Frequency-Domain Characterization of an Up-Converted Complex Second-Order Stationary Process

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Abstract

This document defines the conditions for the **Complex Second-Order Stationarity (SOS)** and the concept of a **Proper Random Process (Proper RP)** for the complex random process $X(t)$, which is fundamental in communication systems. Furthermore, we deeply analyze the statistical characteristics of the real-valued signal $Y(t)$, obtained by modulating (up-converting) the baseband complex SOS process $X(t)$ with a carrier frequency f_c, in the **frequency domain**. Specifically, we derive the **Bi-frequency Spectrum** $S_{YY}(f_1, f_2)$ of $Y(t)$ and prove that $Y(t)$ is generally a **Cyclostationary (CS) process with a cyclic frequency** $\alpha = 2f_c$. We also clearly demonstrate, through its frequency-domain characteristics (the concentration of the Bi-frequency Spectrum along the $f_1 = f_2$ axis), that for $Y(t)$ to become a conventional Second-Order Stationary (SOS) process, the baseband complex signal $X(t)$ must satisfy the Zero-Mean and Proper RP conditions. This analysis provides a crucial foundation for understanding the spectral properties and conditions for stationarity preservation in frequency-translated signals.

1. Stationarity of Random Process

The **Stationarity** of a random process $X(t)$ means that its statistical properties are invariant with respect to time shifts. Depending on the extent of moment invariance, stationarity is categorized into Strict-Sense Stationary (SSS) and Wide-Sense Stationary (WSS). Second-order stationarity defines conditions on statistical moments, specifically the Mean, Autocorrelation, and Complementary Auto-correlation.

1.1. Conventional Definition of Wide-Sense Stationary (WSS)

Wide-Sense Stationarity (WSS) is the most commonly used definition of second-order stationarity, meaning that the first moment (mean) and second moment (auto-correlation function) are partially invariant over time. **The conventional WSS definition for a complex random process** $X(t)$ requires the following two conditions:

- **Invariance of the Mean**: The mean $\mu_X(t)$ of $X(t)$ must be a **constant** regardless of time.

$$\mu_X(t) = E[X(t)] = \mu_X \quad \text{(for all } t\text{)}$$

- **Autocorrelation Function's Dependence on Time Difference**: The autocorrelation function $R_{XX}(t, s)$ of $X(t)$ must depend only on the time difference $\tau = t - s$ between the two time instants.

$$R_{XX}(t, s) = E[X(t)X^*(s)] = R_{XX}(\tau) = R_{XX}(t - s)$$

Note: When $X(t)$ is a **complex** process, the above WSS definition may be insufficient to fully characterize the second-order statistical properties of the complex signal. For real processes, $\tilde{R}_{XX} = R_{XX}$, so there is no issue, but for complex processes, this definition alone cannot account for the signal's improperness. The **Complex Second-Order Stationarity (SOS)** discussed in the next section requires an additional condition on the complementary auto-correlation function for completeness.

2. Complete Mathematical Conditions for Complex Second-Order Stationary Process

For a complex random process $X(t)$ to satisfy the conditions for a **Complete Second-Order Stationary Process (Complex SOS)**, in addition to the two conditions of the conventional WSS definition, the following **Complementary Auto-correlation Function Condition** must also be met.

2.1. Condition on the Mean Path

The mean path $\mu_X(t)$ of the random process $X(t)$ must be a **constant** regardless of time. (Same as the WSS condition)

$$\mu_X(t) = E[X(t)] = \mu_X \quad \text{(for all } t\text{)}$$

Here, μ_X is a constant value, which can generally be **complex**.

2.2. Condition on the Autocorrelation Function

The autocorrelation function $R_{XX}(t, s)$ of $X(t)$ must depend only on the time difference $\tau = t - s$. (Same as the WSS condition)

$$R_{XX}(t, s) = E[X(t)X^*(s)] = R_{XX}(\tau) = R_{XX}(t - s)$$

Here, $X^*(s)$ is the conjugate of $X(s)$, and τ is the Time Lag.

