The Wigner Distribution Function, SCD/CSCD Analysis, and Spectral Relationship of Up-converted Passband Signals for Complex Stochastic Processes: An SOCS Example

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Abstract

This document investigates the spectral analysis of a complex stochastic process $X(t)$ which is a T-periodic **Second-Order Cyclostationary Process (SOCS)**. The analysis begins by defining the process's second-order statistics, namely the **Autocorrelation Function (ACF)** and the **Complementary Autocorrelation Function (CACF)** kernels. These kernels are then used to introduce their time-frequency domain counterparts: the **Wigner Distribution Function (WDF)** and the **Complementary WDF (CWDF)**. By performing a Fourier transform of the WDF and CWDF with respect to the center time t, the document rigorously defines the **Spectral Correlation Density (SCD)** and the **Complementary Spectral Correlation Density (CSCD)**, and establishes their equivalence through coordinate transformation with the Bi-frequency Spectrum representations. The core analysis focuses on the real passband signal $Y(t)$, formed by up-converting $X(t)$ at a carrier frequency f_c. Explicit formulas for the ACF and WDF of $Y(t)$ are derived. Most importantly, the paper presents the relationship where the SCD $S_Y(\alpha, f)$ of the up-converted signal $Y(t)$ is expressed as a linear combination of the baseband SCD $S_X(\alpha, f)$ and CSCD $\tilde{S}_X(\alpha, f)$. Finally, the spectral properties are detailed for the specific case where $X(t)$ is an SOCS process, showing the discrete nature of the SCD and CSCD in terms of Cyclic Power Spectral Density (CPSD) and Complementary CPSD (CCPSD).

1. Definition of Wigner Distribution Function (WDF) and Complementary WDF (CWDF)

The WDF is defined using the ACF kernel as a function of the center time t and time difference τ. The Complementary WDF (CWDF) is defined using the CACF kernel.

1.1. WDF $W_X(t, f)$ of the Complex Baseband Signal $X(t)$

The WDF of $X(t)$ is defined as the Fourier transform of the ACF kernel $R_{XX}(t + \tau/2, t - \tau/2)$ with respect to the time difference τ:

$$W_X(t, f) = \int_{-\infty}^{\infty} R_{XX}(t + \tau/2, t - \tau/2) e^{-j2\pi f \tau} d\tau$$

1.2. Definition of Complementary Wigner Distribution Function (CWDF)

The **CWDF** $\tilde{W}_X(t, f)$ of $X(t)$ is defined as the Fourier transform of the CACF kernel $\tilde{R}_{XX}(t + \tau/2, t - \tau/2)$ with respect to the time difference τ:

$$\tilde{W}_X(t, f) = \int_{-\infty}^{\infty} \tilde{R}_{XX}(t + \tau/2, t - \tau/2) e^{-j2\pi f \tau} d\tau$$

2. Definition of Spectral Correlation Density (SCD) and Complementary SCD (CSCD)

The SCD and CSCD are defined by taking the Fourier transform of the WDF $W_X(t, f)$ and CWDF $\tilde{W}_X(t, f)$, respectively, with respect to the center time t.

2.1. Definition of Spectral Correlation Density (SCD)

The **SCD** $S_X(\alpha, f)$ of $X(t)$ is defined as the Fourier transform of the WDF $W_X(t, f)$ with respect to the time t:

$$S_X(\alpha, f) = \mathcal{F}_t \{W_X(t, f)\} = \int_{-\infty}^{\infty} W_X(t, f) e^{-j2\pi\alpha t} dt$$

where α is the **Cyclic Frequency**.

2.2. Definition of Complementary Spectral Correlation Density (CSCD)

The **CSCD** $\tilde{S}_X(\alpha, f)$ of $X(t)$ is defined as the Fourier transform of the CWDF $\tilde{W}_X(t, f)$ with respect to the time t:

$$\tilde{S}_X(\alpha, f) = \mathcal{F}_t \left\{\tilde{W}_X(t, f)\right\} = \int_{-\infty}^{\infty} \tilde{W}_X(t, f) e^{-j2\pi\alpha t} dt$$

3. Relationship between Bi-frequency Spectrum and SCD/CSCD

The Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ and the Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ are equivalent spectral representations linked to $S_X(\alpha, f)$ and $\tilde{S}_X(\alpha, f)$ through a frequency variable transformation.

