The Wigner-Ville Distribution and Ambiguity Function: Non-Negativity, Duality, and Symmetry

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Abstract

The Wigner-Ville Distribution (WVD) is a cornerstone of time-frequency signal analysis, providing high resolution for instantaneous spectral content. This document addresses two critical mathematical properties of the WVD for a complex signal $x(t)$: its **real-valued nature** and its capacity to assume **negative values**. We rigorously demonstrate that the WVD is always real because its defining kernel exhibits Hermitian symmetry with respect to the delay variable (τ). However, we explain, from an Electrical Engineering perspective, that the WVD's non-negativity constraint violation is due to the inherent **quadratic (bi-linear) nature** of the transform, which generates **cross-terms (interference)** when analyzing multicomponent signals. This prevents the WVD from being strictly interpreted as a true energy or power spectral density.

1 Introduction and Definition of Wigner-Ville Distribution

The Wigner-Ville Distribution (WVD) is defined as a time-frequency representation that attempts to localize a signal's energy simultaneously in both time (t) and frequency (f). It is derived from the instantaneous autocorrelation kernel $\tilde{r}_x(t,\tau)$, which is a bi-linear operation on the signal $x(t)$.

The instantaneous autocorrelation kernel is defined as:

$$\tilde{r}_x(t,\tau) = x(t+\tau/2)x^*(t-\tau/2)$$

The WVD is the Fourier Transform of this kernel with respect to the time lag τ:

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-j2\pi f\tau}d\tau \tag{1}$$

The WVD is a **quadratic transform** because its definition involves the signal $x(t)$ being multiplied by its complex conjugate $x^*(t)$, resulting in an operation that is of the second power (order) with respect to the signal amplitude. This bi-linear nature is the source of its high resolution but also its analytical limitations.

2 Analysis of WVD Properties in Signal Processing

The WVD exhibits two critical, seemingly contradictory properties regarding its range of values: it is always real-valued, but it is not guaranteed to be non-negative.

2.1 The WVD is Always Real-Valued ($\mathbf{W_x(t,f)} \in \mathbb{R}$)

For any complex signal $x(t)$, the WVD always yields a real number. This is a direct consequence of the symmetry of the instantaneous autocorrelation kernel $\tilde{r}_x(t,\tau)$ with respect to the delay variable τ.

1. **Hermitian Symmetry of the Kernel:** The kernel $\tilde{r}_x(t,\tau)$ satisfies the Hermitian symmetry property in τ:

$$\tilde{r}_x^*(t,-\tau) = [x(t-\tau/2)x^*(t+\tau/2)]^* = x^*(t-\tau/2)x(t+\tau/2) = \tilde{r}_x(t,\tau) \tag{2}$$

2. **Real-Valuedness of Fourier Transform:** A fundamental property of the Fourier Transform states that the transform of a Hermitian symmetric function is always real-valued. Since the WVD is the Fourier Transform of the τ-Hermitian kernel $\tilde{r}_x(t,\tau)$, $W_x(t,f)$ is guaranteed to be real:

$$W_x(t,f) = \mathcal{F}_{\tau\to f}\left\{\tilde{r}_x(t,\tau)\right\} \in \mathbb{R} \tag{3}$$

2.2 The Problem of Non-Negativity (Why $\mathbf{W_x(t,f)}$ Can Be Negative)

Unlike the Energy Spectral Density (ESD), $G_x(f) = |X(f)|^2$, which is inherently non-negative, the WVD often yields negative values. This prevents its strict interpretation as a true physical energy or power spectral density in the (t,f) plane. The reason lies in the bi-linear nature of the transform, which generates **cross-terms** or interference.

1. **The Bi-Linear Interference (Cross-Terms):** When the signal $x(t)$ is composed of two distinct components, $x(t) = x_1(t) + x_2(t)$, the WVD is expanded through the distributive property into three terms:

$$W_x(t,f) = W_{x_1}(t,f) + W_{x_2}(t,f) + 2\mathrm{Re}\{W_{x_1x_2}(t,f)\} \tag{4}$$

2. **Negative Values as Interference:** The observed negative values in $W_x(t,f)$ are the troughs of these oscillating cross-terms. They do not signify negative physical energy but rather demonstrate the **coherence** and **relative phase relationship** between $x_1(t)$ and $x_2(t)$. This property forces electrical engineers to treat the WVD as an indicator of instantaneous spectral correlation and coherence, rather than a direct physical measure of energy density.