2.3. Condition on the Complementary Autocorrelation Function (SOS Completeness)

The complementary autocorrelation function $\tilde{R}_{XX}(t, s) = E[X(t)X(s)]$ of $X(t)$ must depend only on the time difference $\tau = t - s$.

$$\tilde{R}_{XX}(t, s) = E[X(t)X(s)] = \tilde{R}_{XX}(\tau) = \tilde{R}_{XX}(t - s)$$

Emphasis: This condition on the complementary auto-correlation $\tilde{R}_{XX}$ is redundant if $X(t)$ is a **real** process, as $\tilde{R}_{XX}(\tau) = R_{XX}(\tau)$ and can be omitted from the WSS definition. However, if $X(t)$ is a **complex random process**, $\tilde{R}_{XX}$ contains **independent statistical information (improperness)** from R_{XX}. Thus, the condition that $\tilde{R}_{XX}(\tau)$ depends only on the time difference must be added to define **Complete Complex Second-Order Stationarity (SOS)**. Omitting this condition in the conventional WSS definition can lead to errors in complex signal analysis.

3. Complex Random Process: Definition of Proper RP

For a complex random process $X(t)$, the second-order moments are defined not only by the autocorrelation function (R_{XX}) and the complementary autocorrelation function ($\tilde{R}_{XX}$), but also by the autocovariance function (C_{XX}) and the complementary autocovariance function ($\tilde{C}_{XX}$).

3.1. Autocovariance and Complementary Autocovariance Functions

The autocovariance function $C_{XX}(t, s)$ and the complementary autocovariance function $\tilde{C}_{XX}(t, s)$ are defined as follows:

$$C_{XX}(t, s) = E[\{X(t) - \mu_X(t)\}\{X(s) - \mu_X(s)\}^*] = R_{XX}(t, s) - \mu_X(t)\mu_X^*(s)$$

$$\tilde{C}_{XX}(t, s) = E[\{X(t) - \mu_X(t)\}\{X(s) - \mu_X(s)\}] = \tilde{R}_{XX}(t, s) - \mu_X(t)\mu_X(s)$$

3.2. Definition of Proper Random Process (Proper RP)

A **Proper Random Process (Proper RP)** is a complex random process for which the complementary autocovariance is zero for all times t and s.

$$\tilde{C}_{XX}(t,s) = 0 \quad \text{(for all } t, s\text{)}$$

If $X(t)$ is **Second-Order Stationary** and satisfies the **Zero-Mean ($\mu_X = 0$)** condition, then $\tilde{R}_{XX}(t,s) = \tilde{C}_{XX}(t,s)$. In this case, the Proper RP condition simplifies to:

$$\tilde{R}_{XX}(\tau) = 0 \quad \text{(Condition for a Zero-Mean, SOS Proper RP)}$$

4. Stationarity Analysis of Real Random Process $Y(t)$

Given a complex SOS process $X(t)$, let us define the real random process $Y(t)$ obtained by up-converting $X(t)$ with a carrier frequency f_c as:

$$Y(t) = \mathrm{Re}\{X(t)e^{\mathrm{j}2\pi f_c t}\} = \frac{1}{2}\left(X(t)e^{\mathrm{j}2\pi f_c t} + X^*(t)e^{-\mathrm{j}2\pi f_c t}\right)$$

We analyze whether $Y(t)$ satisfies the SOS condition in two cases. (SOS conditions: $E[Y(t)]$ is constant, and $R_{YY}(t,s) = R_{YY}(t-s)$)

4.1. Derivation of the General Mean Path $E[Y(t)]$ and Autocorrelation Function $R_{YY}(t,s)$

We derive the mean and autocorrelation function of $Y(t)$ when $X(t)$ is a general SOS process with mean μ_X.

4.1.1 Derivation of the Mean Path $E[Y(t)]$

The mean of $X(t)$ is $\mu_X = \mathrm{Re}\{\mu_X\} + \mathrm{j}\mathrm{Im}\{\mu_X\}$.

$$\begin{aligned}
E[Y(t)] &= E\left[\mathrm{Re}\{X(t)e^{\mathrm{j}2\pi f_c t}\}\right] \\
&= \mathrm{Re}\left\{E[X(t)]e^{\mathrm{j}2\pi f_c t}\right\} \\
&= \mathrm{Re}\left\{\mu_X(\cos(2\pi f_c t) + \mathrm{j}\sin(2\pi f_c t))\right\} \\
&= \mathrm{Re}\{(\mathrm{Re}\{\mu_X\} + \mathrm{j}\mathrm{Im}\{\mu_X\})(\cos(2\pi f_c t) + \mathrm{j}\sin(2\pi f_c t))\} \\
&= \mathrm{Re}\{\mu_X\}\cos(2\pi f_c t) - \mathrm{Im}\{\mu_X\}\sin(2\pi f_c t)
\end{aligned}$$

Since $E[Y(t)]$ is a time-varying function, in the general case where $\mu_X \neq 0$, $Y(t)$ is **not** SOS. Therefore, for $Y(t)$ to be SOS, μ_X must be zero.

