VFT Representation of the Generalized Nyquist Criterion

Gemini AI Assistant and Joon Ho Cho

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Abstract

This paper begins by deriving the **Unit Energy ($||s(t)||^2 = 1$)** condition for the pulse shape function $s(t)$, which is necessary to set the average power of a **random pulse-modulated signal** to a target power P in digital communications. We analyze the simplest example, the Nyquist band-limited Sinc pulse, and verify its energy spectral density $|S(f)|^2$ and total energy in both the time and frequency domains. Crucially, we define the **Vectorized Fourier Transform (VFT)** $\underline{s}(f)$ through the $\mathbf{FRESH(FREquency - SHift)Vectorization}$ of the pulse function $s(t)$.

Subsequently, we demonstrate that the **Square-Root Nyquist Criterion** for a single-stream system using a matched filter receiver is expressed using the VFT norm-square as: $\frac{1}{T}\underline{s}^H(f)\underline{s}(f) = 1$ (within the Nyquist Zone).

Finally, the discussion is extended to a multi-stream system transmitting K streams, where we derive the **Generalized Nyquist Criterion** that simultaneously satisfies **Zero Inter-Symbol Interference (ISI)** and **Zero Inter-Stream Interference (IZI)**. This condition is concisely expressed as a matrix equation, where the Grammian matrix of $\mathbf{S}(f)$ (the matrix whose columns are the VFTs $\underline{s}_k(f)$ of the K pulses) must be the identity matrix within the Nyquist Zone $|\mathbf{f}| \leq \mathbf{\frac{1}{2T}}$:

$$\frac{1}{T}\mathbf{S}^H(f)\mathbf{S}(f) = I_K, \quad \text{for all } |f| \leq \frac{1}{2T}$$

1. Average Power Condition of Random Signal and Unit Energy Pulse Assertion

1.1. Definition of Random Signal $X(t)$ and Nyquist Criterion

The random signal $X(t)$ is defined in the form of a pulse train modulation as:

$$X(t) = A \sum_{n=-\infty}^{\infty} d_n s(t - nT)$$

Here, A is the amplitude constant, and $s(t)$ is the pulse shape function.

To use a Nyquist pulse that satisfies the **Zero Intersymbol Interference (ISI) condition** in a digital communication system, it is generally convenient to normalize the pulse function $s(t)$ to satisfy the **Unit Energy** condition. This means setting the energy of $s(t)$, $||s(t)||^2$, to 1 as follows:

$$||s(t)||^2 = \int_{-\infty}^{\infty} s^*(t)s(t)dt = 1$$

- T: Symbol Period.
- $||s(t)||^2$: Total energy of $s(t)$, assumed to be $||s(t)||^2 = 1$.
- $(d_n)_n$: A sequence with proper-complex, zero-mean, unit-variance, i.i.d. elements, satisfying the following statistical properties:

$$\begin{aligned}
&\mathbb{E}[d_n] = 0 \\
&\mathbb{E}[|d_n|^2] = 1 \\
&\mathbb{E}[d_n d_k^*] = \delta_{n-k} \\
&\mathbb{E}[d_n^2] = 0 \quad \text{(Proper-complex condition)}
\end{aligned}$$

1.2. Derivation of Average Power P_X and Determination of Amplitude A

The average power P_X of $X(t)$ is calculated by the time average $\frac{1}{T}\int_0^T \mathbb{E}[|X(t)|^2]dt$. First, we calculate $\mathbb{E}[|X(t)|^2]$.

$$\begin{aligned}\mathbb{E}[|X(t)|^2] &= \mathbb{E}\left[\left(A\sum_n d_n s(t-nT)\right)\left(A\sum_k d_k^* s^*(t-kT)\right)\right] \\ &= A^2 \sum_n \sum_k s(t-nT)s^*(t-kT)\mathbb{E}[d_n d_k^*]\end{aligned}$$

Due to the property of the sequence d_n, $\mathbb{E}[d_n d_k^*] = \delta_{n-k}$, only terms with $n = k$ remain.

