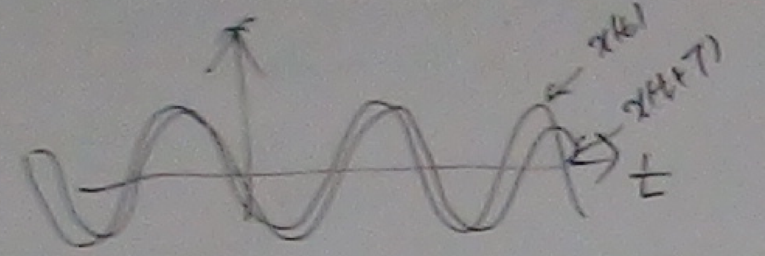
EECE695M CM #2a

□ almost periodic signal

$|x(t) - x(t+T)| = 0, \ \forall t$ ← fully periodic

$|x(t) - x(t+T)| < \varepsilon, \ \forall t$ ← uniformly a.p.



□ multidimensional periodicity

$x(\underline{t}), \quad \underline{t} \in \mathbb{R}^N$

$x : \mathbb{R}^N \to \mathbb{C}$

(ex/ $x(t_1, t_2) = x(t_1+T_1, t_2+T_2), \ \forall \underline{t}$)

Def.

$x(\underline{t})$ is periodic if $\exists$ an N-by-N nonsingular matrix $\mathbf{P} \triangleq [\underline{p}_1, \underline{p}_2, \cdots \underline{p}_N]$ ← periodicity matrix, vectors

s.t.

$x(\underline{t}) = x(\underline{t}+\underline{p}_1) = x(\underline{t}+\underline{p}_2) = \cdots = x(\underline{t}+\underline{p}_N), \ \forall \underline{t}$

See, Intro. to Shannon Sampling & Interpolation Theory by R. J. Mark II

□ multidimensional Fourier Transform

$$X(\underline{f}) = \int_{\underline{t} \in \mathbb{R}^N} x(\underline{t}) \exp(-j 2\pi \underbrace{\underline{f}^T \underline{t}}_{f_1 t_1 + f_2 t_2 \cdots + f_N t_N}) \, d\underline{t}$$

$$x(\underline{t}) = \int_{\underline{f} \in \mathbb{R}^N} X(\underline{f}) \exp(j 2\pi \underline{f}^T \underline{t}) \, d\underline{f}$$