4.1.2 Derivation of the Autocorrelation Function $R_{YY}(t,s)$

Since $Y(t)$ is a real process, $R_{YY}(t,s) \triangleq E[Y(t)Y(s)] = E[Y(t)Y^*(s)]$. We define $\tau = t - s$.

$$\begin{aligned}
R_{YY}(t,s) &= \frac{1}{4}E\left[\left(X(t)e^{\mathrm{j}2\pi f_c t} + X^*(t)e^{-\mathrm{j}2\pi f_c t}\right)\left(X(s)e^{\mathrm{j}2\pi f_c s} + X^*(s)e^{-\mathrm{j}2\pi f_c s}\right)\right] \\
&= \frac{1}{4}\Big(E[X(t)X(s)]e^{\mathrm{j}2\pi f_c(t+s)} + E[X(t)X^*(s)]e^{\mathrm{j}2\pi f_c(t-s)} \\
&\quad + E[X^*(t)X(s)]e^{-\mathrm{j}2\pi f_c(t-s)} + E[X^*(t)X^*(s)]e^{-\mathrm{j}2\pi f_c(t+s)}\Big)
\end{aligned}$$

Substituting the SOS properties of $X(t)$ ($R_{XX}(t,s) = R_{XX}(\tau)$, $\tilde{R}_{XX}(t,s) = \tilde{R}_{XX}(\tau)$):

$$\begin{aligned}
R_{YY}(t,s) &= \frac{1}{4}\left[\tilde{R}_{XX}(\tau)e^{\mathrm{j}2\pi f_c(t+s)} + R_{XX}(\tau)e^{\mathrm{j}2\pi f_c \tau} + R^*_{XX}(\tau)e^{-\mathrm{j}2\pi f_c \tau} + \tilde{R}^*_{XX}(\tau)e^{-\mathrm{j}2\pi f_c(t+s)}\right] \\
&= \frac{1}{2}\left[\mathrm{Re}\left\{\tilde{R}_{XX}(\tau)e^{\mathrm{j}2\pi f_c(t+s)}\right\} + \mathrm{Re}\left\{R_{XX}(\tau)e^{\mathrm{j}2\pi f_c \tau}\right\}\right]
\end{aligned}$$

The first Re term contains the $t+s$ term in addition to τ. Thus, $R_{YY}(t,s)$ is generally **not** a function of $\tau = t - s$ alone.

4.2. Case (i): $X(t)$ is an SOS Process with Zero Mean

We analyze the SOS conditions for $Y(t)$ when $X(t)$ is a zero-mean SOS process, i.e., $E[X(t)] = 0$.

4.2.1 Mean Path

Since $\mu_X = 0$, the derivation in 4.1.1 shows that $E[Y(t)] = 0$. The SOS mean condition is **satisfied**.

4.2.2 Autocorrelation Function

From the $R_{YY}(t,s)$ derivation in 4.1.2, the reason $R_{YY}(t,s)$ is not solely a function of τ is the presence of the $e^{\pm \mathrm{j}2\pi f_c(t+s)}$ terms, which occur when $\tilde{R}_{XX}(\tau) \neq 0$.

$$R_{YY}(t,s) = \frac{1}{2}\left[\mathrm{Re}\left\{\tilde{R}_{XX}(\tau)e^{\mathrm{j}2\pi f_c(t+s)}\right\} + \mathrm{Re}\left\{R_{XX}(\tau)e^{\mathrm{j}2\pi f_c\tau}\right\}\right]$$

For $Y(t)$ to be SOS, the $t+s$ term must be eliminated, which is only possible if $\tilde{R}_{XX}(\tau) = 0$. Therefore, if $\tilde{R}_{XX}(\tau) \neq 0$, $Y(t)$ is not SOS.

