Beyond Properness: The Imperative of Widely-Linear Processing for Improper Man-Made Communication Signals

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Abstract

This document addresses the fundamental statistical limitations encountered when processing **Improper Complex Random Signals** (ICRS), which are prevalent in modern communication systems. We first explore the historical bias towards Properness, showing how the standard complex linear relationship ($\mathbf{y_l(t)} \approx \frac{1}{2}\mathbf{h_l(t)} * \mathbf{x_l(t)}$) in real bandpass systems (Section 1) and the rigorous WSS $\iff$ Proper equivalence (Section 2) marginalized the importance of the conjugate component. The core argument is established in **Section 3**, where we formally prove the necessity of Properness: for a real bandpass signal $X(t)$ to be **Spectrally Uncorrelated** ($\mathbf{S_{XX}(f_1, f_2)} \propto \delta(\mathbf{f_1} - \mathbf{f_2})$), its complex envelope $X_l(t)$ must have an identically zero complementary bi-spectrum ($\tilde{\mathbf{S}}_{\mathbf{X_l X_l}} = \mathbf{0}$). When this spectral mandate fails due to improperness, the signal is statistically asymmetric and spectrally correlated, rendering the conventional Complex Linear Filter (CLF) sub-optimal. The **Widely-Linear Processor (WLP)** is presented as the **optimal linear estimator** for ICRS. By leveraging the augmented complex signal vector $\mathbf{X}_{\text{aug}}(t) = [X_l(t), X_l^*(t)]^T$, the WLP uniquely exploits both the standard and complementary covariance components ($\mathbf{R_{X_l X_l}}$ and $\tilde{\mathbf{R}}_{\mathbf{X_l X_l}}$), ensuring the achievement of the Minimum Mean Square Error (MMSE) performance.

1. The Historical Blind Spot: Why Was WL Processing Ignored?

The concept of Widely-Linear (WL) processing, which is crucial for the optimal use of Improper Complex Random Variables (ICRVs), was historically overlooked in many fields, especially in classical communication system analysis. The primary reason for this neglect lies in the fact that when a **real-valued bandpass signal** passes through a **real-valued bandpass channel**, the resulting baseband-equivalent signal can be accurately modeled using a simple, standard complex convolution, which effectively masks the inherent improperness of the baseband signal.

1.1. Standard Analysis of Real Bandpass Systems

Consider a real-valued input bandpass signal $x(t)$ and a real-valued channel impulse response $h(t)$, both centered around the carrier frequency f_c. They are represented using their respective complex envelopes $x_l(t)$ and $h_l(t)$ as:

$$x(t) = \frac{1}{2}\left(x_l(t)e^{\mathrm{j}2\pi f_c t} + x_l^*(t)e^{-\mathrm{j}2\pi f_c t}\right)$$
$$h(t) = \frac{1}{2}\left(h_l(t)e^{\mathrm{j}2\pi f_c t} + h_l^*(t)e^{-\mathrm{j}2\pi f_c t}\right)$$

The output signal $y(t)$ is the convolution of $h(t)$ and $x(t)$: $y(t) = h(t) * x(t)$.

We aim to prove that the output complex envelope $y_l(t)$ satisfies $y_l(t) = \frac{1}{2}h_l(t) * x_l(t)$ under the **Narrowband Assumption**.

1.1.1. Derivation of the Baseband Equivalent via Frequency Domain

Substituting the complex envelope expressions into the convolution, $y(t) = h(t) * x(t)$, and separating the terms using the linearity of convolution:

$$\begin{aligned}
4y(t) &= \left(h_l(t)e^{\mathrm{j}2\pi f_c t}\right) * \left(x_l(t)e^{\mathrm{j}2\pi f_c t}\right) && \text{(Term 1)} \\
&+ \left(h_l(t)e^{\mathrm{j}2\pi f_c t}\right) * \left(x_l^*(t)e^{-\mathrm{j}2\pi f_c t}\right) && \text{(Term 2)} \\
&+ \left(h_l^*(t)e^{-\mathrm{j}2\pi f_c t}\right) * \left(x_l(t)e^{\mathrm{j}2\pi f_c t}\right) && \text{(Term 3)} \\
&+ \left(h_l^*(t)e^{-\mathrm{j}2\pi f_c t}\right) * \left(x_l^*(t)e^{-\mathrm{j}2\pi f_c t}\right) && \text{(Term 4)}
\end{aligned}$$