3 The Ambiguity Function (AF) and Examples

3.1 Definition of the Ambiguity Function (AF)

The Ambiguity Function ($\chi(\tau,\nu)$) is derived from the same instantaneous autocorrelation kernel $\tilde{r}_x(t,\tau)$ as the WVD, but it is obtained by taking the Fourier Transform with respect to the **average time (t)**. It is defined over the **Time Delay (τ)-Doppler Frequency (ν) plane and is a core tool in radar and sonar systems for analyzing resolution.

$$\chi(\tau,\nu) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-j2\pi\nu t}dt \tag{5}$$

The AF and the WVD are linked by a 2D Fourier Transform, demonstrating their **Duality**:

$$\mathbf{W_x(t,f)} \leftrightarrow \text{2D Fourier Transform}\chi(\tau,\nu)$$

3.2 Example: WVD of a Rectangular Pulse

We analyze the WVD for a rectangular pulse, a common example demonstrating the WVD's properties.

Let the rectangular pulse $x(t)$ be defined as:

$$x(t) = \mathrm{rect}(t/T) = \begin{cases} 1 & \text{if } |t| \le T/2 \\ 0 & \text{if } |t| > T/2 \end{cases}$$

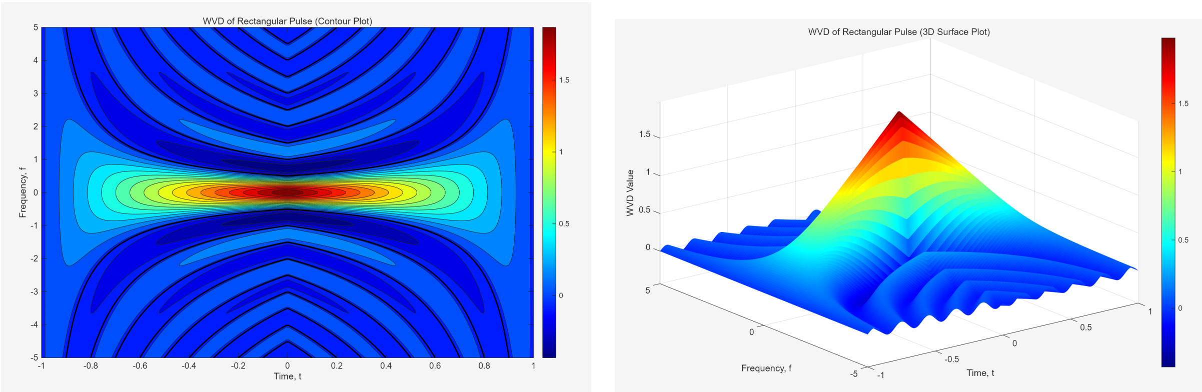
The WVD for the rectangular pulse is calculated as:

$$\mathbf{W_x(t,f)} = (T - 2|t|)\cdot \mathrm{sinc}\left[f(T-2|t|)\right] \quad \text{for } |t| \le T/2 \tag{6}$$

3.3 Observation of Negative Values in WVD

Since $\mathrm{sinc}(x)$ oscillates and takes on **negative values** (e.g., when x is between 1 and 2), the WVD of the simple rectangular pulse $x(t)$ (Equation 6) also exhibits **negative values** for certain combinations of (t,f).

Specifically, negative values appear in the frequency domain f when $f(T-2|t|)$ falls outside the main lobe of the sinc function. This demonstrates that even for a single-component, real signal, the WVD is not non-negative, confirming its nature as a time-frequency correlation function rather than a true energy density.



(a) Contour Plot: WVD of Rectangular Pulse. Negative values appear as oscillating lobes.

(b) 3D Surface Plot: WVD of Rectangular Pulse. Negative values are visible below the Z=0 plane.

