retains enough information about the signal's phase (embedded in the instantaneous autocorrelation kernel) to allow for reconstruction.
- The core inversion is achieved by taking the Inverse Fourier Transform of $W_x(t,f)$ with respect to f to recover the autocorrelation kernel:

$$x(t+\tau/2)x^*(t-\tau/2) = \int_{-\infty}^{\infty} W_x(t,f)e^{j2\pi f\tau}df$$

- **Recovery of Instantaneous Power at $t=0$:** Setting the time $t=0$ and the delay $\tau=0$ in the kernel recovery process allows us to find the initial energy of the signal:

$$|x(0)|^2 = \int_{-\infty}^{\infty} W_x(0,f)df$$

- **Signal Recovery:** More generally, the signal $x(t)$ can be recovered. For instance, setting $\tau = 2t$ and assuming $x(0) \neq 0$ allows recovery:

$$x(2t)x^*(0) = \int_{-\infty}^{\infty} W_x(t,f)e^{j4\pi ft}df$$

because

$$x(t) = \frac{1}{x^*(0)}\int_{-\infty}^{\infty} W_x(t/2,f)e^{j2\pi ft}df \tag{3}$$

This formula is valid only when $\mathbf{x(0) \neq 0}$. The known value of $x(0)$ acts as a **reference point** that fixes the unknown constant phase factor. This demonstrates that $x(t)$ is uniquely determined (up to $e^{j\theta}$) by the knowledge of $W_x(t,f)$.

2.2 Statistical Wigner-Ville Distribution (WVD)

For a random process $X(t)$, the statistical WVD, $W_X(t,f)$, is obtained by taking the ensemble average of the deterministic WVD, or equivalently, by applying the Fourier transform to the transformed autocorrelation function $\tilde{R}_{XX}(t,\tau)$:

$$W_X(t,f) = \mathbb{E}\{W_x(t,f)\} = \mathcal{F}_{\tau\to f}\left\{\tilde{R}_{XX}(t,\tau)\right\} = \int_{-\infty}^{\infty} \tilde{R}_{XX}(t,\tau)e^{-j2\pi f\tau}d\tau \tag{4}$$

The statistical WVD provides the **instantaneous power spectral content** of the random process $X(t)$ at time t.

2.2.1 Relation with PSD

The long-term Power Spectral Density (PSD), $S_{XX}(f)$, which was proved to be the Fourier Transform of the time-averaged autocorrelation function $R_{Avg}(\tau)$, is recovered by averaging the statistical WVD over all time t:

$$S_{XX}(f) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} W_X(t,f)dt \tag{5}$$

This fundamental relationship confirms that the statistical WVD serves as the **Instantaneous Power Spectral Density (IPSD)** from which the long-term average spectrum (PSD) is obtained via time averaging.

2.2.2 Recovery of Autocorrelation Function

Unlike the deterministic WVD, the statistical WVD cannot recover the random process $X(t)$ itself, as the ensemble averaging process inherently removes phase information ($\mathbb{E}\{X(t)\}$). However, $W_X(t, f)$ can perfectly recover the statistical information it was derived from.

The **transformed two-variable autocorrelation function**, $\tilde{R}_{XX}(t, \tau)$, is fully recoverable from $W_X(t, f)$ by applying the Inverse Fourier Transform with respect to the frequency variable f:

$$\tilde{R}_{XX}(t, \tau) = \mathcal{F}^{-1}_{f \to \tau} \{W_X(t, f)\} = \int_{-\infty}^{\infty} W_X(t, f) e^{j2\pi f \tau} df \tag{6}$$

This confirms the direct and reversible mathematical duality between the instantaneous spectral content ($W_X(t, f)$) and the statistical temporal correlation ($\tilde{R}_{XX}(t, \tau)$).

3 Bi-frequency Spectrum

The **Bi-frequency Spectrum**, $S_{XX}(f_1, f_2)$, is a frequency-frequency distribution that expresses the correlation between two frequency components, f_1 and f_2. It is defined as the Symplectic Fourier Transform of $R_{XX}(t_1, t_2)$:

$$S_{XX}(f_1, f_2) = \iint_{-\infty}^{\infty} R_{XX}(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \tag{7}$$

3.1 Frequency Variable Transformation

To clearly link $S_{XX}(f_1, f_2)$ and $W_X(t, f)$, we transform the frequency variables into the **average frequency (f_c)** and **difference frequency (f_d)**:

$$f_c = \frac{f_1 + f_2}{2}$$
$$f_d = f_1 - f_2$$

The Bi-frequency Spectrum expressed in these new variables, $\mathcal{S}_{XX}(f_c, f_d)$, reveals the direct duality with the WVD.

4 Fourier Transform Duality

The WVD and the Bi-frequency Spectrum (in its average/difference frequency form) form a **two-dimensional Fourier Transform pair**.

4.1 Relationship Formula

The Bi-frequency Spectrum $\mathcal{S}_{XX}(f_c, f_d)$ is the Fourier Transform of the WVD $W_X(t, f_c)$ with respect to the **time (t)** variable, provided we align the frequency variables appropriately:

$$\mathcal{S}_{XX}(f_d, f_c) = \mathcal{F}_{t \to f_d} \{W_X(t, f_c)\} = \int_{-\infty}^{\infty} W_X(t, f_c) e^{-j2\pi f_d t} dt \tag{8}$$

4.2 Interpretation of the Duality

- **WVD ($\mathbf{W_X(t, f)}$):** Time-frequency domain representation, showing the **instantaneous power** at frequency f at time t.
- **Bi-frequency Spectrum ($\mathcal{S}_{\mathbf{XX}}(\mathbf{f_d}, \mathbf{f_c})$):** Frequency-frequency domain representation. It shows the spectral content of the WVD's variation over time t. The variable f_d is conjugate to t.
- **Non-Stationarity Measure:** Non-stationary behavior is defined by $W_X(t, f)$ being time-dependent. If $W_X(t, f)$ is time-dependent, its Fourier Transform with respect to t, $\mathcal{S}_{XX}(f_d, f_c)$, will be non-zero for $f_d \neq 0$. Thus, **non-zero off-diagonal terms** ($f_1 \neq f_2$, i.e., $f_d \neq 0$) in the Bi-frequency plane directly quantify the **non-stationarity** of the process.

- **WSS Case:** If the process is WSS, $W_X(t, f)$ is constant with respect to t. Consequently, $\mathcal{S}_{XX}(f_d, f_c)$ becomes a Dirac delta function $\delta(f_d)$, confirming that $S_{XX}(f_1, f_2)$ is non-zero only on the main diagonal $f_1 = f_2$.

5 Summary

The Wigner-Ville Distribution and the Bi-frequency Spectrum are dual representations of the second-order statistics of a random process, connected via a two-dimensional Fourier Transform. They are essential for a complete and complementary understanding of non-stationary signals.