Taking the Fourier Transform (FT) of $y(t)$: Let $Y(f) = \mathcal{F}\{y(t)\}$. We denote $X_l(f) = \mathcal{F}\{x_l(t)\}$ and $H_l(f) = \mathcal{F}\{h_l(t)\}$.

$$\begin{aligned}
4Y(f) = &\underbrace{H_l(f - f_c)X_l(f - f_c)}_{\text{FT of Term 1}} \\
&+ \underbrace{H_l(f - f_c)X_l^*(-f - f_c)}_{\text{FT of Term 2}} \\
&+ \underbrace{H_l^*(-f - f_c)X_l(f - f_c)}_{\text{FT of Term 3}} \\
&+ \underbrace{H_l^*(-f - f_c)X_l^*(-f - f_c)}_{\text{FT of Term 4}}
\end{aligned}$$

1.1.2. Application of the Narrowband Assumption and Final Equivalence

The physical output $Y(f)$ is a real bandpass signal, meaning $Y(f)$ must be non-zero only around $\pm f_c$. The **Narrowband Assumption** requires that the complex envelopes $x_l(t)$ and $h_l(t)$ are low-pass signals, meaning their spectra $X_l(f)$ and $H_l(f)$ are effectively zero for frequencies $|\mathbf{f}| \geq \mathbf{f_c}$. For $Y(f)$ to be non-zero around $f \approx f_c$, Terms 2 and 3 would require $X_l(f)$ or $H_l(f)$ to have significant energy around $f \approx -2f_c$. Since $X_l(f) \approx 0$ and $H_l(f) \approx 0$ for $|f| \geq f_c$, these aliasing terms are set to zero in the passband.

The only remaining components that contribute to the positive and negative frequency bands of $Y(f)$ are the non-aliasing products, which define the desired baseband convolution:

$$4Y(f) \approx H_l(f - f_c)X_l(f - f_c) + H_l^*(-f - f_c)X_l^*(-f - f_c)$$

Let $Y_{\text{base}}(f) = H_l(f)X_l(f)$, which is the FT of $h_l(t) * x_l(t)$.

$$4Y(f) \approx Y_{\text{base}}(f - f_c) + Y_{\text{base}}^*(-f - f_c)$$

Taking the Inverse Fourier Transform $\mathcal{F}^{-1}$ (where $Y_{\text{base}}^*(t)$ is the inverse transform of $Y_{\text{base}}^*(-f)$):

$$4y(t) \approx Y_{\text{base}}(t)e^{\mathrm{j}2\pi f_c t} + Y_{\text{base}}^*(t)e^{-\mathrm{j}2\pi f_c t}$$

The right-hand side is exactly $2\mathrm{Re}\{Y_{\text{base}}(t)e^{\mathrm{j}2\pi f_c t}\}$.

$$y(t) \approx \frac{1}{2}\mathrm{Re}\left\{Y_{\text{base}}(t)e^{\mathrm{j}2\pi f_c t}\right\}$$

Comparing this with the definition of the output complex envelope, $y(t) = \mathrm{Re}\{y_l(t)e^{\mathrm{j}2\pi f_c t}\}$, we derive the baseband equivalence:

$$\mathbf{y_l(t) \approx \frac{1}{2}Y_{\text{base}}(t) = \frac{1}{2}h_l(t) * x_l(t)}$$

1.2. The Consequence: Masking of Improperness

The relationship $\mathbf{y_l(t) = \frac{1}{2}h_l(t) * x_l(t)}$ masked the potential benefit of Widely-Linear Processing (WLP) for the following reasons:

1. **Focus on Standard Complex Processing:** The output $y_l(t)$ is determined solely by the standard complex convolution ($h_l(t) * x_l(t)$). The conjugate components ($x_l^*(t)$ or $h_l^*(t)$) were not explicitly needed for output modeling in a bandpass real-to-real system.