$$\mathbb{E}[|X(t)|^2] = A^2 \sum_n |s(t-nT)|^2$$

Now, we take the time average of $\mathbb{E}[|X(t)|^2]$.

$$P_X = \frac{1}{T}\int_0^T \mathbb{E}[|X(t)|^2]dt = \frac{A^2}{T}\int_0^T \sum_n |s(t-nT)|^2 dt$$

By swapping the order of integration and summation, and using the property that the sum of the integration intervals covers the infinite interval, we get:

$$P_X = \frac{A^2}{T}\int_{-\infty}^{\infty} |s(t)|^2 dt = \frac{A^2}{T}\cdot ||s(t)||^2$$

1.3. Determination of Amplitude A

Applying the condition that the average power P_X of the signal must satisfy the target power P, $P_X = P$, and substituting the condition $||s(t)||^2 = 1$:

$$P = \frac{A^2}{T}\cdot 1$$

Since $A^2 = PT$, the amplitude constant is determined as:

$$A = \sqrt{PT}$$

Therefore, when $s(t)$ is a unit energy pulse, the amplitude constant required to make the average power of the random signal $X(t)$ equal to P is $A = \sqrt{PT}$.

2. Derivation of the Unit Energy Pulse $s(t)$

Now, we continue the discussion by setting

$$X(t) = \sqrt{PT}\sum_{n=-\infty}^{\infty} d_n s(t-nT)$$

For the simplest example, we assume $S(f)$ is a constant (c) within the Nyquist band. The Nyquist bandwidth is $\frac{1}{2T}$.

$$S(f) = \begin{cases} c & \text{for } |f| \le \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases} = c \cdot \mathrm{rect}(fT)$$

2.1. Determination of Amplitude Constant c

Since $s(t)$ must be a unit energy pulse ($||s(t)||^2 = 1$), the integral of $|S(f)|^2$ in the frequency domain must also be 1, according to Parseval's theorem.

$$||s(t)||^2 = \int_{-\infty}^{\infty} |S(f)|^2 df = 1$$

Since $|S(f)|^2 = c^2$:

$$\int_{-1/(2T)}^{1/(2T)} c^2 df = c^2 \left[\frac{1}{2T} - \left(-\frac{1}{2T}\right)\right] = c^2 \cdot \frac{1}{T}$$

This value must be equal to 1:

$$c^2 \cdot \frac{1}{T} = 1 \quad \Rightarrow \quad c^2 = T \quad \Rightarrow \quad c = \sqrt{T}$$

Thus, $S(f)$ satisfying the **Unit Energy condition** is defined as:

$$S(f) = \begin{cases} \sqrt{T} & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

2.2. Derivation of the Time Function $s(t)$

$s(t)$ is derived through the Inverse Continuous-Time Fourier Transform (ICTFT) of $S(f)$.

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df = \int_{-1/(2T)}^{1/(2T)} \sqrt{T} e^{j2\pi ft} df$$

Calculating this integral and applying Euler's formula gives:

$$s(t) = \frac{\sqrt{T}}{2\pi t} \left[e^{j\frac{\pi t}{T}} - e^{-j\frac{\pi t}{T}}\right] = \sqrt{T} \cdot \frac{1}{\pi t} \sin\left(\frac{\pi t}{T}\right)$$

Expressed in terms of the normalized Sinc function $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$:

$$s(t) = \frac{\sqrt{T}}{T} \cdot \text{sinc}\left(\frac{t}{T}\right) = \frac{1}{\sqrt{T}} \text{sinc}\left(\frac{t}{T}\right)$$

3. Energy Spectral Density $|S(f)|^2$

The **Energy Spectral Density (ESD)** $G_s(f)$ for the unit energy pulse $s(t)$ is defined as $|S(f)|^2$.

$$|S(f)|^2 = (\sqrt{T})^2 \cdot \begin{cases} 1 & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

$$|S(f)|^2 = \begin{cases} T & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

4. Verification of Total Energy E_s (Frequency Domain)

The total energy E_s is confirmed to be 1 through integration in the frequency domain.

$$E_s = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_{-1/(2T)}^{1/(2T)} T df$$

$$E_s = T \cdot [f]_{-1/(2T)}^{1/(2T)} = T \cdot \left(\frac{1}{2T} - \left(-\frac{1}{2T}\right)\right) = T \cdot \frac{1}{T} = 1$$

5. Verification of Total Energy E_s (Time Domain)

The total energy E_s is confirmed to be 1 through time domain integration $||s(t)||^2$.