3.1. Relationship between Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ and SCD $S_X(\alpha, f)$

The SCD can be obtained by transforming the Bi-frequency Spectrum to the coordinate system of $\alpha = f_1 - f_2$ and $f = (f_1 + f_2)/2$:

$$S_X(\alpha, f) = S_{XX}\left(f + \alpha/2, f - \alpha/2\right)$$

The Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ is defined as the 2D Fourier transform of the ACF $R_{XX}(t, s)$ with respect to t and s.

3.1.1 Geometric Interpretation of the SCD Transformation

The coordinate transformation from the Bi-frequency plane (f_1, f_2) to the Cyclic Spectrum plane (α, f) is described by the linear mapping:

$$\begin{bmatrix} \alpha \\ f \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Geometrically, this transformation represents a combination of rotation and non-uniform scaling:

1. **Rotation:** The (f_1, f_2) plane is effectively rotated by **45° counter-clockwise**. The α axis becomes perpendicular to the main diagonal ($f_1 = f_2$), and the f axis becomes parallel to the main diagonal.

2. **Scaling:** When decomposed into a 45° clockwise rotation followed by a scaling and reflection:

$$\begin{bmatrix} f \\ \alpha \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

   This implies that after rotation, the graph is scaled by $\mathbf{1/\sqrt{2}}$ along the f direction (mean frequency) and by $-\sqrt{\mathbf{2}}$ along the α direction (cyclic frequency), which includes a reflection (flip) and a scaling by $\sqrt{2}$.

3.2. Relationship between Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ and CSCD $\tilde{S}_X(\alpha, f)$

The CSCD can be obtained by transforming the Complementary Bi-frequency Spectrum:

$$\tilde{S}_X(\alpha, f) = \tilde{S}_{XX}\left(f + \alpha/2, -f + \alpha/2\right)$$

The Complementary Bi-frequency Spectrum $\tilde{S}_{XX}(f_1, f_2)$ is defined as the 2D Fourier transform of the CACF $\tilde{R}_{XX}(t, s)$ with respect to t and s.

3.2.1 Geometric Interpretation of the CSCD Transformation

The transformation from the Bi-frequency plane (f_1, f_2) to the Cyclic Spectrum plane (α, f) is defined by the linear mapping:

$$\begin{bmatrix} \alpha \\ f \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

This matrix can be decomposed as a reflection followed by the standard SCD transformation matrix:

$$\begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Geometrically, the CSCD transformation is achieved by the following sequence of operations on the $\tilde{S}_{XX}(f_1, f_2)$ surface:

1. **Reflection (Flip):** The Complementary Bi-frequency Spectrum is first reflected across the f_1 axis (i.e., $f_2 \to -f_2$). This is equivalent to applying the reflection matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
2. **SCD Transformation:** The standard SCD geometric transformation (rotation by 45° and non-uniform scaling/flip) is then applied to the reflected spectrum.

4. ACF, WDF, and SCD Representation of the Up-converted Signal $Y(t)$

$Y(t)$ is the real passband signal obtained by up-converting $X(t)$ to the carrier frequency f_c:

$$Y(t) = \mathrm{Re}\left\{X(t)e^{j2\pi f_c t}\right\} = \frac{1}{2}\left(X(t)e^{j2\pi f_c t} + X^*(t)e^{-j2\pi f_c t}\right)$$