2. **The Simple Solution was Optimal Enough (Model-Wise):** This simple linear relationship led engineers to believe that the standard complex linear model ($Y_l = H_l X_l$) was sufficient. The possibility of statistical improperness in $x_l(t)$ and the performance gain offered by the conjugate component in a Widely-Linear Filter (WLF) were largely overlooked.

2. The Necessity of WL Processing for Improper Man-Made Signals

The adoption of properness was historically reinforced by fundamental properties of stationary random processes, but this assumption fails for many practical communication signals.

2.1. Properness of the Complex Envelope for WSS Real Bandpass Processes (Time-Domain Proof)

A significant factor that historically masked the relevance of improperness is the fact that the complex envelope of any **Wide-Sense Stationary (WSS) real bandpass random process** is always a (zero-mean) **proper-complex WSS random process**. This led researchers to assume properness in many fundamental scenarios.

First, we prove that a WSS real bandpass process must have zero mean. Let $X(t)$ be a WSS real bandpass random process.

1. Since $X(t)$ is **WSS**, its mean $\mu_X(t) = E[X(t)]$ is a constant value μ_X.
2. The Power Spectral Density (PSD) $S_{XX}(f)$ of a WSS process $X(t)$ is related to its mean by the fact that the PSD contains a term $|\mu_X|^2\delta(f)$, which represents the DC power (power at $f = 0$).
3. Since $X(t)$ is a **bandpass** process, by definition its PSD $S_{XX}(f)$ must be zero for a band of frequencies around $f = 0$ (i.e., $S_{XX}(f) = 0$ for $|f| < f_{\min}$, where $f_{\min} > 0$).
4. For the PSD to be zero at $f = 0$, the delta function term must vanish. Thus, we must have $|\mu_X|^2 = 0$, which implies $\mu_X = 0$.

Therefore, $X(t)$ must be a **zero-mean** process.

Now, we proceed with the proof of properness using $X(t)$'s autocorrelation function. $X(t)$ is a zero-mean, WSS real bandpass random process with carrier frequency f_c. Since $X(t)$ is a real WSS process, its autocorrelation $R_{XX}(t, t-\tau) = R_{XX}(\tau)$ is a real, even function ($R_{XX}(\tau) = R_{XX}(-\tau)$). The complex envelope $X_l(t)$ is defined such that:

$$X(t) = \mathrm{Re}\{X_l(t)e^{j2\pi f_c t}\} = \frac{1}{2}\left(X_l(t)e^{j2\pi f_c t} + X_l^*(t)e^{-j2\pi f_c t}\right)$$

We compute the autocorrelation function of $X(t)$: $R_{XX}(\tau) = E[X(t)X(t-\tau)]$.

$$\begin{aligned}
R_{XX}(\tau) &= E\left[\frac{1}{2}\left(X_l(t)e^{j2\pi f_c t} + X_l^*(t)e^{-j2\pi f_c t}\right)\cdot\frac{1}{2}\left(X_l(t-\tau)e^{j2\pi f_c(t-\tau)} + X_l^*(t-\tau)e^{-j2\pi f_c(t-\tau)}\right)\right] \\
4R_{XX}(\tau) &= E[X_l(t)X_l(t-\tau)]e^{j2\pi f_c(2t-\tau)} + E[X_l(t)X_l^*(t-\tau)]e^{j2\pi f_c\tau} \\
&\quad + E[X_l^*(t)X_l(t-\tau)]e^{-j2\pi f_c\tau} + E[X_l^*(t)X_l^*(t-\tau)]e^{-j2\pi f_c(2t-\tau)}
\end{aligned}$$