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{T}} \cdot \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} \right)^2 dt$$

Applying the substitution $x = \frac{\pi t}{T}$ ($dt = \frac{T}{\pi} dx$):

$$E_s = \frac{1}{T} \int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x} \right)^2 \frac{T}{\pi} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx$$

Substituting the special integral $\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx = \pi$:

$$E_s = \frac{1}{\pi} \cdot \pi = 1$$

6. FRESH Vectorization and Vectorized Fourier Transform (VFT)

6.1. Definition of FRESH Vectorization $\underline{s}(t)$

In a communication system with period T, the **FRESH (FREquency-SHift) vectorization** $\underline{s}(t)$ of the pulse shape function $s(t)$ represents the components of $s(t)$ shifted in frequency by $\frac{1}{T}$ intervals, in vector form. While theoretically infinite-dimensional, it becomes finite-dimensional for band-limited pulses.

The k-th element of $\underline{s}(t)$ is defined as:

$$[\underline{s}(t)]_k = s(t) e^{-j2\pi \frac{k}{T} t}, \quad k \in \mathbb{Z}$$

Thus, the **FRESH vectorization** of $s(t)$ is:

$$\underline{s}(t) = \begin{pmatrix} \cdots \\ s(t) e^{j2\pi \frac{1}{T} t} \\ s(t) \\ s(t) e^{-j2\pi \frac{1}{T} t} \\ \cdots \end{pmatrix}$$

The order of the elements in this vector may vary depending on the definition of k, but it is generally listed symmetrically around $k = 0$. In this paper, we assume the order of k is $\dots, -1, 0, 1, \dots$.

6.2. Definition of Vectorized Fourier Transform (VFT) $\underline{s}(f)$

The **Vectorized Fourier Transform (VFT)** $\underline{s}(f)$ of $\underline{s}(t)$ is the element-wise Fourier transform of each element of $\underline{s}(t)$. (We use $\underline{s}(f)$ for notational consistency.)

$$[\underline{s}(f)]_k = \mathcal{F}\{[\underline{s}(t)]_k\} = \mathcal{F}\left\{ s(t) e^{-j2\pi \frac{k}{T} t} \right\}$$

According to the **Frequency Shift Property** of the Fourier transform, $\mathcal{F}\{s(t)e^{-j2\pi f_0 t}\} = S(f + f_0)$, with $f_0 = \frac{k}{T}$:

$$[\underline{s}(f)]_k = S\left(f + \frac{k}{T}\right)$$

The **VFT** of $s(t)$ is:

$$\underline{s}(f) = \begin{pmatrix} \cdots \\ S(f + \frac{1}{T}) \\ S(f) \\ S(f - \frac{1}{T}) \\ \cdots \end{pmatrix}$$

6.3. Example Revisited: ESD Representation using VFT

The special case derived in Section 3, where the Energy Spectral Density is constant only within the Nyquist bandwidth, is:

$$|S(f)|^2 = \begin{cases} T & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

Dividing this equation by $\frac{1}{T}$ represents the **frequency domain characteristic of a unit energy pulse**:

$$\frac{1}{T}|S(f)|^2 = \begin{cases} 1 & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases}$$

Using the VFT $\underline{s}(f)$, the above special case condition is expressed as:

$$\frac{1}{T}\underline{s}^H(f)\underline{s}(f) = 1, \quad \text{for all } |f| \leq \frac{1}{2T}$$

This shows that the VFT's norm square, scaled by $(1/T)$, is uniformly 1 within the Nyquist Zone.