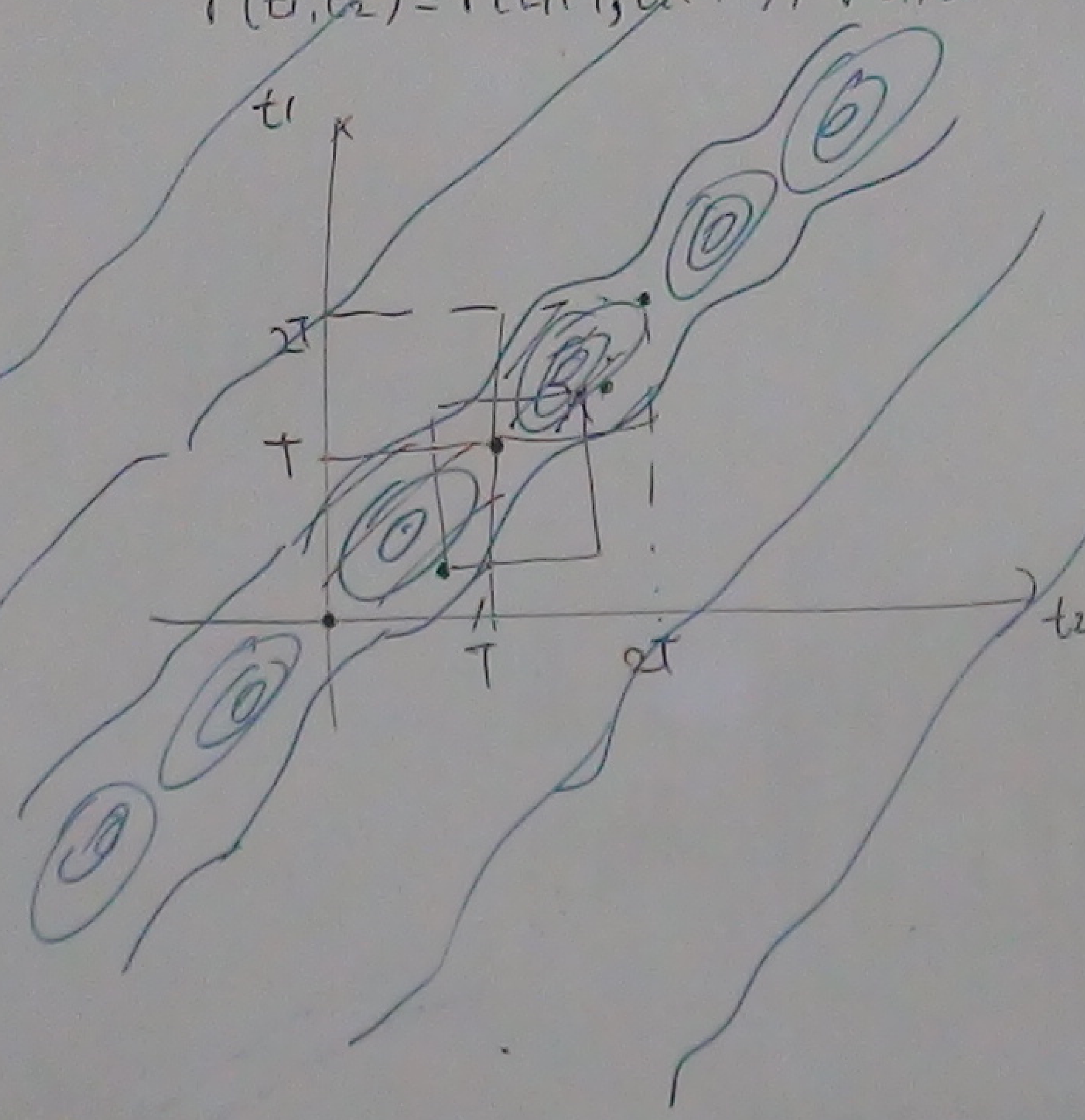
□ 2-D Fourier TF

$$R(f_1, f_2) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(t_1, t_2) \exp(-j 2\pi f_1 t_1) \exp(-j 2\pi f_2 t_2) \, dt_1 \, dt_2$$

