The Origin of Fourier Transform with Respect to Delay in Spectral Analysis

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Abstract

This document explores the fundamental origin of the **Fourier Transform with respect to the delay variable (τ)** in classical spectral analysis. We establish this principle by examining the analysis of a deterministic, finite-energy signal $x(t)$. The **Energy Spectral Density (ESD)**, $E_{xx}(f)$, is initially defined in the frequency domain as the magnitude squared of the signal's Fourier Transform. Through the application of the **Convolution Theorem** and the properties of time reversal and conjugation, we rigorously prove the equivalence of this definition to the Fourier Transform of the **time-averaged autocorrelation function**, $r_{xx}(\tau)$. This relationship, known as the **Deterministic Wiener-Khinchine Theorem**, provides the essential logical link: spectral content ($E_{xx}(f)$) is inherently dual to time correlation ($r_{xx}(\tau)$), thereby dictating the necessary transformation from the time-lag domain to the frequency domain for all subsequent spectral definitions, including the Power Spectral Density (PSD) for random processes.

1 Equivalence of ESD and Autocorrelation Fourier Transform

For a deterministic signal $x(t)$ with finite energy $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$, the equivalence between the Energy Spectral Density (ESD) and the Fourier Transform of the Time-Averaged Autocorrelation Function forms the foundation for the Power Spectral Density (PSD) extension.

1.1 Definition of Energy Spectral Density (ESD)

The ESD, $E_{xx}(f)$, is defined as the magnitude squared of the signal's Fourier Transform $X(f) = \mathcal{F}\{x(t)\}$:

$$E_{xx}(f) = |X(f)|^2 = X(f)X^*(f) = \left| \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \right|^2 \tag{1}$$

1.2 Definition of Time-Averaged Autocorrelation Function

The time-averaged autocorrelation function, $r_{xx}(\tau)$, measures the self-similarity of the signal $x(t)$ at a time lag τ:

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)x^*(t)dt = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)dt \tag{2}$$

1.3 Derivation of the Equivalence (Deterministic Wiener-Khinchine Theorem)

We use the Fourier Transform property that multiplication in the frequency domain corresponds to convolution in the time domain.

1. **Apply Inverse Fourier Transform to ESD:** The inverse Fourier Transform of the ESD $E_{xx}(f) = X(f)X^*(f)$ must be the convolution of $x(t)$ with the time-reversed conjugate of $x(t)$. We use the property $\mathcal{F}^{-1}\{X^*(f)\} = x^*(-t)$.

$$\mathcal{F}^{-1}\{E_{xx}(f)\} = \mathcal{F}^{-1}\{X(f)X^*(f)\} = x(t) * x^*(-t)$$

2. **Evaluate the Convolution:** The convolution $x(t) * x^*(-t)$ is defined as:

$$x(t) * x^*(-t) = \int_{-\infty}^{\infty} x(\lambda)x^*(-(t-\lambda))d\lambda = \int_{-\infty}^{\infty} x(\lambda)x^*(\lambda - t)d\lambda$$

3. **Identify the Autocorrelation:** By comparing the result of the convolution with the definition of the time-averaged autocorrelation $r_{xx}(\tau)$ (Equation 2), we see that the convolution result is $r_{xx}(t)$.

$$\mathcal{F}^{-1}\{E_{xx}(f)\} = r_{xx}(t)$$

4. **Conclusion of Equivalence:** Since $E_{xx}(f)$ is the Fourier Transform of $r_{xx}(t)$, we establish the fundamental equivalence:

$$E_{xx}(f) = \mathcal{F}\{r_{xx}(\tau)\} = \int_{-\infty}^{\infty} r_{xx}(\tau)e^{-j2\pi f\tau}d\tau \tag{3}$$

1.4 Summary

The result demonstrates that for deterministic, finite-energy signals, the Energy Spectral Density (ESD) defined by the magnitude squared of the Fourier Transform is equivalent to the Fourier Transform of the time-averaged autocorrelation function. This is the deterministic precursor to the Wiener-Khinchine Theorem for random processes.