The Bi-Frequency Spectrum of Non-Stationary Processes: Based on the Symplectic Fourier Transform

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Abstract

This document explores the **Bi-frequency Spectrum** ($S_{XX}(f_1, f_2)$), a critical analytical tool for characterizing **non-stationary random processes**. Defined as the two-dimensional Symplectic Fourier Transform of the two-variable autocorrelation function $R_{XX}(t_1, t_2)$, this spectrum serves as a measure of non-stationarity; significant non-zero values off the main diagonal ($f_1 \neq f_2$) directly indicate time-varying statistical properties. The spectrum is also known as the **Generalized Spectral Density (GSD)**.

We rigorously derive the general input-output relationship for a non-stationary process $X(t)$ propagating through a **Linear Time-Varying (LTV) system** with impulse response $h(t, \tau)$. The output spectrum $S_{YY}(f_1, f_2)$ is shown to be governed by a complex **integral relationship** involving the input spectrum $S_{XX}(f_1, f_2)$ and the system's frequency-time response $\mathcal{H}(f, \tau)$. Crucially, the simpler **multiplicative product form** ($S_{YY} = HS_{XX}H^*$) is generally **not valid** for arbitrary LTV systems, only holding under the constraint of a Linear Time-Invariant (LTI) system or within specialized analytical frameworks that adopt the product relation by definition.

1 Definition via Symplectic Fourier Transform

The analysis of non-stationary random processes utilizes the two-variable autocorrelation function, $R_{XX}(t_1, t_2)$, which is the expected value of the product of the process at two time instants: $R_{XX}(t_1, t_2) = \mathbb{E}\{X(t_1)X^*(t_2)\}$.

The **Bi-frequency Spectrum**, $S_{XX}(f_1, f_2)$, is defined as the two-dimensional Fourier Transform of $R_{XX}(t_1, t_2)$, often expressed in a form related to the Symplectic Fourier Transform, which uses opposing signs in the exponent for the two time variables.

1.1 Formal Definition

The Bi-frequency Spectrum $S_{XX}(f_1, f_2)$ is given by the following integral transform:

$$S_{XX}(f_1, f_2) = \iint_{-\infty}^{\infty} R_{XX}(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \tag{1}$$

This two-dimensional transform maps the time-domain correlation information into the frequency domain, where f_1 and f_2 are the two independent frequency variables.

1.2 Bi-Frequency Spectrum under WSS Condition

The Bi-frequency Spectrum takes a remarkably simple form when the random process $X(t)$ is **Wide-Sense Stationary (WSS)**. The WSS condition stipulates that the autocorrelation function $R_{XX}(t_1, t_2)$ depends only on the time difference $\tau = t_1 - t_2$, i.e., $R_{XX}(t_1, t_2) = R_{XX}(\tau)$.

To demonstrate the relationship $S_{XX}(f_1, f_2) = S_{XX}(f_1)\delta(f_1 - f_2)$, we substitute the WSS condition into the formal definition (Eq. 1):

$$S_{XX}(f_1, f_2) = \iint_{-\infty}^{\infty} R_{XX}(t_1 - t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2$$

We introduce the variable change $\tau = t_1 - t_2$ (and $t_1 = t_2 + \tau$).

$$
\begin{aligned}
S_{XX}(f_1, f_2) &= \iint_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi(f_1(t_2+\tau)-f_2 t_2)} d\tau dt_2 \\
&= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi f_1 \tau} d\tau \cdot \int_{-\infty}^{\infty} e^{-j2\pi(f_1 t_2 - f_2 t_2)} dt_2 \\
&= \left[\int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi f_1 \tau} d\tau\right] \cdot \left[\int_{-\infty}^{\infty} e^{-j2\pi(f_1-f_2)t_2} dt_2\right]
\end{aligned}
$$

We recognize the two bracketed terms:

- The first term is the standard Power Spectral Density (PSD) of the WSS process, $S_{XX}(f_1)$.
- The second term is the definition of the Dirac Delta function, $\delta(f_1 - f_2)$.

Therefore, for a WSS process, the Bi-frequency Spectrum simplifies to:

$$
S_{XX}(f_1, f_2) = S_{XX}(f_1) \cdot \delta(f_1 - f_2) \tag{2}
$$

This result confirms that for a WSS process, the spectrum is strictly non-zero only along the main diagonal where $f_1 = f_2$, implying that there is no correlation or power coupling between different frequency components.

2 Alternative Names

The Bi-frequency Spectrum is known by several names, particularly when discussing non-stationary processes:

- **Generalized Spectral Density (GSD):** This is perhaps the most common name, as it extends the concept of the standard Power Spectral Density (PSD) to non-stationary cases.
- **Two-Frequency Spectrum:** A descriptive name emphasizing the two independent frequency variables, f_1 and f_2.
- **Complex Spectral Density:** Used in some literature, particularly when $X(t)$ is a complex-valued process.

3 Key Applications

The primary application of $S_{XX}(f_1, f_2)$ is the rigorous analysis and characterization of **non-stationary random processes**.