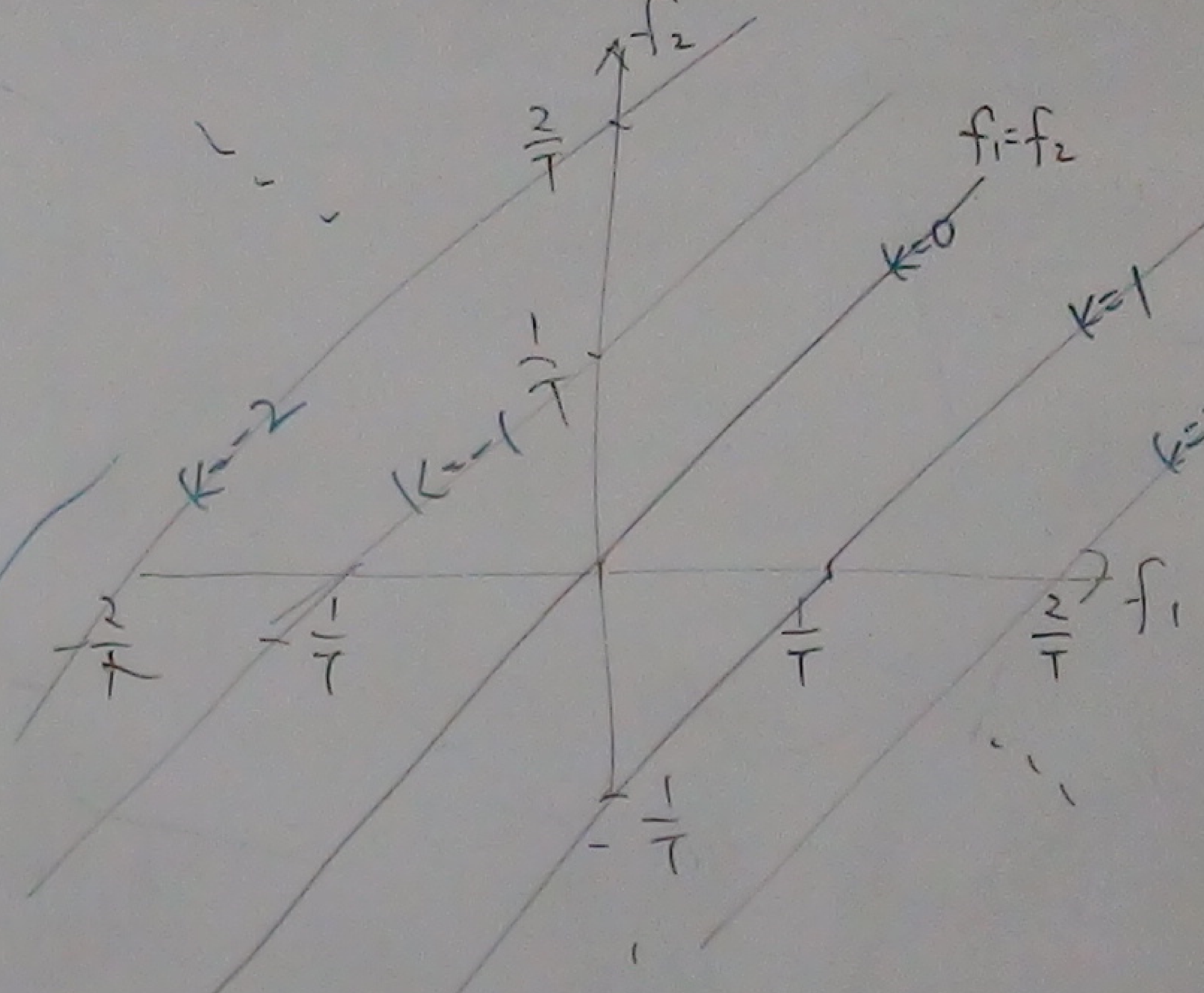
$$r(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(f_1, f_2) \exp(j 2\pi f_1 t_1) \exp(-j 2\pi f_2 t_2) \, df_1 \, df_2$$

□ 2-D periodic-like(?) signal

$r(t_1, t_2) = r(t_1+T, t_2+T), \ \forall t_1, t_2 \in \mathbb{R}$



□ 2-D FTF of $r(t_1, t_2) = r(t_1+T, t_2+T), \ \forall t_1, t_2 \in \mathbb{R}$



$R(f_1, f_2)$

$$= \sum_{k=-\infty}^{\infty} R^{(k)}\left(f_1 - \frac{k}{T}\right) \delta\left(f_1 - f_2 - \frac{k}{T}\right)$$

$$= R^{(0)}(f_1) \delta(f_1 - f_2) + R^{(1)}\left(f_1 - \frac{1}{T}\right) \delta\left(f_1 - f_2 - \frac{1}{T}\right) + \cdots$$