We define the complementary autocorrelation $\tilde{R}_{X_lX_l}(\tau) \triangleq E[X_l(t)X_l(t-\tau)]$ and the autocorrelation $R_{X_lX_l}(\tau) \triangleq E[X_l(t)X_l^*(t-\tau)]$.

$$4R_{XX}(\tau) = \tilde{R}_{X_lX_l}(\tau)e^{j2\pi f_c(2t-\tau)} + R_{X_lX_l}(\tau)e^{j2\pi f_c\tau} + R^*_{X_lX_l}(\tau)e^{-j2\pi f_c\tau} + \tilde{R}^*_{X_lX_l}(\tau)e^{-j2\pi f_c(2t-\tau)}$$

Since $R_{XX}(\tau)$ must be independent of time t (WSS property), the time-dependent term $e^{\pm j2\pi f_c(2t)}$ must cancel out, which is only possible if the coefficients of the terms containing $e^{\pm j4\pi f_c t}$ are zero. This requires:

$$\tilde{\mathbf{R}}_{\mathbf{X_lX_l}}(\tau) = \mathbf{0} \quad \text{for all } \tau$$

Since the complementary autocorrelation is zero, the complex envelope $X_l(t)$ is proven to be a **Proper Complex Random Process**. This WSS $\rightarrow$ Proper property provided a historical blind spot against utilizing the conjugate component.

2.1.1. Consequence: The Misguided Reliance on Properness

The rigorous result that the complex envelope of a WSS real bandpass process is (zero-mean) proper-complex WSS provided a strong mathematical justification for focusing exclusively on **Proper Complex Signals** in classical communication theory and random process analysis. This implied that for systems dealing with natural noise (often modeled as WSS real bandpass noise) or highly symmetrical modulation (often assumed WSS), the use of the conjugate component (X_l^*) was statistically unnecessary.

This fundamental property, combined with the masking effect shown in Section 1.2, strongly discouraged the exploration of Widely-Linear techniques, creating a historical gap in optimal signal processing.

2.2. Propagation of Improperness Through Real LTI Channels

If the input signal $X_l(t)$ is generated by a man-made source that is inherently non-WSS or statistically asymmetrical (Improper), the improperness is preserved by the channel.

We proved that the complementary autocorrelation (complementary covariance) of the output $Y_l(t)$ is a linear transformation of the input complementary autocorrelation $\tilde{R}_{X_l X_l}$:

$$\tilde{R}_{Y_l Y_l}(t_1, t_2) = \iint H_l(\tau_1) H_l(\tau_2) \tilde{R}_{X_l X_l}(t_1 - \tau_1, t_2 - \tau_2) d\tau_1 d\tau_2$$

Since $\tilde{R}_{X_l X_l} \neq 0$ for an Improper input, the output must remain **Improper Complex Random Process ($\mathbf{\tilde{R}_{Y_l Y_l}(t_1, t_2) \neq 0}$)**, demonstrating that the channel does not remove the improperness.

2.3. Superiority of Widely-Linear Processing at the Receiver

Since the received signal $Y_l(t)$ is Improper, it contains two essential statistical components: **Autocorrelation** ($R_{Y_l Y_l}$) and **Complementary Autocorrelation** ($\tilde{R}_{Y_l Y_l}$).

- A **Standard Complex Linear Filter (CLF)** utilizes only $Y_l(t)$ and thus ignores the exploitable $\tilde{R}_{Y_l Y_l}$.
- A **Widely-Linear Filter (WLF)** utilizes both $Y_l(t)$ and $Y_l^*(t)$, enabling it to capture and utilize **both $R_{Y_l Y_l}$ and $\tilde{R}_{Y_l Y_l}$**, guaranteeing optimal linear estimation performance (Minimum Mean Square Error).

