Power Spectral Density of General Non-Stationary Processes :Derivation and Limitations of the Wiener-Khinchine Extension

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Abstract

This document presents a step-by-step rigorous derivation of the Power Spectral Density (PSD) for a general, second-order non-stationary random process, building upon the definition via the time-truncated Fourier transform. The derivation validates the **Wiener-Khinchine Theorem**, and reveals that the resulting PSD, $S_{XX}(f)$, is the Fourier transform of the **Time-Averaged Autocorrelation Function**, $\mathcal{R}_{XX}(\tau)$, which is obtained by averaging the two-variable autocorrelation function, $R_{XX}(t_1, t_2)$, over time $t = (t_1 + t_2)/2$ with $\tau = t_1 - t_2$. We conclude that while this definition provides a mathematically sound measure of average power distribution, the mandatory time-averaging step inherently **removes the process's non-stationary information**. Consequently, the standard PSD fails to capture the evolution of frequency content over time, effectively reducing the analysis to that of an equivalent Wide-Sense Stationary (WSS) process. This limitation highlights the necessity of time-frequency distributions for true non-stationary analysis.

1 Definition of PSD via Time-Truncated Fourier Transform

For a second-order random process $X(t)$, we define the time-truncated version $X_T(t)$ over the interval $[-T, T]$:

$$X_T(t) = \begin{cases} X(t) & \text{if } |t| \leq T \\ 0 & \text{if } |t| > T \end{cases}$$

The Fourier Transform of $X_T(t)$ is:

$$\mathcal{F}\{X_T(t)\} = \int_{-T}^{T} X(t)e^{-j2\pi ft}dt$$

The Power Spectral Density (PSD) $S_{XX}(f)$ is defined as the limit of the ensemble average of the magnitude squared of this Fourier transform, divided by the observation time $2T$:

$$S_{XX}(f) = \lim_{T\to\infty} \frac{1}{2T}\mathbb{E}\left[\left|\int_{-T}^{T} X(t)e^{-j2\pi ft}dt\right|^2\right] \tag{1}$$

This definition is consistent with interpreting the PSD as the average power at frequency f.

2 Derivation using Two-Variable Autocorrelation Function

2.1 Expanding the Magnitude Squared Term

We expand the magnitude squared term in Equation 1:

$$\begin{aligned}\left|\int_{-T}^{T} X(t)e^{-j2\pi ft}dt\right|^2 &= \int_{-T}^{T}\int_{-T}^{T} X(t_1)e^{-j2\pi ft_1}X^*(t_2)e^{j2\pi ft_2}dt_1dt_2 \\ &= \int_{-T}^{T}\int_{-T}^{T} X(t_1)X^*(t_2)e^{-j2\pi f(t_1-t_2)}dt_1dt_2\end{aligned}$$

2.2 Applying Ensemble Average

Substituting this back into the PSD definition and exchanging the expectation and integration (since the bounds are finite):

$$S_{XX}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} \mathbb{E}\left[X(t_1)X^*(t_2)\right] e^{-j2\pi f(t_1-t_2)} dt_1 dt_2$$

The ensemble average is the **two-variable autocorrelation function**, $R_{XX}(t_1, t_2)$.

$$S_{XX}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) e^{-j2\pi f(t_1-t_2)} dt_1 dt_2$$

2.3 Transformation to Average Time (t) and Delay (τ)

To separate the variables, we change the integration variables:

$$\begin{aligned} \tau &= t_1 - t_2 \quad \text{(Delay)} \\ t &= \frac{t_1 + t_2}{2} \quad \text{(Average Time)} \end{aligned}$$

The autocorrelation function becomes $\tilde{R}_X(t, \tau) = R_{XX}(t + \tau/2, t - \tau/2)$. The integral bounds also change, where t and τ are coupled.

$$S_{XX}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} e^{-j2\pi f\tau} \left(\int_{t\in D(\tau)} \tilde{R}_X(t, \tau) dt \right) d\tau$$

where $D(\tau)$ denotes the restricted domain of t for a fixed τ to keep the integration within the original square $[-T, T] \times [-T, T]$.