3.1 Characterizing Non-Stationarity

The Bi-frequency Spectrum serves as a direct measure of non-stationarity:

- For a **Wide-Sense Stationary (WSS)** process, $R_{XX}(t_1, t_2)$ depends only on the time difference $\tau = t_1 - t_2$. In this case, $S_{XX}(f_1, f_2)$ is non-zero only along the main diagonal where $f_1 = f_2$:

  $$S_{XX}(f_1, f_2) = S_{XX}(f_1) \cdot \delta(f_1 - f_2)$$

  where $S_{XX}(f_1)$ is the standard Power Spectral Density (PSD).
- The presence of significant non-zero values **off the main diagonal** ($f_1 \neq f_2$) indicates that the process is **non-(wide-sense)stationary**.

3.2 Linear Time-Varying (LTV) Systems

The Bi-frequency Spectrum is critical in analyzing how non-stationary signals propagate through LTV systems. The system's input spectrum $S_{XX}(f_1, f_2)$ is related to the output spectrum $S_{YY}(f_1, f_2)$ via the system's **Bi-frequency Transfer Function**, $H(f_1, f_2)$.

4 System and Process Definitions

Let $X(t)$ be a second-order random process, and let $Y(t)$ be the output of a **Linear Time-Varying (LTV) system** with time-varying impulse response $h(t, \tau)$.

4.1 LTV System Input-Output Relation

The output $Y(t)$ is given by the convolution integral:

$$Y(t) = \int_{-\infty}^{\infty} h(t, \tau) X(\tau) d\tau \tag{3}$$

4.2 Autocorrelation and Bi-Frequency Spectrum

- The autocorrelation function of the output $Y(t)$ is: $R_{YY}(t_1, t_2) = \mathbb{E}\{Y(t_1)Y^*(t_2)\}$.
- The Bi-frequency Spectrum of $Y(t)$ is defined as the Symplectic Fourier Transform of $R_{YY}(t_1, t_2)$:

$$S_{YY}(f_1, f_2) = \iint_{-\infty}^{\infty} R_{YY}(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \tag{4}$$

5 Derivation of the Output Autocorrelation Function (R_{YY})

Substitute the LTV input-output relation (Eq. 3) into the definition of $R_{YY}(t_1, t_2)$:

$$\begin{aligned} R_{YY}(t_1, t_2) &= \mathbb{E}\left\{ \left[\int_{-\infty}^{\infty} h(t_1, \tau_1) X(\tau_1) d\tau_1 \right] \left[\int_{-\infty}^{\infty} h(t_2, \tau_2) X(\tau_2) d\tau_2 \right]^* \right\} \\ &= \mathbb{E}\left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1, \tau_1) X(\tau_1) X^*(\tau_2) h^*(t_2, \tau_2) d\tau_1 d\tau_2 \right\} \end{aligned}$$

Since the expectation operator $\mathbb{E}\{\cdot\}$ is linear and the impulse response h is deterministic, we can interchange the expectation and integration:

$$\begin{aligned} R_{YY}(t_1, t_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1, \tau_1) \mathbb{E}\{X(\tau_1) X^*(\tau_2)\} h^*(t_2, \tau_2) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1, \tau_1) R_{XX}(\tau_1, \tau_2) d\tau_1 h^*(t_2, \tau_2) d\tau_2 \end{aligned} \tag{5}$$

6 Derivation of the Output Bi-Frequency Spectrum (S_{YY})

The Bi-frequency Spectrum $S_{YY}(f_1, f_2)$ is obtained by taking the Symplectic Fourier Transform of $R_{YY}(t_1, t_2)$ (Eq. 5):

$$\begin{aligned} S_{YY}(f_1, f_2) &= \iint_{t_1, t_2} R_{YY}(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \\ &= \iint_{t_1, t_2} \left[\iint_{\tau_1, \tau_2} h(t_1, \tau_1) R_{XX}(\tau_1, \tau_2) h^*(t_2, \tau_2) d\tau_1 d\tau_2 \right] e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \end{aligned}$$

Rearranging the terms to group related variables:

$$\begin{aligned} S_{YY}(f_1, f_2) &= \iint_{\tau_1, \tau_2} R_{XX}(\tau_1, \tau_2) \left[\iint_{t_1, t_2} h(t_1, \tau_1) e^{-j2\pi f_1 t_1} h^*(t_2, \tau_2) e^{j2\pi f_2 t_2} dt_1 dt_2 \right] d\tau_1 d\tau_2 \\ &= \iint_{\tau_1, \tau_2} \left[\int_{t_1} h(t_1, \tau_1) e^{-j2\pi f_1 t_1} dt_1 \right] R_{XX}(\tau_1, \tau_2) \left[\int_{t_2} h^*(t_2, \tau_2) e^{j2\pi f_2 t_2} dt_2 \right] d\tau_1 d\tau_2 \end{aligned}$$