2.4. The Prevalence of Improper Man-Made Signals

Many man-made communication signals are inherently **Improper** due to statistical asymmetry between the In-phase (I) and Quadrature (Q) components. Examples include:

- **Amplitude Modulation (AM, DSB, DSB-SC):** The complex envelope $X_l(t)$ is often purely real, making it **Maximally Improper**.
- **Pulse Amplitude Modulation (PAM) / On-Off Keying (OOK):** Result in purely real complex envelopes.
- **Transceiver Imperfections (IQ Imbalance):** Cause ideal zero-mean proper signals (like QPSK) to become Improper in practice.

3. The Spectral Mandate: Why Uncorrelation Implies Properness (and Why It Fails)

The initial focus on Complex Linear Filtering (CLF) stemmed from the seemingly sufficient model used to analyze real bandpass systems. This section rigorously establishes the fundamental equivalence between the traditional time-domain stationarity assumption (Wide-Sense Stationary, WSS) and the powerful frequency-domain uncorrelation property. Crucially, we demonstrate that a real process can only be spectrally uncorrelated if its complex envelope is **Proper**. This strong mathematical mandate highlights why most **Man-Made Communication Signals** fail this condition, rendering them spectrally correlated (Improper), and thus justifying the necessity of Widely-Linear Processing (WLP) for optimal performance.

3.1. Definition of Spectrally Uncorrelated Real Random Process

A real random process $X(t)$ is **Spectrally Uncorrelated** if its Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ is non-zero only along the main diagonal $f_1 = f_2$, i.e.,

$$S_{XX}(f_1, f_2) = S_{XX}(f_1)\delta(f_1 - f_2)$$

3.2. The Fundamental Equivalence for Real Bandpass Processes

To claim:

$$X(t) \text{ is Real Bandpass WSS} \iff X(t) \text{ is Spectrally Uncorrelated}$$
$$\iff X_l(t) \text{ is Zero-Mean Proper-Complex WSS}$$

3.2.1. Proof of Equivalence ($X(t)$ WSS $\iff$ $X(t)$ Spectrally Uncorrelated)

The proof relies on the 2D Fourier Transform of $R_{XX}(t, s)$, showing that dependence only on $\tau = t - s$ is equivalent to dependence only on $f_1 = f_2$.

3.2.2. Proof of Equivalence ($X(t)$ Spectrally Uncorrelated $\iff$ $X_l(t)$ Zero-Mean Proper-Complex WSS)

This equivalence was proven by demonstrating that the WSS condition ($R_{XX}(t, s) = R_{XX}(\tau)$) forces the complementary autocorrelation $\tilde{R}_{X_l X_l}(\tau)$ to zero (Properness), and vice versa, linking the time-domain WSS property to the time-domain Properness property. If $X_l(t)$ has a non-zero mean, then $R_{XX}(t, s) = R_{XX}(\tau)$ has $\Re\{\mu e^{j2\pi f_c t}\}\Re\{\mu e^{j2\pi f_c s}\}$ component, which translates in $S_{XX}(t, s)$ as four terms proportional to $\delta(f_1 \pm f_c)\delta(f_2 \pm f_c)$ and results in strong correlation on $f_2 = f_1 \pm 2f_c$.

3.3. Definition of Spectrally Uncorrelated Complex Random Process

A complex random process $X_l(t)$ is defined as **Spectrally Uncorrelated** if and only if both of the following conditions are met:

1. **Standard Spectrum (WSS Equivalent):** The standard Bi-frequency Spectrum must be confined to the main diagonal (Standard Spectral Uncorrelation):

$$\mathbf{S_{X_l X_l}(f_1, f_2) = S_{X_l X_l}(f_1)\delta(f_1 - f_2)}$$

2. **Complementary Spectrum (Properness Equivalent):** The Complementary Bi-frequency Spectrum must be identically zero everywhere (Elimination of Anti-Diagonal Correlation):

$$\mathbf{\tilde{S}_{X_l X_l}(f_1, f_2) = 0} \quad \text{for all } \mathbf{f_1, f_2}$$

The second condition enforcing **Properness**, which is necessary to ensure the underlying real signal $X(t)$ is also Spectrally Uncorrelated.