3 Wiener-Khinchine Theorem for Non-Stationary Processes and Limitation

For a general non-stationary process, $\tilde{R}_X(t, \tau)$ depends on both t and τ. We define the term inside the parenthesis as the **Time-Averaged Autocorrelation Function**, $\mathcal{R}_{XX}(\tau)$:

$$\mathcal{R}_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{t\in D(\tau)} \tilde{R}_X(t, \tau) dt$$

3.1 Conclusion of the Logical Flow

Substituting this definition back into the PSD equation yields:

$$S_{XX}(f) = \int_{-\infty}^{\infty} \mathcal{R}_{XX}(\tau) e^{-j2\pi f\tau} d\tau \tag{2}$$

Interpretation and Limitation:

- **General Result:** The PSD of a general non-stationary process is the **Fourier Transform** of its **Time-Averaged Autocorrelation Function** $\mathcal{R}_{XX}(\tau)$.
- **Loss of Time Resolution:** This result is structurally identical to the classic Wiener-Khinchine Theorem, but the PSD $S_{XX}(f)$ is now the spectrum of the time-averaged statistics. The $\lim_{T\to\infty} \frac{1}{2T} \int \cdots dt$ operation **removes the time dependency** (t) from the autocorrelation function $\tilde{R}_X(t, \tau)$.
- **The Dilemma:** The standard PSD definition, while mathematically rigorous for power signals, cannot capture the **evolution of frequency content over time** for a non-stationary process. It only provides the overall, long-term average power distribution across frequencies, effectively reducing the non-stationary process to an equivalent Wide-Sense Stationary (WSS) process with autocorrelation $\mathcal{R}_{XX}(\tau)$.

4 Wiener-Khinchine Theorem for WSS Processes

4.1 Variable Transformation and WSS Assumption

We now assume the process $X(t)$ is **Wide-Sense Stationary (WSS)**. This means $R_{XX}(t_1, t_2)$ depends only on the **time difference** $\tau = t_1 - t_2$. Thus, $R_{XX}(t_1, t_2) = R_{XX}(\tau)$.

We change variables:

- Delay (τ): $\tau = t_1 - t_2$
- Average Time (t): $t = t_2$ (or $t = (t_1 + t_2)/2$ can be used)

The double integral transforms, and the constant $R_{XX}(\tau)$ can be pulled out of the internal integral (which is an integral over t):

$$S_{XX}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} R_{XX}(\tau) e^{-j2\pi f\tau} \left(\int_{D(\tau)} dt \right) d\tau$$

where $D(\tau)$ represents the length of the internal integration domain, constrained by the original limits. For $|\tau| < 2T$, the length $\int_{D(\tau)} dt$ is exactly $2T - |\tau|$.

$$S_{XX}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} R_{XX}(\tau) e^{-j2\pi f\tau} (2T - |\tau|) d\tau$$

Dividing by $2T$:

$$S_{XX}(f) = \lim_{T\to\infty} \int_{-2T}^{2T} R_{XX}(\tau) e^{-j2\pi f\tau} \left(1 - \frac{|\tau|}{2T} \right) d\tau$$

4.2 Taking the Limit and Completing the Wiener-Khinchine Theorem

Taking the limit as $T \to \infty$, the term $\frac{|\tau|}{2T}$ goes to zero, and the integration bounds expand to infinity.

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi f\tau} d\tau$$

5 Summary of Logical Flow

The derivation follows this logical sequence:

- **PSD Definition:** Defined $S_{XX}(f)$ as the $T \to \infty$ limit of the ensemble average of the magnitude squared of the time-truncated Fourier spectrum, normalized by time $2T$.
- **Autocorrelation Introduction:** The $\mathbb{E}[\cdot]$ operation was moved inside the integral to introduce the **two-variable autocorrelation function** $R_{XX}(t_1, t_2)$.

The result demonstrates that the PSD is the Fourier Transform of the time-averaged autocorrelation function $\mathcal{R}_{XX}(\tau)$.

However, this approach of spectral analysis of a 2nd-order random process may lead to a serious limitation: **By necessity, the standard PSD loses all temporal information.** This is due to the inherent time-averaging process that must be performed for the PSD to be defined:

- **Core Operation:** The definition includes the time-averaging operation ($\lim_{T\to\infty} \frac{1}{2T} \int \cdots dt$).
- **Data Collapse:** This operation collapses the **two-variable autocorrelation function** ($R_X(t_1, t_2)$) into a single function of delay, $R_{Avg}(\tau)$ (or $\mathcal{R}_{XX}(\tau)$).
- **Effect of Smoothing:** This time-averaging effectively **smooths out** any time-varying characteristics of the process, preventing the detection or analysis of the evolution of the signal's frequency content over time.
- **Non-Stationary Failure:** Since the evolution of spectral content is the defining feature of non-stationary behavior, the standard PSD is fundamentally incapable of capturing the transient or evolving nature of the signal's spectrum.