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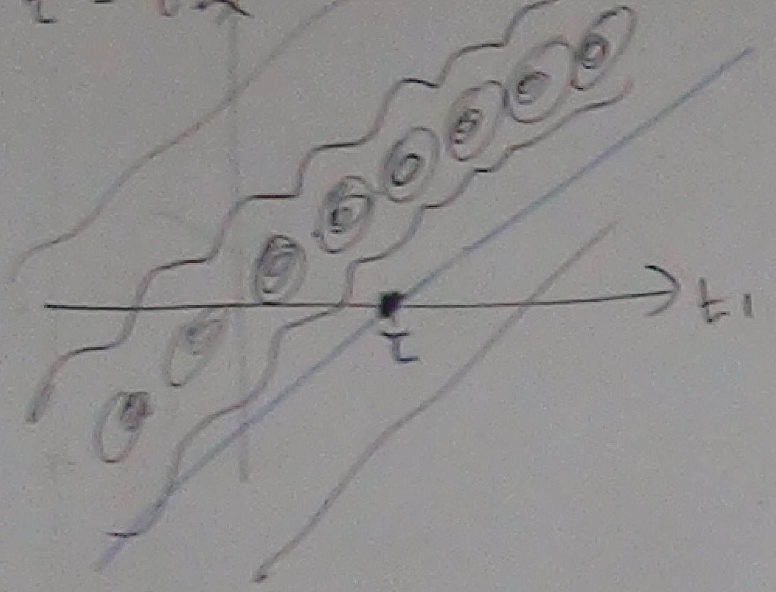
□ Derivation

MR (Mental Representation)

$t = t_2 x$

$r(t_1, t_2) = r(t_1+T, t_2+T), \ \forall t_1, t_2$



$\hat{r}(t, \tau) \triangleq r(t+\tau, t)$

$= \hat{r}(t+T, \tau), \ \forall t$

$= \sum_k \boxed{r^{(k)}(\tau)} \exp\left(j \frac{2\pi}{T} k t\right)$

where $r^{(k)}(\tau) = \frac{1}{T} \int_{<T>} \hat{r}(t, \tau) \exp\left(-j \frac{2\pi k t}{T}\right) dt$

$r(t_1, t_2) = \sum_k r^{(k)}(t_1 - t_2) \exp\left(j \frac{2\pi k t_2}{T}\right)$

cf) $x(t) = x(t+T)$

$= \sum_k \boxed{C^{(k)}} \exp\left(j \frac{2\pi k t}{T}\right)$

where $C^{(k)} = \frac{1}{T} \int_{<T>} x(t) \exp\left(-j \frac{2\pi k}{T} t\right) dt$

$\tau = t_1 - t_2$

$R(f_1, f_2) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(t_1, t_2) \exp(-j 2\pi f_1 t_1) \exp(j 2\pi f_2 t_2)\, dt_1\, dt_2$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_k r^{(k)}(t_1 - t_2) \exp\left(j \frac{2\pi k t_2}{T}\right) \boxed{\exp(-j 2\pi f_1 t_1)} \exp(j 2\pi f_2 t_2)\, dt_1\, dt_2$

$t_1 - t_2 = \tau \quad t_1 = t_2 + \tau$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_k r^{(k)}(\tau) \exp\left(j \frac{2\pi k t_2}{T}\right) \boxed{\exp(-j 2\pi f_1 t_2) \exp(-j 2\pi f_1 \tau)} \exp(j 2\pi f_2 t_2)\, d\tau\, dt_2$