4.1. Autocorrelation Function (ACF) of $Y(t)$

Under the zero-mean assumption, the ACF of $Y(t)$, $R_{YY}(t, s) = E[Y(t)Y(s)]$, is expressed in terms of the baseband ACF R_{XX} and CACF $\tilde{R}_{XX}$:

$$R_{YY}(t + \tau/2, t - \tau/2) = \frac{1}{2}\mathrm{Re}\left\{R_{XX}(t + \tau/2, t - \tau/2)e^{j2\pi f_c \tau} + \tilde{R}_{XX}(t + \tau/2, t - \tau/2)e^{j4\pi f_c t}\right\}$$

4.2. Relationship between $W_Y(t, f)$ and $W_X, \tilde{W}_X$

The WDF of $Y(t)$, $W_Y(t, f)$, is expressed in terms of the WDF $W_X(t, f)$ and CWDF $\tilde{W}_X(t, f)$ of $X(t)$ as:

$$\begin{aligned} W_Y(t, f) = &\frac{1}{4}\left[W_X(t, f - f_c) + W_X^*(t, f + f_c)\right] \\ &+ \frac{1}{4}\left[\tilde{W}_X(t, f)e^{j4\pi f_c t} + \tilde{W}_X^*(t, -f)e^{-j4\pi f_c t}\right] \end{aligned}$$

4.3. SCD Expression of $Y(t)$ (Relationship with S_X and $\tilde{S}_X$)

Taking the Fourier transform of the $W_Y(t, f)$ relationship with respect to time t, the SCD of $Y(t)$ is expressed using the SCD and CSCD of $X(t)$:

$$\begin{aligned} S_Y(\alpha, f) = &\frac{1}{4}\left[S_X(\alpha, f - f_c) + S_X^*(-\alpha, f + f_c)\right] \\ &+ \frac{1}{4}\left[\tilde{S}_X(\alpha - 2f_c, f) + \tilde{S}_X^*(-\alpha + 2f_c, -f)\right] \end{aligned}$$

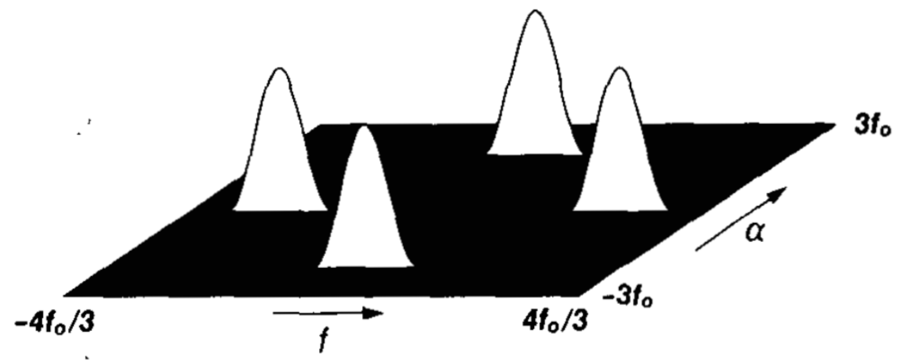

Fig. 1. Spectral correlation magnitude surface for an amplitude-modulated sine wave with carrier frequency f_0.

Figure 1: Fig.1 in [1]

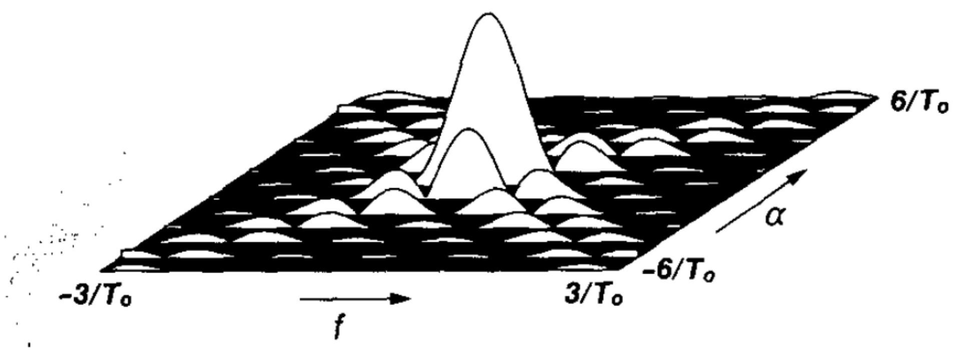

Fig. 2. Spectral correlation magnitude surface for a pulse-amplitude modulated pulse train with pulse rate $1/T_0$.