Recall that the Bi-frequency Transfer Function $H(f_1, f_2)$ is the 2D Fourier Transform of the impulse response $h(t_1, t_2)$:

$$H(f_1, f_2) = \iint_{t_1, t_2} h(t_1, t_2) e^{-j2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2$$

Defining the generalized frequency response $\mathcal{H}(f,\tau)$ as the Fourier Transform of $h(t,\tau)$ with respect to the first variable t:

$$\mathcal{H}(f,\tau) = \int_t h(t,\tau)e^{-j2\pi ft}dt$$

Then, the inner terms become:

- The first term is $\mathcal{H}(f_1,\tau_1)$.
- The third term is $\mathcal{H}^*(f_2,\tau_2)$ (due to the conjugate of $h(t_2,\tau_2)$ and the sign change in the exponential).

Substituting these back, we get:

$$S_{YY}(f_1,f_2) = \iint_{\tau_1,\tau_2} \mathcal{H}(f_1,\tau_1)R_{XX}(\tau_1,\tau_2)\mathcal{H}^*(f_2,\tau_2)d\tau_1 d\tau_2 \tag{6}$$

where $\mathcal{H}(f,\tau)$ is the system's **frequency-time response**, which is the Fourier Transform of $h(t,\tau)$ with respect to t.

For a general Linear Time-Varying (LTV) system with impulse response $h(t,\tau)$ and a non-stationary input $X(t)$ with autocorrelation $R_{XX}(\tau_1,\tau_2)$, the output Bi-frequency Spectrum, $S_{YY}(f_1,f_2)$, is generally given by the above **integral relationship**:

This complex integral form arises because the LTV system mixes or couples the input's frequency components and correlation across different time instances (τ_1 and τ_2).

7 Conditions for the Multiplicative Relation

The simpler **multiplicative product form** often desired in spectral analysis is:

$$S_{YY}(f_1,f_2) = H(f_1,f_2)S_{XX}(f_1,f_2)H^*(f_1,f_2) \tag{7}$$

This product relation is generally **not valid** for arbitrary LTV systems and non-stationary inputs. It only holds under specific, restrictive constraints:

7.1 Constraint 1: The System Must Be LTI

The most straightforward case where the product form holds is when the LTV system reduces to a **Linear Time-Invariant (LTI) system**.

- **LTI Condition:** For an LTI system, the impulse response depends only on the time difference, $h(t,\tau) = h(t-\tau)$.
- **Spectral Consequence:** The Bi-frequency Transfer Function simplifies to:

$$H(f_1,f_2) = H(f_1)\delta(f_1 - f_2)$$

If the input $X(t)$ is also **Wide-Sense Stationary (WSS)**, then $S_{XX}(f_1,f_2)$ is also a delta function on the diagonal. Substituting these delta functions into the general integral relation (or the product relation) leads to the standard PSD filtering equation: $S_{YY}(f) = |H(f)|^2 S_{XX}(f)$, which is the LTI equivalent of the product form. **The multiplicative relation is therefore consistent with LTI/WSS analysis.**

7.2 Constraint 2: Adoption of a Specialized Framework (Generalized PSD)

In some non-stationary process literature, the multiplicative relation (Eq. 7) is adopted as a matter of definition or convenience within a specialized analytical framework known as the **Generalized Spectral Density (GSD)** approach.

- **Specialized Definition:** In this approach, $H(f_1,f_2)$ is not directly derived from the LTV impulse response $h(t,\tau)$ via simple Fourier Transform, but is rather defined as a **spectral operator** that acts multiplicatively on $S_{XX}(f_1,f_2)$.
- **Analytical Goal:** The purpose is often to retain the simplicity of the LTI spectral analysis (Output = Transfer Function × Input) while formally extending it to LTV systems. This requires **assuming** or **defining** the LTV system's spectral properties in a way that respects the multiplication rule, even though the underlying time-domain physics is described by the complex integral (Eq. 6).

8 Conclusion

The statement emphasizes that because LTV systems possess the general ability to **couple different frequency components** (i.e., non-zero off-diagonal terms in S_{XX} and H), the simple product $\mathbf{S_{YY}} = \mathbf{H S_{XX} H}^*$ is only rigorously valid when the system's LTV characteristics are highly constrained (i.e., when it approximates an LTI system) or when a simplified spectral framework is explicitly adopted.

8.1 Connection to Time-Frequency Analysis

While $S_{XX}(f_1, f_2)$ directly addresses frequency-domain correlation, it is fundamentally related to time-frequency representations. A common practice is to first transform $R_{XX}(t_1, t_2)$ into the variables of **average time $t = (t_1 + t_2)/2$** and **time difference $\tau = t_1 - t_2$** (yielding $\tilde{R}_{XX}(t, \tau)$). The Fourier Transform of $\tilde{R}_{XX}(t, \tau)$ with respect to τ results in the **Instantaneous Spectrum** $S_{XX}(t, f)$, which forms the basis for time-frequency distributions like the Wigner-Ville Distribution (WVD).