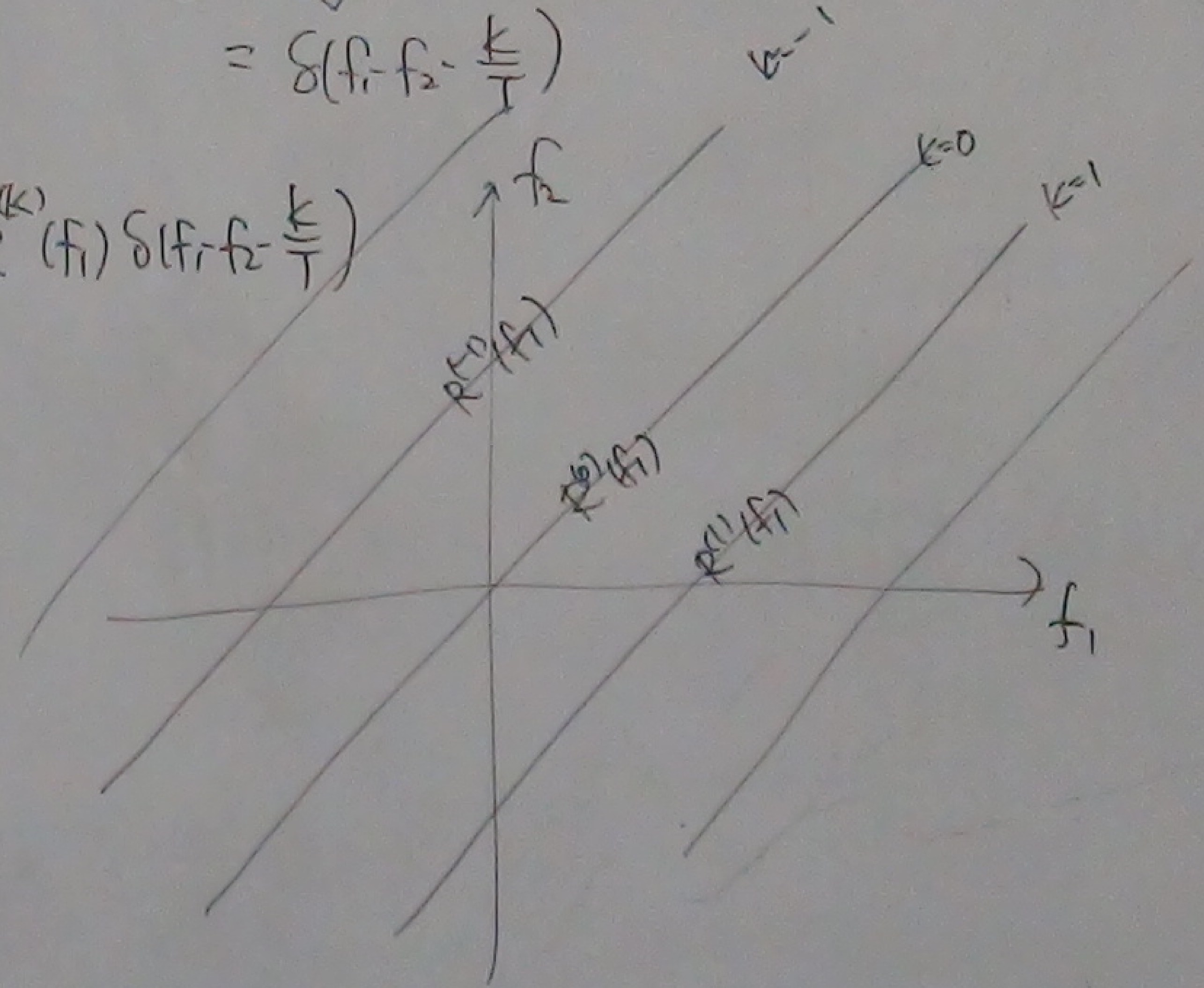
$= \sum_k \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} r^{(k)}(\tau) \exp(-j 2\pi f_1 \tau)\, d\tau \right)}\exp\left(-j 2\pi \left(f_1 - f_2 - \frac{k}{T}\right) t_2\right) dt_2$

$\int_{-\infty}^{\infty} \exp(j 2\pi f_0 t)\, dt = \delta(f_0)$

$= \sum_k R^{(k)}(f_1) \underbrace{\int_{-\infty}^{\infty} \exp\left(j 2\pi \left(f_1 - f_2 - \frac{k}{T}\right) t_2\right) dt_2}_{= \delta\left(f_1 - f_2 - \frac{k}{T}\right)}$