Figure 2: Fig.2 in [1]

5. Characteristics of the T-periodic SOCS Complex Stochastic Process $X(t)$ and Characteristics of $Y(t)$

$X(t)$ is defined as a T-periodic SOCS process, where its second-order statistics are T-periodic. This characteristic determines the properties of the up-converted signal $Y(t)$.

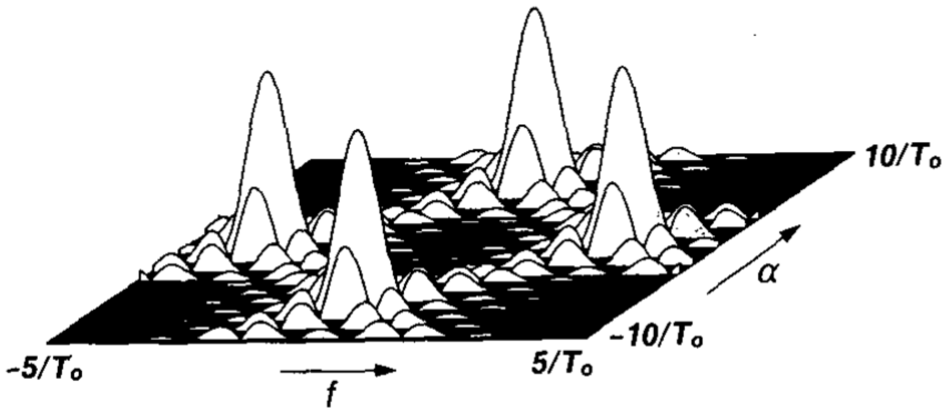

Fig. 3. Spectral correlation magnitude surface for amplitude-shift keying (and binary phase-shift keying) with carrier frequency = $3.3/T_0$.

Figure 3: Fig.3 in [2]

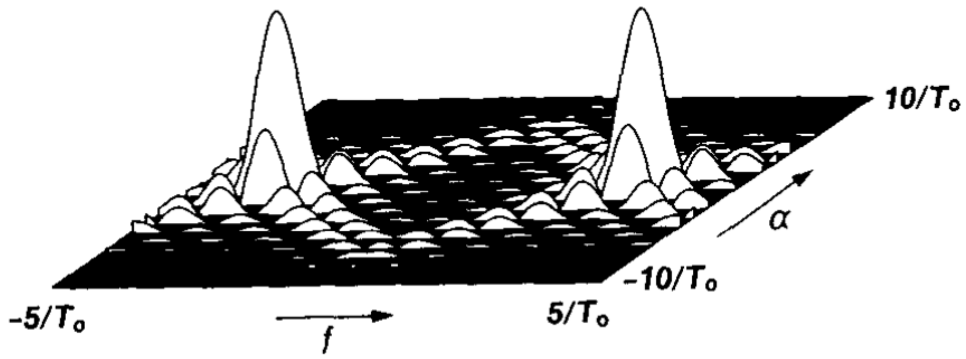

Fig. 4. Spectral correlation magnitude surface for balanced amplitude-phase-shift keying (and quaternary phase-shift keying) with carrier frequency = $3.3/T_0$.

Figure 4: Fig. 4 in [2]

5.1. $X(t)$'s Second-Order Statistics and Spectral Characteristics

5.1.1 Autocorrelation Function (ACF) and Complementary Autocorrelation Function (CACF)

The ACF $R_{XX}(t,s) = E[X(t)X^*(s)]$ and CACF $\tilde{R}_{XX}(t,s) = E[X(t)X(s)]$ of $X(t)$ are both T-periodic:

$$R_{XX}(t,s) = R_{XX}(t+T, s+T), \quad \tilde{R}_{XX}(t,s) = \tilde{R}_{XX}(t+T, s+T)$$

The ACF and CACF kernels, $R_{XX}(t+\tau/2, t-\tau/2)$ and $\tilde{R}_{XX}(t+\tau/2, t-\tau/2)$, exhibit a **Jointly Periodic** characteristic with respect to the center time t.

