Translation Series Representation (TSR) of a Cyclostationary Random Process

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Abstract

This document explores the mathematical representation and properties of a **Zero-Mean Proper-Complex Wide-Sense Cyclostationary (WSCS)** random process $X(t)$ using the **Translation Series Representation (TSR)**. The core concept is the **stationarizability** of $X(t)$ by transforming it into a **Wide-Sense Stationary (WSS) vector process** $\underline{X}(t)$ via the FRESH vectorizer. The resulting Power Spectral Density (PSD) matrix $S_{\underline{X}\underline{X}}(f)$ of $\underline{X}(t)$ is shown to be Hermitian symmetric and positive semi-definite, allowing for an **Eigen-Value Decomposition (EVD)** into a sum of M rank-one components, where M is the maximum number of non-zero eigenvalues. This EVD provides the TSR of $X(t)$ as a sum of M uncorrelated cyclostationary components, highlighting its **non-unique representation forms**. Finally, the document establishes the link between the TSR and the **Shift-Invariant Subspace (SIS)** theory, demonstrating that the complexity of $X(t)$, defined by the rank M, corresponds precisely to the minimum number of generator functions required to span the T-periodic SIS containing all realizations of the random process.

1. Zero-Mean Proper-Complex WSCS

Suppose that a complex-valued random process $X(t)$ is in the form of

$$X(t) = \sum_{n=-\infty}^{\infty} d_n p(t - nT),$$

where $\{d_n\}$ is a **Zero-Mean Proper-Complex WSS symbol sequence**, and $p(t)$ is a deterministic pulse function with bandwidth $B = W/2$ [Hz]. It can be shown that this process is cyclostationary with period T.

1.1. Theorem: Stationarizability of $X(t)$

The **Zero-Mean Proper-Complex WSCS** random process $X(t)$ can be **stationarized** (i.e., transformed into a WSS process) as:

$$\underline{X}(t) = \sum_{n=-\infty}^{\infty} d_n \underline{p}(t - nT)$$

where $\underline{p}(t)$ is the output of the **FRESH vectorizer** for parameter pair $(B, 1/T)$ given by

$$\underline{p}(t) = \begin{bmatrix} p_{-L}(t) \\ p_{-L+1}(t) \\ \vdots \\ p_L(t) \end{bmatrix}$$

with the vector elements defined as:

$$p_k(t) = p(t)e^{-j2\pi kt/T} * h_{LPF}(t), \quad k \in \{-L, \dots, L\}$$

where $h_{LPF}(t)$ represents the low-pass filtering operation (LPF) that restricts the frequency content to the **Nyquist zone** (i.e., $|f| < 1/(2T)$), and L is the number of required FRESH basis functions:

$$L = \lceil BT - 1/2 \rceil .$$

The resulting vector $\underline{p}(t)$ has a dimension of $\mathbf{2L+1}$. This dimension is the **minimum required dimension (Minimally-Oversampled)** determined by the bandwidth B and the period T, which is necessary to perfectly represent $X(t)$ and achieve the WSS vector process $\underline{X}(t)$.

The resulting vector random process $\underline{X}(t)$ is **Zero-Mean Proper-Complex WSS** with respect to t.

1.2. Theorem: Spectral Decomposition for $\underline{X}(t)$

Let $S_{\underline{X}\underline{X}}(f)$ be the **Power Spectral Density (PSD) matrix** of the WSS vector random process $\underline{X}(t)$.

Let $\lambda(f)$ be the **Discrete-Time PSD** of the WSS symbol sequence $\{d_n\}$, given by $\lambda(f) = S_{dd}(e^{\mathrm{j2\pi fT}})$, and let $\underline{P}(f)$ be the **Fourier Transform** of the vector pulse function $\underline{p}(t)$, i.e., $\underline{P}(f) = \mathcal{F}\{\underline{p}(t)\}$.