$\sum_k R^{(k)}\left(f_1 - \cancel{\frac{k}{T}}\right) \delta\left(f_1 - f_2 - \frac{k}{T}\right)$

$= \sum_k R^{(k)}(f_1)\, \delta\left(f_1 - f_2 - \frac{k}{T}\right)$



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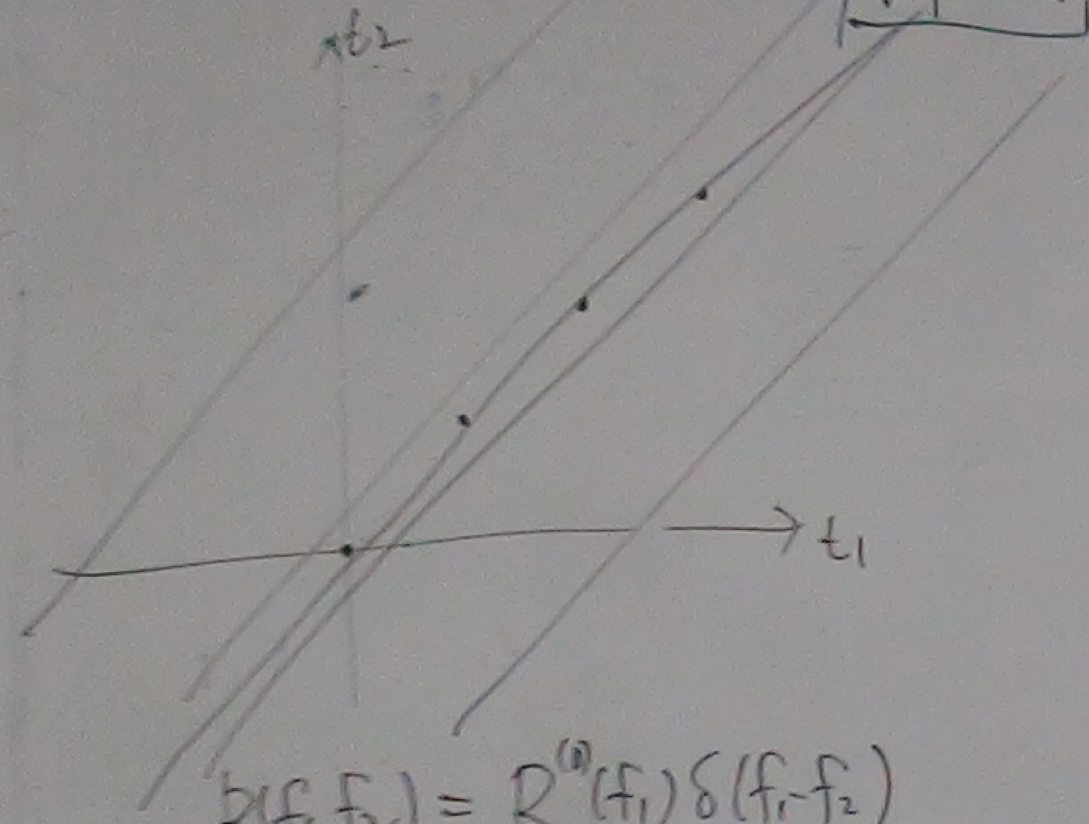
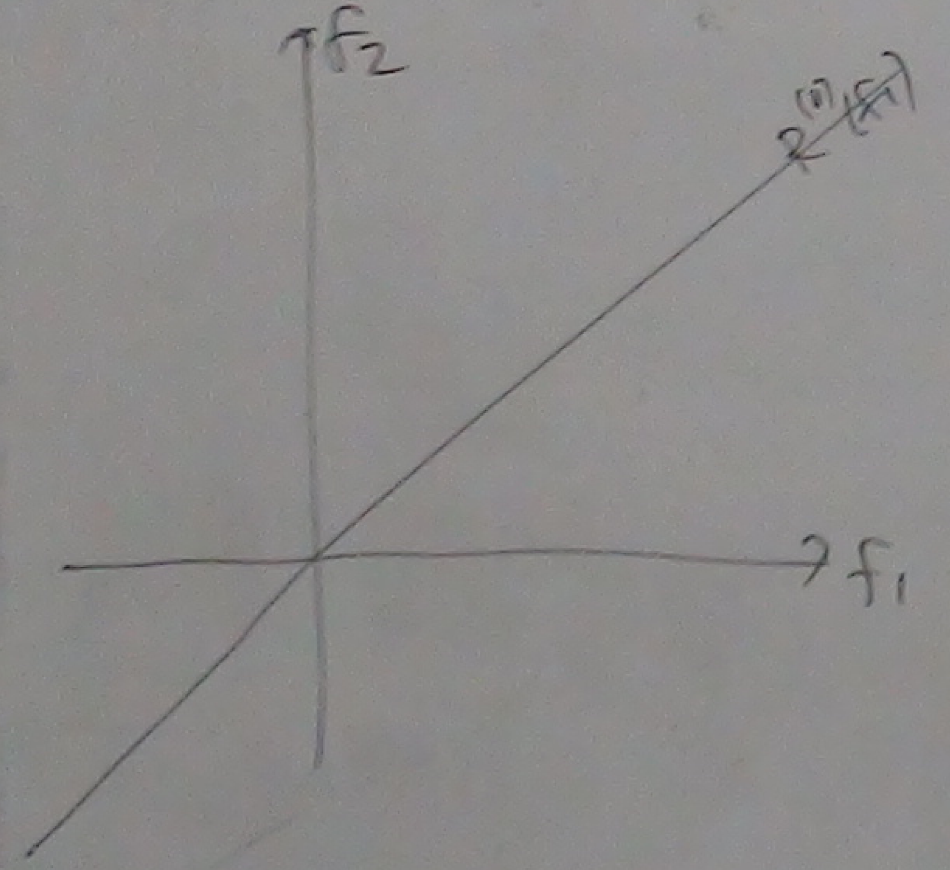
□ Special case