5.1.2 Fourier Series Expansion of ACF and CACF Kernels

Since the kernel of the T-periodic SOCS process has a period T with respect to the center time t, it can be expanded into a Fourier series as follows.

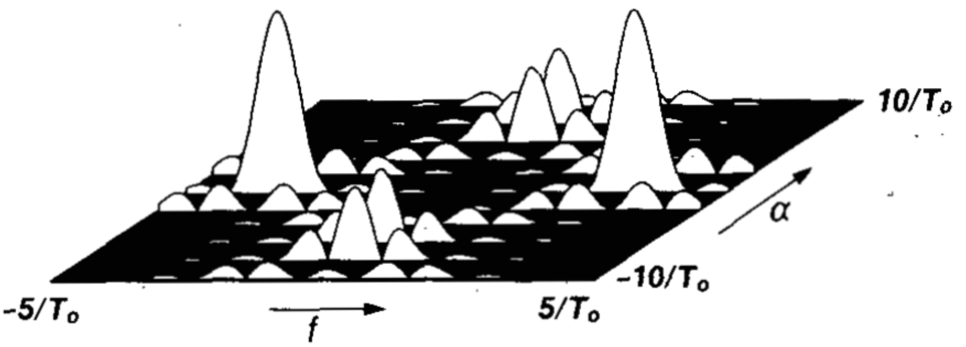

Fig. 5. Spectral correlation magnitude surface for staggered quarternary-phase-shift keying with carrier frequency $= 3.3/T_0$.

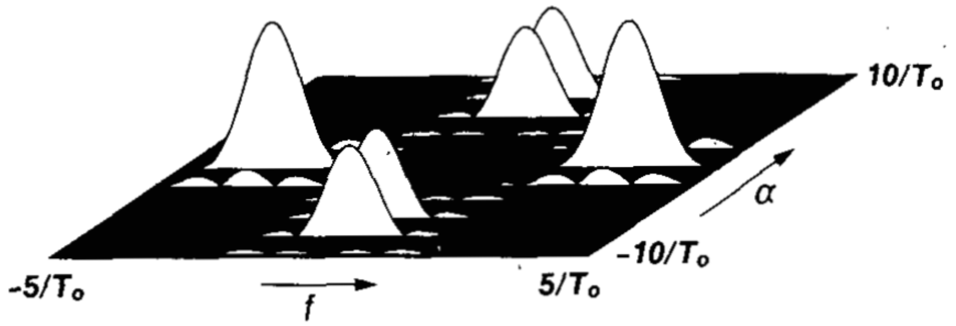

Fig. 6. Spectral correlation magnitude surface for minimum-shift keying with carrier frequency $= 3.3/T_0$.

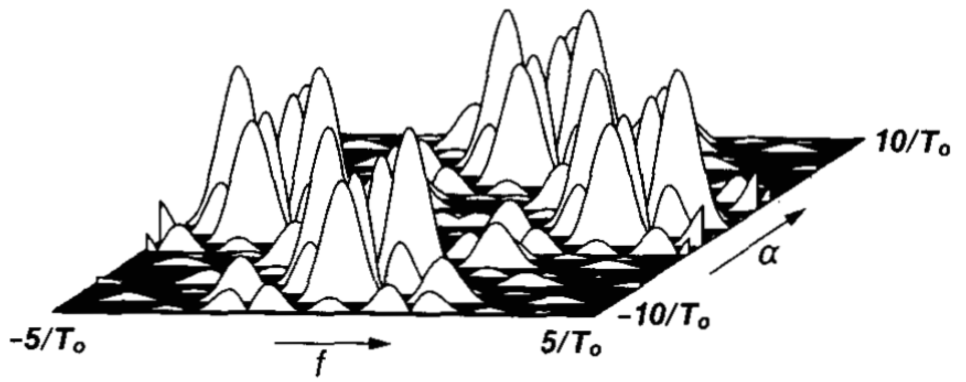

Fig. 7. Spectral correlation magnitude surface for Manchester-encoded binary phase-shift keying with carrier frequency $= 3.3/T_0$.

