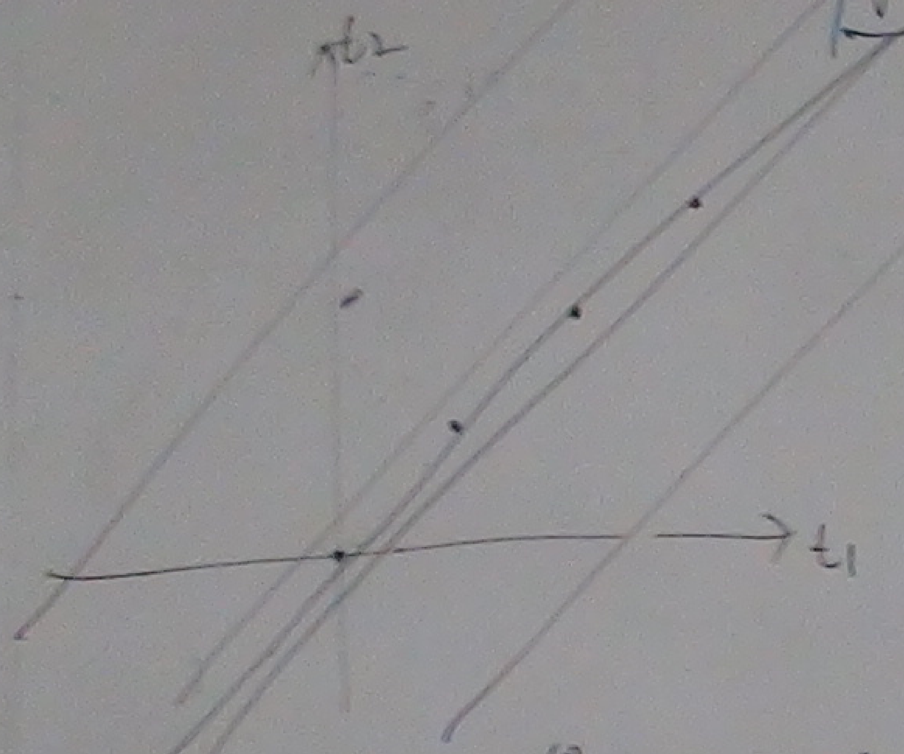
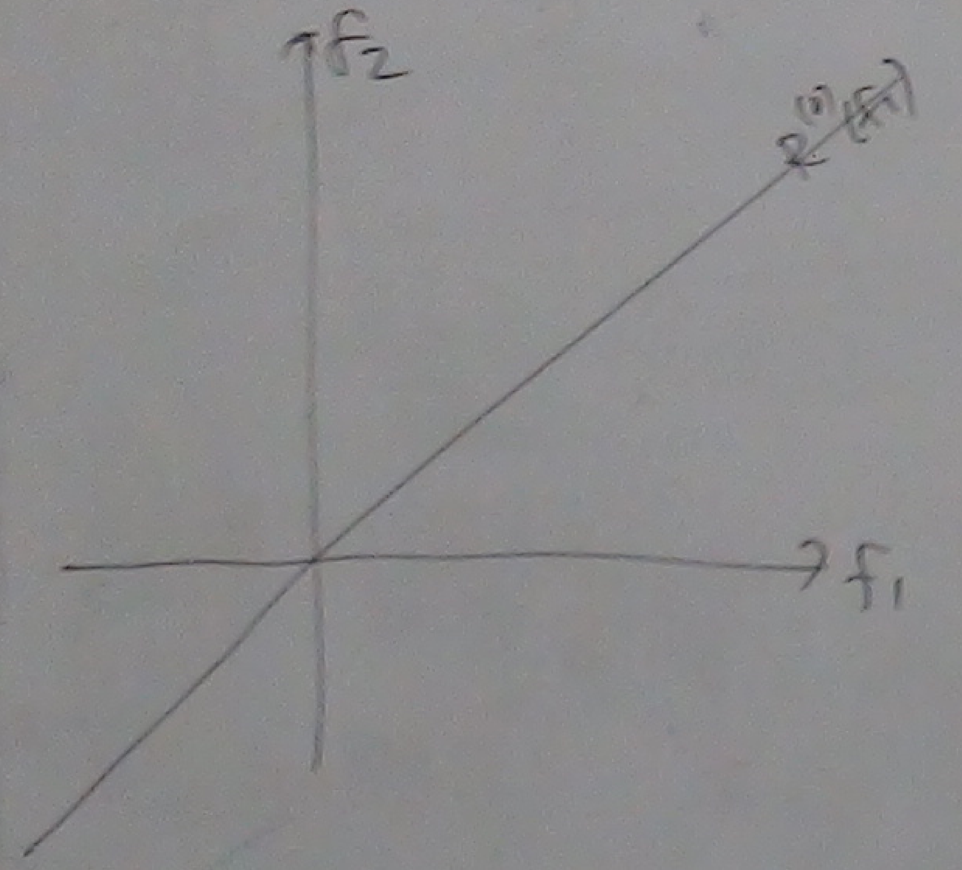
EECE695M CM#2c