$r(t_1,t_2) = r(t_1+T, t_2+T)$, $\forall t_1,t_2$ $\boxed{\forall T \in \mathbb{R}}$



$R(f_1,f_2) = R^{(1)}(f_1)\,\delta(f_1-f_2)$



complex-valued second-order → finite power

□ random process $X(t)$, $t \in \mathbb{R}$

(i) mean process

$\mu_{X(t)} \triangleq E[X(t)]$

$E[X(t_0)] = \mu_X(t_0)$

(ii) auto-correlation function

$r_{XX}(t_1,t_2) \triangleq E[X(t_1)X^*(t_2)]$

$r_{XY}(t_1,t_2) \triangleq E[X(t_1)Y^*(t_2)]$

cross-correlation function

(iii) pseudo- / complementary } auto-correlation

$\tilde{r}_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$

(iv) auto-covariance function

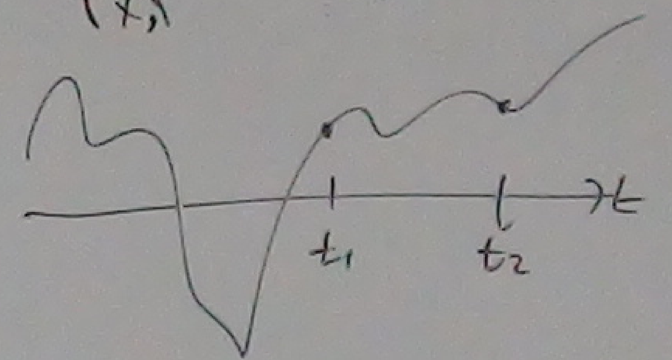
$C_{XX}(t_1,t_2) \triangleq E\left[\left(X(t_1)-\mu_X(t_1)\right)\left(X(t_2)-\mu_X(t_2)\right)^*\right]$