Conclusion: Generally, $Y(t)$ is not SOS. (No)

4.3. Case (ii): $X(t)$ is a Zero-Mean and Proper RP SOS Process

$X(t)$ being zero-mean and Proper RP means that $\mu_X = 0$ and $\tilde{R}_{XX}(\tau) = 0$ (refer to Section 3). We substitute this condition into the $R_{YY}(t,s)$ equation:

4.3.1 Mean Path

Since $\mu_X = 0$, $E[Y(t)] = 0$, and the SOS mean condition is **satisfied**.

4.3.2 Autocorrelation Function

Substituting $\tilde{R}_{XX}(\tau) = 0$ into the $R_{YY}(t,s)$ equation from 4.1.2:

$$\begin{aligned} R_{YY}(t,s) &= \frac{1}{2}\left[\mathrm{Re}\left\{0 \cdot e^{\mathrm{j}2\pi f_c(t+s)}\right\} + \mathrm{Re}\left\{R_{XX}(\tau)e^{\mathrm{j}2\pi f_c\tau}\right\}\right] \\ &= \frac{1}{2}\mathrm{Re}\left\{R_{XX}(\tau)e^{\mathrm{j}2\pi f_c\tau}\right\} \end{aligned}$$

The result shows that $R_{YY}(t,s)$ is expressed solely as a function of $\tau = t - s$. Since both the mean condition ($E[Y(t)] = 0$) and the autocorrelation condition ($R_{YY}(t,s) = R_{YY}(\tau)$) are satisfied, $Y(t)$ is SOS.

Conclusion: When $X(t)$ is a Zero-Mean Proper RP, $Y(t)$ is SOS. (Yes)

4.4. Verification of Cyclostationarity Condition

For $Y(t)$ to be a second-order Cyclostationary (CS) process, the mean path $E[Y(t)]$ and the autocorrelation function $R_{YY}(t,s)$ must be periodic, sharing an identical or common period T.

4.4.1 Periodicity of the Mean Path $E[Y(t)]$

The mean path derived in Section 4.1.1 is:

$$E[Y(t)] = \mathrm{Re}\{\mu_X\}\cos(2\pi f_c t) - \mathrm{Im}\{\mu_X\}\sin(2\pi f_c t)$$

This function is periodic with a period of $T_E = 1/f_c$.

4.4.2 Periodicity of the Autocorrelation Function $R_{YY}(t,s)$

The autocorrelation function $R_{YY}(t,s)$ is a periodic function with a period of $T_R = 1/(2f_c)$ in both t and s.

$$R_{YY}\left(t + \frac{1}{2f_c}, s + \frac{1}{2f_c}\right) = R_{YY}(t,s)$$

This shows that $R_{YY}(t,s)$ is a periodic function with period $T_R = 1/(2f_c)$, proving that the real-valued modulated signal $Y(t)$ satisfies the conditions of a second-order Cyclostationary process.

4.4.3 Determination of the Fundamental Period T_0 of Cyclostationarity

The fundamental period T_0 of the Cyclostationary process $Y(t)$ is determined by the **Least Common Multiple (LCM)** of the mean path period T_E and the autocorrelation function period T_R. Here, $T_E = 1/f_c$ and $T_R = 1/(2f_c)$, so $T_E = 2T_R$.

- **Case A: Non-Zero Mean ($\mu_X \neq 0$)** The mean path $E[Y(t)]$ has a period of $T_E = 1/f_c$ and is non-zero. Thus, $T_0 = \text{LCM}(T_E, T_R) = \text{LCM}(1/f_c, 1/(2f_c)) = \mathbf{1/f_c}$.
- **Case B: Zero Mean ($\mu_X = 0$)** Since $E[Y(t)] = 0$, the period of the mean path can be any $T > 0$. In this case, the periodicity is determined only by the autocorrelation function $R_{YY}(t,s)$, and $T_0 = T_R = \mathbf{1/(2f_c)}$.

4.5. Summary

The analysis in this section clearly identified the stationarity and cyclostationarity conditions for the real signal $Y(t)$ obtained by up-converting a baseband zero-mean SOS complex process $X(t)$.