7. VFT Representation of the Square-Root Nyquist Criterion

7.1. Time Domain Definition of the Square-Root Nyquist Criterion

A pulse shape function $s(t)$ satisfies the **Square-Root Nyquist Criterion** in digital communications if the **composite pulse $g(t)$**, defined by the convolution of $s(t)$ with its complex conjugate time-reversed pulse $s^*(-t)$, satisfies the **Nyquist Criterion**, even if $s(t)$ itself does not satisfy the ISI-free condition.

$$g(t) = s(t) * s^*(-t)$$

Assuming the unit energy condition for $s(t)$, the composite pulse $g(t)$ satisfies the ISI-free condition when sampled at intervals of kT: it is 1 only for $k = 0$ and 0 for other $k \neq 0$.

$$g(kT) = [s(t) * s^*(-t)]\big|_{t=kT} = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \in \mathbb{Z}, k \neq 0 \end{cases} = \delta_k$$

Here, $g(t)$ is equivalent to the **Autocorrelation Function** of $s(t)$, and $g(0) = \int_{-\infty}^{\infty} |s(t)|^2 dt = ||s(t)||^2 = 1$.

7.2. Frequency Domain Representation of the Square-Root Nyquist Criterion

The Fourier transform $G(f)$ of the composite pulse $g(t)$ relates to the Fourier transform $S(f)$ of $s(t)$ as follows:

$$G(f) = \mathcal{F}\{s(t) * s^*(-t)\} = S(f)S^*(f) = |S(f)|^2$$

The frequency domain condition for $g(t)$ to satisfy the Nyquist Criterion is that the sum of its components shifted by $\frac{1}{T}$ intervals is equal to the constant $\frac{1}{T}$.

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} G\left(f + \frac{k}{T}\right) = \frac{1}{T}, \quad \text{for all } f \in \mathbb{R}$$

Substituting $G(f) = |S(f)|^2$ yields the frequency domain condition for the **Square-Root Nyquist Criterion** of $s(t)$:

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} \left|S\left(f + \frac{k}{T}\right)\right|^2 = 1, \quad \text{for all } |f| \leq \frac{1}{2T}$$

7.3. VFT Representation of the Square-Root Nyquist Criterion

Using the **Vectorized Fourier Transform (VFT)** $\underline{s}(f)$ of $s(t)$, the frequency domain summation condition of the Square-Root Nyquist Criterion is expressed in the form of the vector norm-square.

The norm-square of the VFT, $\underline{s}^H(f)\underline{s}(f)$, exactly matches the periodic version of $|S(f)|^2$ of $s(t)$.

$$\underline{s}^H(f)\underline{s}(f) = \sum_{k=-\infty}^{\infty} S^*\left(f+\frac{k}{T}\right) S\left(f+\frac{k}{T}\right) = \sum_{k=-\infty}^{\infty} \left|S\left(f+\frac{k}{T}\right)\right|^2$$

Therefore, the frequency domain condition of the Square-Root Nyquist Criterion is concisely expressed using the VFT as:

$$\frac{1}{T}\underline{s}^H(f)\underline{s}(f) = 1, \quad \text{for all } |f| \leq \frac{1}{2T}$$

This means that the $\frac{1}{T}$-scaled Periodic Spectral Density of the pulse function $s(t)$ must be uniformly 1 within the Nyquist Zone.

8. Generalized Nyquist Criterion for Multi-Stream Systems

8.1. Definition of Multi-Stream Signal $X(t)$ and Receiver Structure

In a multi-stream system transmitting K streams, the transmitted signal $X(t)$ is defined as the sum of K orthogonal pulse trains:

$$X(t) = \sum_{k=1}^{K}\sum_{n=-\infty}^{\infty} d_k[n]s_k(t-nT)$$

Here, $d_k[n]$ is the n-th symbol of the k-th stream, and $s_k(t)$ is the pulse shape function used for the k-th stream.