Then, for the elementary case ($M = 1$), the PSD matrix of the stationarized vector process is given by the decomposition:

$$S_{\underline{X}\underline{X}}(f) = \frac{1}{T}\lambda(f)\underline{P}(f)\underline{P}^H(f)$$

where $\underline{P}^H(f)$ denotes the **Hermitian (Conjugate Transpose)** of the vector $\underline{P}(f)$. The general form of the decomposition is addressed in Section 2.

2. Spectral Properties and Non-Unique Representation Forms

2.1. Theorem: Spectral Properties of the Stationarized PSD Matrix

The **Power Spectral Density (PSD) matrix** $S_{\underline{X}\underline{X}}(f)$ of the Zero-Mean WSS vector random process $\underline{X}(t)$ defined in (1.1) satisfies the following properties:

1. **Hermitian Symmetry:** $S_{\underline{X}\underline{X}}(f) = S^H_{\underline{X}\underline{X}}(f)$, for all frequencies f.
2. **Positive Semi-Definite:** The matrix $S_{\underline{X}\underline{X}}(f)$ is **Hermitian Symmetric positive semi-definite** for every f in the **Nyquist zone** ($|f| < 1/(2T)$).

Since $S_{\underline{X}\underline{X}}(f)$ is a Hermitian matrix, it can be decomposed by the **Orthonormal Eigen-Value Decomposition (EVD)** for each frequency f. In the general case where the cyclostationary process $X(t)$ is modeled as a sum of M independent elementary cyclostationary components, the PSD matrix can be represented as a sum of rank-one matrices based on its EVD:

$$S_{\underline{X}\underline{X}}(f) = \sum_{m=1}^{M} \frac{1}{T}\hat{\lambda}_m(f)\hat{\underline{P}}_m(f)\hat{\underline{P}}^H_m(f) = \sum_{m=1}^{M} \frac{1}{T}\lambda_m(f)\underline{P}_m(f)\underline{P}^H_m(f)$$

where M is the number of **maximum non-zero eigenvalues** over the **Nyquist zone** ($|f| < 1/(2T)$), and $\{\hat{\lambda}_m(f), \hat{\underline{P}}_m(f)\}$ represent the eigenvalue and orthonormal eigenvector, respectively, for the m-th component. The second equality represents the generic form of the rank-one decomposition derived from the EVD, where $\lambda_m(f)$ and $\underline{P}_m(f)$ are arbitrary scaled versions of the eigenvalues and eigenvectors, respectively.

2.2. Component WSS Vector Processes

Based on the spectral decomposition, the stationarized vector random process $\underline{X}(t)$ can be viewed as the sum of M uncorrelated component WSS vector random processes, $\underline{X}_m(t)$:

$$\underline{X}(t) = \sum_{m=1}^{M} \underline{X}_m(t)$$

Each component process $\underline{X}_m(t)$ can be represented by non-unique representation forms:

$$\underline{X}_m(t) = \sum_{n=-\infty}^{\infty} \hat{d}_{m,n}\hat{\underline{p}}(t-nT) = \sum_{n=-\infty}^{\infty} d_{m,n}\underline{p}(t-nT)$$

where:

- $(\hat{d}_{m,n})_n$ and $(d_{m,n})_n$ are **zero-mean proper-complex WSS symbol sequences**.
- The first representation uses the orthonormal eigenvectors $\underline{\hat{P}}_m(f)$ (Fourier Transform of $\underline{\hat{p}}(t)$) and the eigenvalues $\hat{\lambda}_m(f)$ (PSD of $(\hat{d}_{m,n})_n$).
- The second representation uses scaled pulse functions $\underline{P}_m(f)$ (Fourier Transform of $\underline{p}(t)$) and corresponding PSDs $\lambda_m(f)$ (PSD of $(d_{m,n})_n$).

2.3. TSR (Non-Unique) of $X(t)$

The decomposition derived from the EVD of the PSD matrix (as shown in Section 2.1) suggests an interpretation regarding the structure of the original cyclostationary process $X(t)$.

