Mean-Square (MS) Periodic vs. Wide-Sense Periodic (WSP) Random Processes

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Abstract

This document explores the properties and implications of a **Mean-Square (MS) Periodic Random Process** within the framework of **second-order stochastic processes**. An MS periodic process $\{X(t), t \in \mathbb{R}\}$ with period T is defined by the stringent condition that the mean-square error between $X(t+T)$ and $X(t)$ is zero, implying $X(t+T) = X(t)$ **almost surely**. This strong constraint ensures that the process's realizations are virtually deterministic periodic functions. We rigorously demonstrate that MS periodicity guarantees the periodicity of its fundamental statistical moments: the **mean function** $m_X(t)$ and the **autocorrelation function** $R_X(t_1, t_2)$, with the same period T. Finally, we clarify that the MS periodic condition is a significantly stronger requirement than, yet is a sufficient condition for, the **Wide-Sense Periodic (WSP)** property, highlighting the consistency and structural rigidity imposed on the process's sample paths.

1 Prerequisites: Second-Order Random Process

A random process $\{X(t), t \in \mathbb{R}\}$ must be a **second-order random process** to satisfy the conditions for MS periodicity.

1.1 Second-Order Condition

The process $X(t)$ is second-order if its second moment is finite for all t:

$$\mathbb{E}\{|X(t)|^2\} < \infty \quad \text{for all } t \in \mathbb{R}$$

This condition ensures that the mean function $m_X(t)$ and the autocorrelation function $R_X(t_1, t_2)$ are well-defined and finite.

2 Definition of an MS Periodic Random Process

A second-order random process $\{X(t), t \in \mathbb{R}\}$ is said to be **periodic in the Mean-Square (MS) sense** with period $T > 0$ if the following strong condition holds for all time t.

2.1 Mean-Square Convergence Condition

The mean-square error between the random variable at time $t + T$ and the random variable at time t is zero.

$$\mathbb{E}[|X(t+T) - X(t)|^2] = 0 \quad \text{for all } t \in \mathbb{R}$$

Interpretation: Almost Sure Equality The MS condition $\mathbb{E}[|Y|^2] = 0$ for a random variable Y implies that Y is equal to zero **with probability 1** (almost surely). Therefore, the MS periodic condition is equivalent to:

$$X(t+T) = X(t) \quad \text{with probability 1 (a.s.)}$$

This is a very strong requirement: almost every sample path (realization) of an MS periodic process is a **deterministic periodic function** with period T.

3 Key Statistical Properties (Consequences of MS Periodicity)

The MS condition forces all key statistical functions of the process to be periodic with the same period T.

3.1 Periodicity of the Mean Function

The mean function $m_X(t) = \mathbb{E}[X(t)]$ is periodic with period T.

$$m_X(t) = m_X(t+T)$$

Proof Sketch: This result follows directly from the MS condition by applying the Schwarz inequality (or Jensen's inequality for the absolute value):

$$|\mathbb{E}[X(t+T)] - \mathbb{E}[X(t)]|^2 = |\mathbb{E}[X(t+T) - X(t)]|^2 \leq \mathbb{E}[|X(t+T) - X(t)|^2] = 0$$

3.2 Periodicity of the Autocorrelation Function

The autocorrelation function $R_X(t_1, t_2) = \mathbb{E}[X(t_1)X^*(t_2)]$ is periodic with period T in both arguments, t_1 and t_2.

$$R_X(t_1, t_2) = R_X(t_1 + T, t_2) = R_X(t_1, t_2 + T)$$

Derivation: Since $X(t_1 + T) = X(t_1)$ almost surely, substituting this into the expectation yields:

$$R_X(t_1 + T, t_2) = \mathbb{E}[X(t_1 + T)X^*(t_2)] = \mathbb{E}[X(t_1)X^*(t_2)] = R_X(t_1, t_2)$$

A similar argument holds for the second argument t_2.

3.3 Periodicity of the Autocovariance Function

The autocovariance function $C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)m_X^*(t_2)$ is also periodic in both arguments:

$$C_X(t_1, t_2) = C_X(t_1 + T, t_2) = C_X(t_1, t_2 + T)$$

4 Relation to Wide-Sense Periodic (WSP) Processes

The MS periodic condition is a much stronger requirement than that for a WSP process.

- **Wide-Sense Periodic Process (WSP):** A second-order process $X(t)$ is WSP if it satisfies the periodicity of the mean and autocorrelation functions:
  1. $m_X(t) = m_X(t+T)$
  2. $R_X(t_1, t_2) = R_X(t_1 + T, t_2)$
- **Key Distinction (MS vs. WSP):** An **MS periodic process is always a WSP process**. However, a WSP process is **not** necessarily MS periodic. A WSP random process only requires the mean function to be periodic and the autocorrelation function to be periodic in t_1 and t_2. The fundamental difference lies in the constraint on sample paths.
  - For instance, consider replacing the values of an MS periodic process $X(t)$ at $t = nT$ (where n is an integer) with **independent and identically distributed (i.i.d.) random variables** Z_n.
  - The resulting process remains **WSP** because its statistical averages (mean and autocorrelation) are preserved over the period T.
  - However, the individual sample paths are no longer periodic because $X(nT)$ is not equal to $X((n+1)T)$ (i.e., $Z_n \neq Z_{n+1}$ with high probability). This violation of periodicity in the sample paths confirms that the new process **fails to satisfy the MS periodic condition**.
- **Conclusion:** An **MS periodic process is always a WSP process**. However, a WSP process is **not** necessarily MS periodic, as the WSP condition does not impose any constraints on the individual sample paths, allowing them to be non-periodic, provided their statistical averages satisfy the periodic rules. The MS condition guarantees a high degree of consistency across the realizations.

  The MS condition guarantees a high degree of consistency across the realizations by requiring almost sure equality, while the WSP condition only requires periodic statistical averages.