The receiver is assumed to use **Matched Filter** receivers to demodulate each stream. To demodulate the k'-th stream, the matched filter $s^*_{k'}(-t)$ corresponding to the pulse $s_{k'}(t)$ is used. The output $y_{k'}(t)$ of the matched filter is:

$$y_{k'}(t) = X(t) * s^*_{k'}(-t) = \sum_{k=1}^{K}\sum_{n=-\infty}^{\infty} d_k[n]\left[s_k(t-nT) * s^*_{k'}(-t)\right]$$

The output signal $y_{k'}(t)$ is sampled at intervals of the symbol period T to obtain the symbol $\hat{d}_{k'}[m] = y_{k'}(mT)$.

8.2. Zero ISI and Zero Inter-Stream Interference Conditions

The condition for simultaneously satisfying **Zero Inter-Symbol Interference (ISI)** and **Zero Inter-Stream Interference (IZI)** is that the sampled output $\hat{d}_{k'}[m]$ must contain only the target symbol $d_{k'}[m]$, and no symbols from other streams ($k \neq k'$) or from other times ($n \neq m$). That is:

$$\hat{d}_{k'}[m] = y_{k'}(mT) = d_{k'}[m], \quad \text{for all } k'$$

The sampling condition for the convolution term to satisfy this is:

$$\left[s_k(t-nT) * s^*_{k'}(-t)\right]\Big|_{t=mT} = \delta_{k,k'}\delta_{m,n}$$

Substituting $t' = t - nT$ and $t = mT$ yields:

$$\left[s_k(t') * s^*_{k'}(-t')\right]\Big|_{t'=(m-n)T} = \delta_{k,k'}\delta_{m,n}$$

Thus, a value must exist only when $m = n$, and it must be 1 only when $k = k'$.

- **ISI-free Condition ($k = k'$):** The composite pulse $g_k(t) = s_k(t) * s^*_k(-t)$ must satisfy the Nyquist Criterion.

$$g_k(nT) = \delta_{n,0} \quad \Rightarrow \quad \frac{1}{T}\sum_m |S_k(f+\frac{m}{T})|^2 = 1$$

- **IZI-free Condition ($k \neq k'$):** The composite pulse $h_{k,k'}(t) = s_k(t) * s^*_{k'}(-t)$ must be 0 when sampled at T intervals.

$$h_{k,k'}(nT) = 0, \quad \text{for all } n \in \mathbb{Z} \quad \Rightarrow \quad \sum_m S_k(f+\frac{m}{T})S^*_{k'}(f+\frac{m}{T}) = 0$$

8.3. VFT Matrix Representation of the Generalized Nyquist Criterion

These two conditions (ISI-free and IZI-free) can be integrated into a single matrix equation within the Nyquist Zone $|f| \leq \frac{1}{2T}$ using the VFT.

Defining the VFT of the K pulses $s_1(t), \ldots, s_K(t)$ as $\underline{s}_k(f)$, the orthogonality and unit energy conditions are determined by the value of $\frac{1}{T}\underline{s}_{k'}^H(f)\underline{s}_k(f)$.

$$\frac{1}{T}\underline{s}_{k'}^H(f)\underline{s}_k(f) = \begin{cases} 1 & \text{for } k = k' \quad \text{(ISI-free)} \\ 0 & \text{for } k \neq k' \quad \text{(IZI-free)} \end{cases}, \quad \text{for all } |f| \leq \frac{1}{2T}$$

We define the K-column matrix $\mathbf{S}(f)$ as $\mathbf{S}(f) = \begin{pmatrix} \underline{s}_1(f) & \underline{s}_2(f) & \ldots & \underline{s}_K(f) \end{pmatrix}$, where the K VFT vectors are its column vectors. The (k', k)-th element of the Grammian matrix $\mathbf{S}^H(f)\mathbf{S}(f)$ is exactly $\underline{s}_{k'}^H(f)\underline{s}_k(f)$.

Therefore, the above K^2 identities can be expressed by stating that the Grammian of $\mathbf{S}(f)$ must be the identity matrix I_K within the Nyquist Zone.

$$\frac{1}{T}\mathbf{S}^H(f)\mathbf{S}(f) = I_K, \quad \text{for all } |f| \leq \frac{1}{2T}$$

This is the VFT representation of the **Generalized Nyquist Criterion** for multi-stream systems.