Figure 1: Visualizations of the Wigner-Ville Distribution (WVD) for a Rectangular Pulse.

3.4 Comparison of WVD and AF for Rectangular Pulse

The AF for the rectangular pulse is correctly derived from the WVD using the symmetry relation $\chi_x(\tau,\nu) = \frac{1}{2}W_x(\frac{\tau}{2},\frac{\nu}{2})$. The corresponding form is:

$$\chi(\tau,\nu) = \frac{1}{2}(T-|\tau|)\cdot \text{sinc}\left[\frac{\nu}{2}(T-|\tau|)\right] \quad \text{for } |\tau| \leq T \tag{7}$$

The analytical forms are similar, but their domains and interpretations are distinct. The following figure visually compares the AF and WVD distributions.

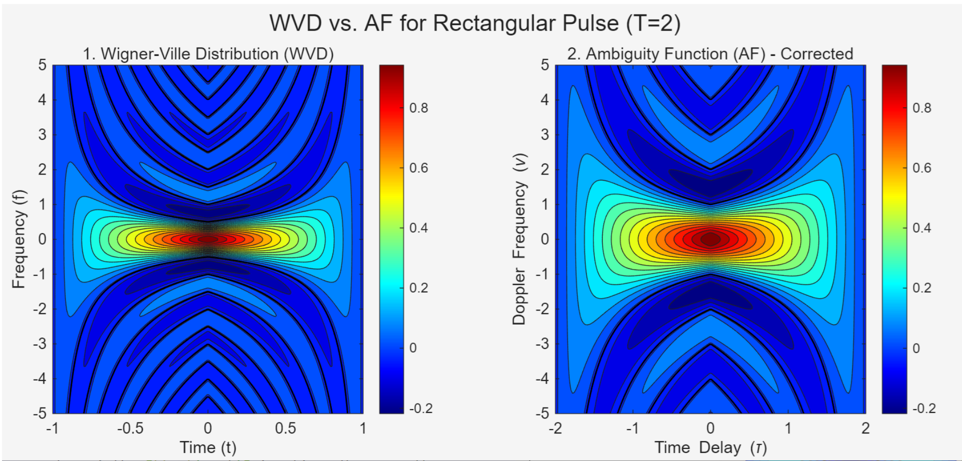


Figure 2: Comparison of the Wigner-Ville Distribution (WVD) and the Ambiguity Function (AF) for a Rectangular Pulse ($T = 2$). The WVD is supported over $t \in [-1, 1]$, while the AF is supported over $\tau \in [-2, 2]$. Both show characteristic sinc oscillations (negative lobes).

Feature	**WVD ($\mathbf{W_x(t,f)}$)**	**AF** ($\chi(\tau,\nu)$)
Domain	Time (t) and Frequency (f)	Delay (τ) and Doppler (ν)
Time/Delay Support	$t \in [-T/2, T/2]$ (**1/2** pulse width)	$\tau \in [-T, T]$ (**1** pulse width)
Sinc Argument	$f \cdot (T - 2\|t\|)$	$\frac{\nu}{\mathbf{2}} \cdot (\mathbf{T} - \|\tau\|)$
Amplitude	$(T - 2\|t\|)$	$\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{T} - \|\tau\|)$
Interpretation	Instantaneous spectral content (Signal structure)	Range-Doppler resolution (System performance)

The similarity in form is due to the simple symmetry of the rect function. However, the difference in the support width ($T/2$ vs. T) and the variables used confirm that they are distinct functions operating in different physical coordinate systems.

3.5 The Duality Relation: Why AF is not a Simple Variable Substitution of WVD

It is important to clarify that the structural similarity between the WVD and AF for the rectangular pulse (Equations 6 and 7) is an **exception due to the high symmetry of the rect function**, not the general rule.

