The Fourier Duality between Statistical Wigner-Ville Distribution and Statistical Ambiguity Function (SAF)

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Abstract

This document establishes the fundamental mathematical relationship between the Statistical Wigner-Ville Distribution ($W_X(t, f)$) and the Statistical Ambiguity Function (SAF: $\chi_X(\tau, \nu)$) for a random process $X(t)$. We rigorously demonstrate that these two functions, both derived from the transformed two-variable autocorrelation function $\tilde{R}_{XX}(t, \tau)$, form a **two-dimensional Fourier Transform pair**. This duality is crucial in linking the **Instantaneous Power Spectral Density** (t, f domain) to the **Statistical Delay-Doppler** (τ, ν domain) characteristics of a random signal, providing a complete framework for analyzing and modeling time-varying communication and radar environments.

1 Introduction and Definition of Statistical Functions

In the analysis of random processes, quadratic time-frequency representations must be ensemble-averaged. Both the Wigner-Ville Distribution (WVD) and the Ambiguity Function (AF) are extended statistically by replacing the instantaneous autocorrelation kernel with the ensemble-averaged transformed autocorrelation function, $\tilde{R}_{XX}(t, \tau)$.

The transformed two-variable autocorrelation function is defined as:

$$\tilde{R}_{XX}(t, \tau) = \mathbb{E}\{X(t + \tau/2)X^*(t - \tau/2)\} \tag{1}$$

where $t = (t_1 + t_2)/2$ is the average time and $\tau = t_1 - t_2$ is the time lag (delay).

1.1 Statistical Wigner-Ville Distribution ($\mathbf{W_X(t, f)}$)

The Statistical WVD is obtained by taking the Fourier Transform of $\tilde{R}_{XX}(t, \tau)$ with respect to the time lag τ. It represents the **Instantaneous Power Spectral Density (IPSD)** in the **Time (t)-Frequency (f)** domain.

$$W_X(t, f) = \mathcal{F}_{\tau \to f}\left\{\tilde{R}_{XX}(t, \tau)\right\} = \int_{-\infty}^{\infty} \tilde{R}_{XX}(t, \tau)e^{-j2\pi f\tau}d\tau \tag{2}$$

1.2 Statistical Ambiguity Function ($\chi_{\mathbf{X}}(\tau, \nu)$)

The Statistical Ambiguity Function (SAF) is obtained by taking the Fourier Transform of $\tilde{R}_{XX}(t, \tau)$ with respect to the average time t. It represents the statistical distribution in the **Time Delay (τ)-Doppler Frequency (ν)** domain.

$$\chi_X(\tau, \nu) = \mathcal{F}_{t \to \nu}\left\{\tilde{R}_{XX}(t, \tau)\right\} = \int_{-\infty}^{\infty} \tilde{R}_{XX}(t, \tau)e^{-j2\pi\nu t}dt \tag{3}$$

2 The Two-Dimensional Fourier Transform Duality

The definitions in Equations 2 and 3 establish that $W_X(t, f)$ and $\chi_X(\tau, \nu)$ are related through a **two-dimensional Fourier Transform**, as they share the same kernel $\tilde{R}_{XX}(t, \tau)$.

2.1 Derivation: SAF from Statistical WVD

The AF can be viewed as the Inverse Fourier Transform of $W_X(t,f)$ with respect to t, followed by a Fourier Transform with respect to τ (though the variables are interchanged due to the definitions). A simpler and more standard derivation is achieved by transforming the Inverse Fourier Transform of $W_X(t,f)$.

Starting with the Inverse Fourier Transform of the Statistical WVD (Equation 2 inversion):

$$\tilde{R}_{XX}(t,\tau) = \int_{-\infty}^{\infty} W_X(t,f)e^{j2\pi f\tau}df$$

Now, substitute this expression for $\tilde{R}_{XX}(t,\tau)$ into the definition of the SAF (Equation 3):

$$\begin{aligned}
\chi_X(\tau,\nu) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} W_X(t,f)e^{j2\pi f\tau}df\right]e^{-j2\pi\nu t}dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_X(t,f)e^{j2\pi f\tau}e^{-j2\pi\nu t}dtdf \\
\chi_X(\tau,\nu) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_X(t,f)e^{-j2\pi(\nu t-f\tau)}dtdf
\end{aligned}$$

This confirms that the SAF is the 2D Fourier Transform of the Statistical WVD, mapping the (t,f) domain to the (ν,τ) domain.

2.2 Derivation: Statistical WVD from SAF

Conversely, the Statistical WVD can be recovered from the SAF using the 2D Inverse Fourier Transform of the relationship derived above.

Starting with the definition of the SAF (Equation 3) inversion:

$$\tilde{R}_{XX}(t,\tau) = \int_{-\infty}^{\infty} \chi_X(\tau,\nu)e^{j2\pi\nu t}d\nu$$

Substitute this into the definition of the Statistical WVD (Equation 2):

$$\begin{aligned}
W_X(t,f) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \chi_X(\tau,\nu)e^{j2\pi\nu t}d\nu\right]e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_X(\tau,\nu)e^{j2\pi\nu t}e^{-j2\pi f\tau}d\tau d\nu \\
W_X(t,f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_X(\tau,\nu)e^{j2\pi(\nu t-f\tau)}d\tau d\nu
\end{aligned}$$

3 Interpretation of the Statistical Duality

The established duality highlights how the statistical properties of a random process can be interchangeably viewed in two critical domains:

- **Statistical WVD $(\mathbf{t},\mathbf{f})$:** Provides insight into **how the power spectrum statistically changes over time**. This is useful for analyzing systems where the spectrum is rapidly varying (non-stationary).
- **Statistical AF (τ,ν):** Provides insight into **how the process's correlation properties depend on time delay and Doppler frequency**. This is essential for modeling realistic fading channels in communication (Time-Frequency Dispersive Channels).

In essence, the SAF quantifies the amount of correlation between two frequency components that have been shifted by ν and observed with a time lag of τ. This mathematical relationship ensures that the information contained within the instantaneous spectral domain (W_X) is identical to the statistical delay-Doppler domain (χ_X).