In the following subsubsections, we prove the necessity by showing the relation between spectral uncorrelation of the real signal and the complementary bi-spectrum of the complex envelope. That is, for a real bandpass random process $X(t)$ (upconverted from $X_l(t)$ at carrier f_c) to be **Spectrally Uncorrelated** (i.e., Wide-Sense Stationary or WSS), its complex baseband envelope $X_l(t)$ must satisfy both the WSS and Properness conditions in the frequency domain.

3.3.1. The General Relationship: Real and Complex Bi-spectra

The Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ of the real bandpass signal $X(t)$ is generally related to the two Bi-frequency Spectra of its complex envelope $X_l(t)$ as follows:

$$\begin{aligned} 4S_{XX}(f_1, f_2) = {} & S_{X_l X_l}(f_1 - f_c, f_2 - f_c) \quad (\text{Term I: } +f_c, +f_c \text{ band}) \\ & + S^*_{X_l X_l}(-f_1 - f_c, -f_2 - f_c) \quad (\text{Term II: } -f_c, -f_c \text{ band}) \\ & + \tilde{S}_{X_l X_l}(f_1 - f_c, -f_2 - f_c) \quad (\text{Term III: } +f_c, -f_c \text{ band, Complementary}) \\ & + \tilde{S}^*_{X_l X_l}(-f_1 - f_c, f_2 - f_c) \quad (\text{Term IV: } -f_c, +f_c \text{ band, Complementary}) \end{aligned} \tag{1}$$

3.3.2. Imposing the Spectrally Uncorrelated Condition ($X(t)$ WSS)

For $X(t)$ to be WSS (Spectrally Uncorrelated), $S_{XX}(f_1, f_2)$ must be non-zero only along the main diagonal $f_1 = f_2$: $\mathbf{S_{XX}(f_1, f_2)} \propto \delta(\mathbf{f_1} - \mathbf{f_2})$.

Condition 1: Elimination of Complementary Correlation ($\tilde{\mathbf{S}}_{\mathbf{X_l X_l}} = \mathbf{0}$) Terms III and IV are the Fourier Transforms of the complementary autocorrelation terms: $\tilde{R}_{X_l X_l}(t, s)e^{\pm j2\pi f_c(t+s)}$. These terms are problematic because they are multiplied by a factor dependent on the sum of times ($\mathbf{t} + \mathbf{s}$), meaning the real signal $X(t)$ is generally **not WSS** but rather Cyclostationary.

In the frequency domain, the presence of the $\mathbf{e^{\pm j2\pi f_c(t+s)}}$ factor causes the energy of $\tilde{S}_{X_l X_l}$ to be shifted onto the spectral axes:

- Term III shifts energy to the axis defined by $f_2 = f_1 - 2f_c$.
- Term IV shifts energy to the axis defined by $f_2 = f_1 + 2f_c$.

Since the WSS condition demands that $S_{XX}(f_1, f_2)$ must exist **only on the main diagonal $\mathbf{f_1} = \mathbf{f_2}$**, any non-zero energy on the $\mathbf{f_2} = \mathbf{f_1} \pm \mathbf{2f_c}$ axes constitutes an **off-diagonal correlation** that fundamentally violates the definition of a Spectrally Uncorrelated process.

The only way to guarantee the elimination of all off-diagonal correlation at $f_2 = f_1 \pm 2f_c$ is to ensure that the source of that energy, the complementary bi-spectrum, is identically zero everywhere:

$$\tilde{\mathbf{S}}_{\mathbf{X_l X_l}}(\mathbf{f_1}, \mathbf{f_2}) = \mathbf{0} \quad \text{for all } \mathbf{f_1}, \mathbf{f_2}$$

This is the spectral domain requirement for $X_l(t)$ to be **Zero-Mean Proper**, proving that Zero-Mean Properness is a prerequisite for the real bandpass signal to be WSS.