- $Y(t)$ is generally **not Second-Order Stationary (SOS)** because its autocorrelation function includes the $t+s$ term.
- $Y(t)$ is a **Second-Order Cyclostationary (CS) process with a cyclic frequency** $\alpha = 2f_c$ due to $R_{YY}(t,s)$. Its fundamental period T_0 is:
  - If $X(t)$ is non-zero mean ($\mu_X \neq 0$): $T_0 = 1/f_c$
  - If $X(t)$ is zero mean ($\mu_X = 0$): $T_0 = 1/(2f_c)$
- The necessary and sufficient condition for $Y(t)$ to be SOS is that the baseband signal $X(t)$ must satisfy the **Zero-Mean Proper RP** condition ($\mu_X = 0, \tilde{R}_{XX}(\tau) = 0$). Only under this condition does $R_{YY}(t,s)$ simplify to a function of $\tau = t - s$ alone.

These results emphasize the importance of the Proper RP condition when applying complex baseband modeling to the analysis of real-valued modulated signals and serve as a starting point for understanding non-stationarity.

5. Bi-frequency Spectrum of a Complex Random Process

5.1. Definition of Bi-frequency Spectrum for Non-Stationary Processes

To analyze the statistical characteristics of a non-stationary complex random process $X(t)$ in the frequency domain, two forms of the Bi-frequency Spectrum are defined.

Loève Bi-frequency Spectrum $S_{XX}(f_1, f_2)$

Defined as the two-dimensional Fourier Transform of the autocorrelation function $R_{XX}(t,s)$.

$$S_{XX}(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t,s)e^{-\mathrm{j}2\pi(f_1 t - f_2 s)}dtds$$

Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$

Defined as the two-dimensional Fourier Transform of the complementary autocorrelation function $\tilde{R}_{XX}(t,s)$.

$$\tilde{S}_{XX}(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{R}_{XX}(t,s)e^{-\mathrm{j}2\pi(f_1 t + f_2 s)}dtds$$

5.2. Simplification of the Bi-frequency Spectrum for an SOS Process

When the random process $X(t)$ satisfies the conditions of an **SOS process**, $R_{XX}(t,s)$ and $\tilde{R}_{XX}(t,s)$ depend only on the time difference $\tau = t - s$. Consequently, the Bi-frequency Spectrum simplifies into a form involving the one-dimensional Spectrum Density (PSD) and the Dirac delta function.

Simplification of the Loève Bi-frequency Spectrum

Substituting the SOS condition $R_{XX}(t,s) = R_{XX}(t-s) = R_{XX}(\tau)$ into the definition of $S_{XX}(f_1, f_2)$ and using $\tau = t - s$, s as integration variables ($t = \tau + s$):

$$S_{XX}(f_1, f_2) = \left(\int_{-\infty}^{\infty} R_{XX}(\tau) e^{-\mathrm{j}2\pi f_1 \tau} d\tau \right) \left(\int_{-\infty}^{\infty} e^{-\mathrm{j}2\pi (f_1 - f_2)s} ds \right)$$

The first term is the **Power Spectral Density (PSD)** $S_{XX}(f_1)$, defined as the Fourier Transform of the autocorrelation function $R_{XX}(\tau)$, and the second term is the **Dirac Delta Function** $\delta(f_1 - f_2)$.

$$S_{XX}(f_1) \triangleq \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-\mathrm{j}2\pi f_1 \tau} d\tau$$

Thus, the Loève Bi-frequency Spectrum for an SOS process simplifies to:

$$S_{XX}(f_1, f_2) = S_{XX}(f_1)\delta(f_1 - f_2)$$

Simplification of the Complementary Bi-frequency Spectrum

Similarly, substituting the SOS condition $\tilde{R}_{XX}(t,s) = \tilde{R}_{XX}(t-s) = \tilde{R}_{XX}(\tau)$ into the definition of $\tilde{S}_{XX}(f_1, f_2)$:

$$\tilde{S}_{XX}(f_1, f_2) = \left(\int_{-\infty}^{\infty} \tilde{R}_{XX}(\tau) e^{-\mathrm{j}2\pi f_1 \tau} d\tau \right) \left(\int_{-\infty}^{\infty} e^{-\mathrm{j}2\pi (f_1 + f_2)s} ds \right)$$

Therefore, the Complementary Bi-frequency Spectrum for an SOS process simplifies to:

$$\tilde{S}_{XX}(f_1, f_2) = \tilde{S}_{XX}(f_1)\delta(f_1 + f_2)$$

6. Derivation of the Bi-frequency Spectrum $S_{YY}(f_1, f_2)$ of the Real Modulated Signal $Y(t)$

The Bi-frequency Spectrum $S_{YY}(f_1, f_2)$ of $Y(t)$ is defined as the two-dimensional Fourier Transform of the autocorrelation function $R_{YY}(t,s)$.