The rank-one terms $\frac{1}{T}\lambda_m(f)\underline{P}_m(f)\underline{P}_m^H(f)$ can be interpreted as the PSD matrix of a component process $\underline{X}_m(t)$. As seen in Section 1, this implies that the decomposition provides a model for $\underline{X}(t)$ as a sum of M zero-mean WSS sub-processes.

Therefore, under certain conditions (i.e., when $\frac{1}{T}\lambda_m(f)$ are interpreted as the PSD of a discrete WSS sequence and $\underline{P}_m(f)$ as the Fourier Transform of the vector pulse), we can perform FRESH scalarization with parameter pair (B, 1/T) to obtain the cyclostationary process $X(t)$ can be represented as the sum of M zero-mean uncorrelated cyclostationary components:

$$X(t) = \sum_{m=1}^{M} \left(\sum_{n=-\infty}^{\infty} d_{m,n} p_m(t - nT) \right)$$

where $(d_{m,n})_n$ is a zero-mean proper-complex WSS symbol sequence with PSD $S_{d_m d_m}(e^{j2\pi fT}) = \lambda_m(f)$, and $\underline{p}_m(t)$ is the corresponding vector pulse function whose Fourier Transform is $\underline{P}_m(f)$. This is the **Translation Series Representation (TSR) of $X(t)$**, which is **not unique** due to the scaling ambiguity of the rank-one factors as well as matching of the m-th component in f_1 and the m'-th component in $f_2 \neq f_1$.

A crucial point regarding the non-uniqueness is the scaling freedom between $\lambda_m(f)$ and $\underline{P}_m(f)$. Specifically, if $\hat{\lambda}_m(f) \neq 0$ for all f, we can choose $\lambda_m(f)$ such that $\lambda_m(f) = 1$ for all f. This choice implies that the symbol sequence $(d_{m,n})_n$ can be chosen to be a **zero-mean WSS white random process** ($S_{d_m d_m}(e^{j2\pi fT}) = 1$) while simultaneously defining the corresponding vector pulse function $\underline{P}_m(f)$ to absorb the original spectral coloring.

3. Relation to KL Expansion

The **Translation Series Representation (TSR)**, derived from the Eigen-Value Decomposition (EVD) of the PSD matrix $S_{\underline{X}\underline{X}}(f)$ of the stationary vector process $\underline{X}(t)$, has a profound link to the **Karhunen-Loève (KL) Expansion**.

- **Wiener-Khintchine Theorem for Vector RP:** For a Wide-Sense Stationary (WSS) vector random process $\underline{X}(t)$, the PSD matrix $S_{\underline{X}\underline{X}}(f)$ and the autocorrelation matrix $R_{\underline{X}\underline{X}}(\tau)$ form a Fourier pair:
$$R_{\underline{X}\underline{X}}(\tau) = \mathcal{F}^{-1}\left\{S_{\underline{X}\underline{X}}(f)\right\} = \int_{-\infty}^{\infty} S_{\underline{X}\underline{X}}(f) e^{j2\pi f\tau} df$$
- **KL Expansion and EVD:** The classical **Karhunen-Loève Expansion** for a random process defined over a finite time interval $[0, T_0]$ is obtained by solving the homogeneous integral equation involving the autocorrelation function $R(\tau_1, \tau_2)$. The TSR, which is based on the EVD of the **spectral density** $S_{\underline{X}\underline{X}}(f)$ (the spectral domain equivalent of the autocorrelation matrix), provides a **canonical decomposition** for the WSS vector process $\underline{X}(t)$ and, consequently, for the cyclostationary scalar process $X(t)$. This decomposition essentially represents the process using a basis of uncorrelated components, which is the defining characteristic of the KL Expansion.

4. Shift-invariant subspaces and TSR

The **Translation Series Representation (TSR)** of a zero-mean proper-complex **Wide-Sense Cyclostationary Random Process (WSCS RP)** $X(t)$ is intrinsically linked to the concept of a **Shift-Invariant

Subspace (SIS)**. The number of components M in the TSR represents the minimum number of generator functions required to span the functional space containing all possible realizations of the random process.