1. **Mathematical Connection is an Integral Transform:** Generally, the Ambiguity Function and the Wigner-Ville Distribution are connected by a **2D Fourier Transform** (FT), which is an integral operation. It is **not** possible to obtain one from the other via a simple algebraic variable substitution (e.g., $t \to \tau$ or $f \to \nu$). The relationship is defined by:

$$\chi(\tau, \nu) = \iint_{-\infty}^{\infty} W_x(t, f) e^{j2\pi(\nu t - \tau f)} dt df \tag{8}$$

2. **The Role of Symmetry:** The near-identical analytical form for the rectangular pulse arises because its instantaneous autocorrelation kernel, $\tilde{r}_x(t, \tau)$, is separable and symmetric enough that the FT operation over t (yielding the AF) produces a result structurally similar to the FT operation over τ (yielding the WVD).

3. **General Case Difference:** For complex signals like a Linear FM (Chirp) signal, the WVD is localized along a diagonal line in the t-f plane, while the AF is localized along a diagonal line in the τ-ν plane with a different slope. Their forms are clearly distinct, confirming that the rect function's similarity is merely a special case of signal symmetry.

4. **Special Case of Real Even Signals:** Despite the general rule, a direct transformation relationship exists if the signal $x(t)$ is a **real even function** ($x(t) = x(-t)$), such as the rectangular pulse. In this specific scenario, the Ambiguity Function can be obtained from the WVD by a combination of variable scaling and a constant factor:

$$\chi_x(\tau, \nu) = \frac{1}{2} W_x(\frac{\tau}{2}, \frac{\nu}{2}) \tag{9}$$

   This scaling arises because the integration variable change $t = t'/2$ in the AF definition (Equation 5) introduces a factor of $1/2$ while aligning the remaining variables ($\tau/2 \to t$, $\nu/2 \to f$) with the definition of the WVD. This relation confirms that the core link is indeed through their shared kernel and the properties of the Fourier Transform.

   Derivation: The Ambiguity Function (AF) for a real signal ($x^* = x$) is defined as:

$$\chi_x(\tau, \nu) = \int_{-\infty}^{\infty} x(t + \frac{\tau}{2}) x(t - \frac{\tau}{2}) e^{-j2\pi\nu t} dt$$

   To transform this into the WVD form, we perform the following substitution on the integration variable t:

$$\text{Let } \mathbf{t}' = 2t$$
$$\implies \mathbf{t} = \frac{t'}{2}$$
$$\implies \mathbf{dt} = \frac{1}{2} dt'$$

   Substituting these into the χ_x integral:

$$\begin{aligned} \chi_x(\tau, \nu) &= \int_{-\infty}^{\infty} x(\frac{t'}{2} + \frac{\tau}{2}) x(\frac{t'}{2} - \frac{\tau}{2}) e^{-j2\pi\nu\frac{t'}{2}} \cdot \frac{1}{2} dt' \\ &= \frac{1}{2} \int_{-\infty}^{\infty} x(\frac{\tau}{2} + \frac{t'}{2}) x(\frac{\tau}{2} - \frac{t'}{2}) e^{-j2\pi(\frac{\nu}{2})t'} dt' \end{aligned}$$

   Comparing the final integral form with the WVD definition:

$$W_x(\mathbf{t}, \mathbf{f}) = \int_{-\infty}^{\infty} x(\mathbf{t} + \frac{\tau'}{2}) x(\mathbf{t} - \frac{\tau'}{2}) e^{-j2\pi\mathbf{f}\tau'} d\tau'$$

We can establish the following correspondence by setting the integration variable $\tau' = t'$:

$$\mathbf{t} = \frac{\tau}{2} \quad \text{(WVD fixed time variable)}$$
$$\mathbf{f} = \frac{\nu}{2} \quad \text{(WVD frequency variable)}$$

Therefore, the χ_x integral simplifies to:

$$\chi_x(\tau, \nu) = \frac{1}{2} W_x(\frac{\tau}{2}, \frac{\nu}{2})$$

This relation highlights that for real even functions, the AF is a scaled and compressed version of the WVD in the time/delay and frequency/Doppler dimensions.

4 Conclusion

The WVD's unique combination of being real-valued (due to Hermitian symmetry) yet capable of taking on negative values (due to cross-term interference) defines its role in signal processing. While its sharp resolution is invaluable, the presence of negative interference terms necessitates the use of smoothed Wigner-Ville variants (like the Spectrogram or other Cohen's Class distributions) in applications where a strictly non-negative energy interpretation is required.