(v) complementary

$\tilde{C}_{XX}(t_1,t_2) \triangleq E\left[\left(X(t_1)-\mu_X(t_1)\right)\left(X(t_2)-\mu_X(t_2)\right)\right]$

X, Y $X(t_1)$ $X(t_2)$

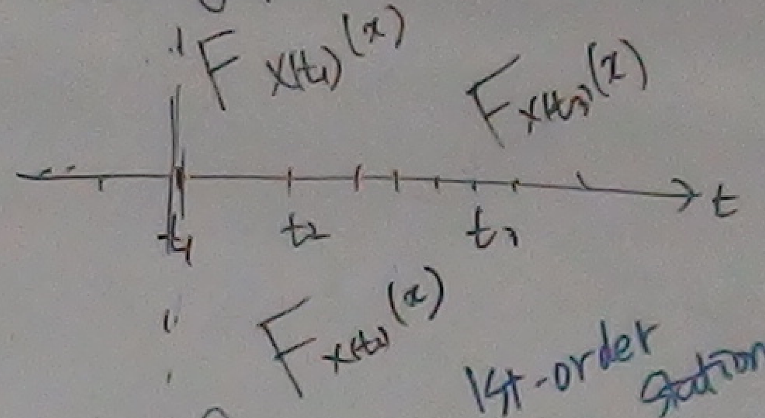
$r_{X,X}$ $r_{X(t_1)X(t_2)}$



X, Y $(X-\mu_X)$ $(Y-\mu_Y)$

□ Classification of r. processes.

(i) Stationary process



1st-order stationary

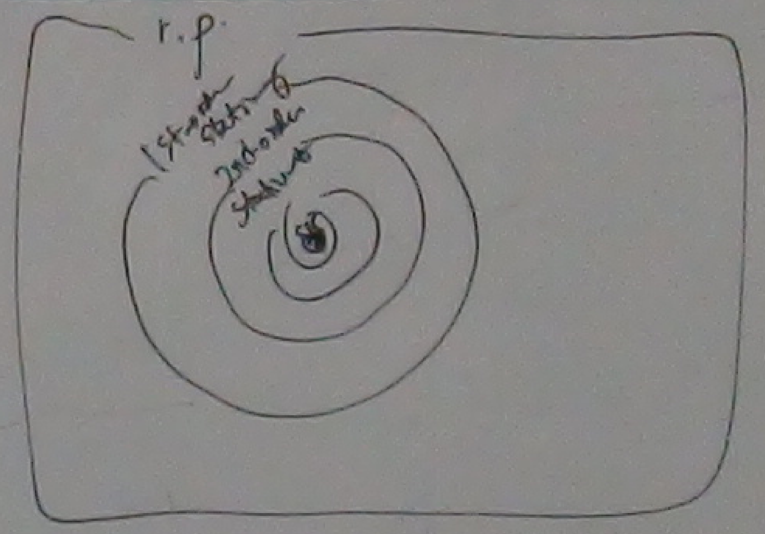
$\left(F_{X(t)}(x) = F_{X(t+\tau)}(x),\ \forall x\right)\ \forall \tau$

2nd-order stationary

$\left(F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(t_1+\tau),X(t_2+\tau)}(x_1,x_2),\ \forall x_1,x_2\right)\ \forall \tau$

Strict-sense stationary

$\forall N \in \mathbb{N}$, $\forall \underline{t} \in \mathbb{R}^N$ $\left(F_{X(t_1),X(t_2),\dots,X(t_N)}(x_1,\dots,x_N) = F_{X(t_1+\tau),\dots,X(t_N+\tau)}(x_1,\dots,x_N)\ \forall x_1 \dots x_N\right)\ \forall \tau$



(ii) cyclostationary process

$\forall \tau \to T(\cdot)$