Condition 2: Standard Spectrum Diagonal Restriction ($\mathbf{S_{X_l X_l}} \propto \delta(\mathbf{f_1} - \mathbf{f_2})$) Assuming $\tilde{S}_{X_l X_l} = 0$, Term III and IV vanish. We are left with the requirement that the sum of Terms I and II must be proportional to $\delta(f_1 - f_2)$.

Since Term I is $S_{X_l X_l}$ shifted by f_c in both arguments, for this term to be proportional to $\delta(f_1 - f_2)$, the original complex spectrum $S_{X_l X_l}(f_1', f_2')$ must be proportional to $\delta(f_1' - f_2')$:

$$S_{X_l X_l}(f_1 - f_c, f_2 - f_c) \propto \delta((f_1 - f_c) - (f_2 - f_c)) = \delta(f_1 - f_2)$$

This implies the complex envelope must satisfy the spectral condition for WSS:

$$\mathbf{S_{X_l X_l}(f_1, f_2)} = \mathbf{S_{X_l X_l}(f_1)}\delta(\mathbf{f_1} - \mathbf{f_2})$$

3.3.3. Conclusion

For the real bandpass process $X(t)$ to be Spectrally Uncorrelated (WSS), the complex envelope $X_l(t)$ must be both **WSS** (Condition 2) and **Proper** (Condition 1). Remind that we have already shown that $X_l(t)$ being improper-complex second-order stationary (SOS) leads to

$$\tilde{S}_{X_l X_l}(f_1, f_2) = \tilde{S}_{X_l X_l}(f_1)\delta(f_1 + f_2).$$

Substituting this in (1) leads to strong correlation in $S_{XX}(f_1, f_2)$ on the lines $f_2 = f_1 \pm 2f_c$.

3.4. The Imperative for Widely-Linear Processing

The rigorous mathematical equivalence established in the previous subsections demonstrates the direct link between the zero-mean properness of the complex envelope and the stationarity (Spectral Uncorrelation) of the real signal.

- If $X_l(t)$ is **Zero-Mean Proper** ($\tilde{\mathbf{S}}_{\mathbf{X_l X_l}} = \mathbf{0}$), the real signal $X(t)$ is WSS and Spectrally Uncorrelated. In this ideal scenario, all statistically exploitable information is contained within the standard (conjugate) covariance $R_{X_l X_l}$, justifying the use of a conventional **Complex Linear Filter (CLF)**.
- However, for many man-made communication signals, $X_l(t)$ is **Improper** ($\tilde{\mathbf{S}}_{\mathbf{X_l X_l}} \neq \mathbf{0}$). As proven in Section 3.3, this immediately implies the breakdown of the equivalence: $X(t)$ is **not WSS** and is **Spectrally Correlated** (via the complementary terms in $S_{XX}(f_1, f_2)$).

When improperness exists, the signal $X_l(t)$ is characterized by **two components**: the standard covariance ($\mathbf{R_{X_l X_l}}$) and the complementary covariance ($\tilde{\mathbf{R}}_{\mathbf{X_l X_l}}$).

A **Standard CLF** is blind to $\tilde{R}_{X_l X_l}$ because it only operates on $X_l(t)$ and not its conjugate, $X_l^*(t)$. Consequently, a CLF performs sub-optimally when processing improper signals.

The **Widely-Linear Processor (WLP)** utilizes the augmented complex signal vector $\mathbf{X}_{\text{aug}}(t) = [X_l(t), X_l^*(t)]^T$. By jointly processing both the signal and its conjugate, the WLP is capable of capturing and exploiting **both $R_{X_l X_l}$ and $\tilde{R}_{X_l X_l}$**. This comprehensive statistical exploitation ensures that the WLP achieves the **Minimum Mean Square Error (MMSE)** performance—the optimal linear estimation achievable for improper complex signals.