Figure 5: Fig. 4 in [2]

ACF Kernel:

$$R_{XX}(t+\tau/2, t-\tau/2) = \sum_{k=-\infty}^{\infty} R_X^{(k)}(\tau) e^{j2\pi \frac{k}{T} t}$$

where $R_X^{(k)}(\tau)$ are the Fourier coefficients for the k-th **Cyclic Autocorrelation Function (CAF)**, and their Fourier transform is the **Cyclic Power Spectral Density (CPSD)** $S_X^{(k)}(f)$.

CACF Kernel:

$$\tilde{R}_{XX}(t+\tau/2, t-\tau/2) = \sum_{k=-\infty}^{\infty} \tilde{R}_X^{(k)}(\tau) e^{j2\pi \frac{k}{T} t}$$

where $\tilde{R}_X^{(k)}(\tau)$ are the Fourier coefficients for the k-th **Complementary Cyclic Autocorrelation Function (CCACF)**, and their Fourier transform is the **Complementary Cyclic Power Spectral Density (CCPSD)** $\tilde{S}_X^{(k)}(f)$.

5.2. Derivation of $W_Y(t, f)$ based on SOCS and Cyclic Spectrum Expression

If the condition $f_c = m_0/(2T)$ is satisfied, $Y(t)$ satisfies the T-periodic WSCS process, and its WDF $W_Y(t, f)$ is expressed as a Fourier series with frequencies $\alpha_l = l/T$:

$$W_Y(t, f) = \sum_{l=-\infty}^{\infty} S_Y^{(l)}(f) e^{j2\pi \frac{l}{T} t}$$

Here, the l-th cyclic spectral density function $S_Y^{(l)}(f)$ is composed of the sum of the baseband signal's CPSD $S_X^{(k)}(f)$ and CCPSD $\tilde{S}_X^{(k)}(f)$, existing discretely at the cyclic frequency $\alpha = l/T$:

$$S_Y^{(l)}(f) = \frac{1}{4}\left[S_X^{(l)}(f - f_c) + (S_X^{(-l)}(f + f_c))^* + \tilde{S}_X^{(l-m_0)}(f) + (\tilde{S}_X^{(-l-m_0)}(-f))^*\right]$$

5.3. Relationship between SCD/CSCD and CPSD/CCPSD

When $X(t)$ is a T-periodic SOCS process, the SCD and CSCD exist only at discrete cyclic frequencies $\alpha = k/T$, and are determined by the CPSD $S_X^{(k)}(f)$ and CCPSD $\tilde{S}_X^{(k)}(f)$, respectively:

$$S_X(\alpha, f) = \sum_{k=-\infty}^{\infty} S_X^{(k)}(f) \delta\left(\alpha - \frac{k}{T}\right)$$

$$\tilde{S}_X(\alpha, f) = \sum_{k=-\infty}^{\infty} \tilde{S}_X^{(k)}(f) \delta\left(\alpha - \frac{k}{T}\right)$$

References

[1] W. A. Gardner, "Spectral correlation of modulated signals: part I - analog modulation," *IEEE Trans. Commun.*, vol. 35, no. 6, pp. 584–594, June 1987. `https://ieeexplore.ieee.org/document/1096820`

[2] W. A. Gardner, W. A. Brown, III, and C.-K. Chen, "Spectral correlation of modulated signals: part II - digital modulation," *IEEE Trans. Commun.*, vol. 35, no. 6, pp. 595–601, June 1987. `https://ieeexplore.ieee.org/document/1096816`