6.1. Term-by-Term Derivation of the Bi-frequency Spectrum

Applying the Fourier Transform to each term of $R_{YY}(t,s)$ results in the sum of four terms:

Term 1: Transform of $\frac{1}{4}\tilde{R}_{XX}(\tau)e^{\mathrm{j}2\pi f_c (t+s)}$

$$\mathrm{FT}_1 = \frac{1}{4}\tilde{S}_{XX}(f_1 - f_c)\delta(f_1 - f_2 - 2f_c)$$

Term 2: Transform of $\frac{1}{4}R_{XX}(\tau)e^{\mathrm{j}2\pi f_c \tau}$

$$\mathrm{FT}_2 = \frac{1}{4}S_{XX}(f_1 - f_c)\delta(f_1 - f_2)$$

Term 3: Transform of $\frac{1}{4}R^*_{XX}(\tau)e^{-\mathrm{j}2\pi f_c \tau}$

$$\mathrm{FT}_3 = \frac{1}{4}S_{XX}(-(f_1 + f_c))\delta(f_1 - f_2)$$

Term 4: Transform of $\frac{1}{4}\tilde{R}^*_{XX}(\tau)e^{-\mathrm{j}2\pi f_c (t+s)}$

$$\mathrm{FT}_4 = \frac{1}{4}\tilde{S}^*_{XX}(-(f_1 + f_c))\delta(f_1 - f_2 + 2f_c)$$

6.2. Final Bi-frequency Spectrum $S_{YY}(f_1, f_2)$

Since $S_{YY}(f_1, f_2) = \sum_{i=1}^{4} \mathrm{FT}_i$, the final Bi-frequency Spectrum of $Y(t)$ is given by:

$$\begin{aligned} S_{YY}(f_1, f_2) = \frac{1}{4}\Big[&\tilde{S}_{XX}(f_1 - f_c)\delta(f_1 - f_2 - 2f_c) + S_{XX}(f_1 - f_c)\delta(f_1 - f_2) \\ &+ S_{XX}(-(f_1 + f_c))\delta(f_1 - f_2) + \tilde{S}^*_{XX}(-(f_1 + f_c))\delta(f_1 - f_2 + 2f_c)\Big] \end{aligned}$$

Interpretation

Terms FT_1 and FT_4 show that spectral components exist on the axes $f_2 = f_1 \pm 2f_c$. This signifies that $R_{YY}(t, s)$ contains components dependent on the sum $t + s$ of t and s. These components indicate that $Y(t)$ is generally **not Second-Order Stationary (SOS)** and represents a **Cyclostationary process** with a cyclic frequency $\alpha = 2f_c$. (Refer to Section 4.4.2)

6.3. Characteristics of $S_{YY}(f_1, f_2)$ under the Proper RP Condition ($\tilde{S}_{XX}(f) = 0$)

If $X(t)$ is a Proper RP, then $\tilde{R}_{XX}(\tau) = 0$, which implies $\tilde{S}_{XX}(f) = 0$. In this case, terms FT_1 and FT_4 vanish, and $S_{YY}(f_1, f_2)$ simplifies to:

$$\begin{aligned} S_{YY}(f_1, f_2)|_{\text{Proper RP}} &= \frac{1}{4}\Big[S_{XX}(f_1 - f_c)\delta(f_1 - f_2) + S_{XX}(-(f_1 + f_c))\delta(f_1 - f_2)\Big] \\ &= \frac{1}{4}\Big[S_{XX}(f_1 - f_c) + S_{XX}(-(f_1 + f_c))\Big]\delta(f_1 - f_2) \end{aligned}$$

When $f_2 = f_1$, the conventional Power Spectral Density $S_{YY}(f)$ is:

$$S_{YY}(f) = \frac{1}{4}\Big[S_{XX}(f - f_c) + S_{XX}(-(f + f_c))\Big]$$

This result re-confirms that $Y(t)$ is SOS only under the Proper RP condition, as the spectrum exists only on the $f_2 = f_1$ axis. Furthermore, it shows that the PSD of the real modulated signal is the sum of the PSD of the baseband complex signal $X(t)$ shifted to f_c and $-f_c$.