4.1. Shift-Invariant Subspace (SIS) Concept

A **Shift-Invariant Subspace (SIS)** is a function space $V \subset L^2(\mathbb{R})$ that is closed under the discrete time-shift operator $\mathcal{S}_T$.

- **Shift Operator ($\mathcal{S}_T$):** For any function $f(t) \in V$, the discrete shift by T is defined as:

$$\mathcal{S}_T f(t) = f(t - T)$$

- **Shift Invariance:** The space V is an SIS if for every $f(t) \in V$, the shifted function $\mathcal{S}_T f(t) = f(t - T)$ is also contained in V.

Specifically, an **SIS with M generators** $V(\{\phi_m\}_{m=1}^{M})$ is defined as the closure of all functions formed by linear combinations of integer shifts of M **generator functions** $\{\phi_m(t)\}_{m=1}^{M}$:

$$V = \left\{ f(t) = \sum_{m=1}^{M} \sum_{n=-\infty}^{\infty} c_{m,n} \phi_m(t - nT) \mid \{c_{m,n}\} \in l^2(\mathbb{Z}) \right\}$$

where $\{c_{m,n}\}$ is a discrete sequence of coefficients.

4.2. Relationship between SIS and TSR

The TSR of a zero-mean proper-complex WSCS RP $X(t)$ is explicitly an instance of the general form of an SIS with M generators:

$$X(t) = \sum_{m=1}^{M} \left(\sum_{n=-\infty}^{\infty} d_{m,n} p_m(t - nT) \right)$$

- **TSR Components are SIS Generators:** The pulse functions $\{p_m(t)\}_{m=1}^{M}$ in the TSR act as the **generator functions** $\{\phi_m(t)\}_{m=1}^{M}$ of the underlying SIS V. Thus, the process $X(t)$ is an element of the SIS V.
- **TSR Coefficients are SIS Sequence:** The WSS symbol sequences $(d_{m,n})_n$ in the TSR correspond to the discrete coefficient sequences $\{c_{m,n}\}_n$ in the SIS definition.
- **M as Dimensionality:** The **number of components M in the TSR** (which equals the rank of the stationarized PSD matrix $S_{\underline{XX}}(f)$) precisely specifies the minimum number of generator functions required to span the SIS V. This connection demonstrates that the structure and complexity (M) of the WSCS RP are entirely determined by the dimensionality of the underlying T-periodic SIS.

In conclusion, the **TSR provides a canonical decomposition of a WSCS RP, demonstrating that $X(t)$ can be modeled as a stochastic signal whose realizations belong to an M-dimensional T-periodic SIS V**. The rank M derived from the PSD matrix EVD determines the multi-generator structure of this SIS.

5. Generalization for Zero-Mean Improper-Complex SOCS Random Process

The methodology presented in this document for **Zero-Mean Proper-Complex WSCS** processes can be extended to analyze **Zero-Mean Improper-Complex Spectrally-Correlated Cyclostationary (SOCS) Random Processes**.

- **p-FRESH Vectorization:** An Improper-Complex SOCS process $X(t)$ can be converted into a **Zero-Mean Proper-Complex WSCS vector random process** $\underline{X}_p(t)$ using the **p-FRESH vectorizer**. This vectorization augments the standard FRESH vector with the conjugate terms, effectively proper-complexifying the process.
- **TSR Derivation:** The resulting proper-complex WSCS vector process $\underline{X}_p(t)$ can then be analyzed using the exact methods outlined in Sections 2 and 3: the PSD matrix $S_{\underline{X}_p\underline{X}_p}(f)$ is computed and subjected to **Eigen-Value Decomposition (EVD)**. This yields the TSR for the augmented vector process $\underline{X}_p(t)$.
- **p-FRESH Scalarization:** Finally, applying the **p-FRESH scalarization** operation to the TSR of $\underline{X}_p(t)$ will recover the TSR for the original scalar process $X(t)$. This process, however, involves **complex selection procedures** to correctly map the components from the augmented vector space back to the original scalar signal, particularly due to the potential coupling introduced by improperness.