□ Special case

$r(t_1,t_2) = r(t_1+T, t_2+T)$, $\forall t_1,t_2$ $\boxed{\forall T \in \mathbb{R}}$



$R(f_1,f_2) = R^{(1)}(f_1)\,\delta(f_1-f_2)$



complex-valued second-order → finite power

□ random process $X(t)$, $t \in \mathbb{R}$

(i) mean process

$\mu_X(t) \triangleq E[X(t)]$

$E[X(t_0)] = \mu_X(t_0)$

(ii) auto-correlation function

$r_{XX}(t_1,t_2) \triangleq E[X(t_1)X^*(t_2)]$

$r_{XY}(t_1,t_2) \triangleq E[X(t_1)Y^*(t_2)]$

cross-correlation function

(iii) pseudo- / complementary } auto-correlation

$\tilde{r}_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$

(iv) auto-covariance function

$C_{XX}(t_1,t_2) \triangleq E\left[\left(X(t_1)-\mu_X(t_1)\right)\left(X(t_2)-\mu_X(t_2)\right)^*\right]$

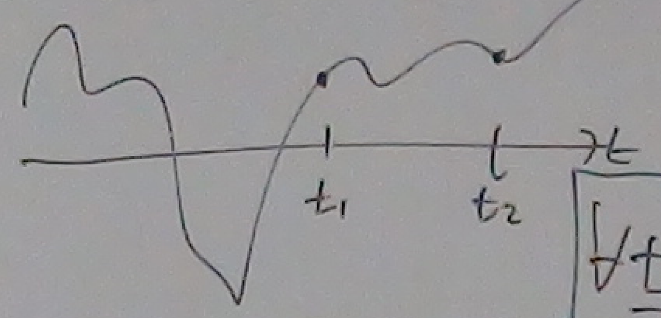
(v) complementary

$\tilde{C}_{XX}(t_1,t_2) \triangleq E\left[\left(X(t_1)-\mu_X(t_1)\right)\left(X(t_2)-\mu_X(t_2)\right)\right]$

X, Y $\quad X(t_1)\ X(t_2)$

$r_{X,Y}$ $\quad r_{X(t_1),X(t_2)}$

$N=1$, $\boxed{\forall t_1}$

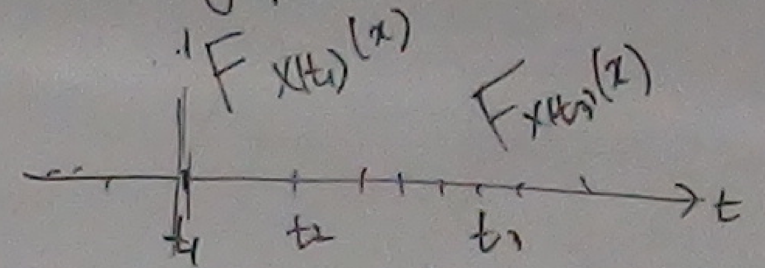


$\boxed{\forall \underline{t} \in \mathbb{R}^2}$

X, Y $\quad (X-\mu_X)\ (Y-\mu_Y)$

□ Classification of r. processes

(i) Strict-sense Stationary process



$F_{X(t_1)}(x)$

1st-order stationary

$\left(F_{X(t_1)}(x) = F_{X(t_1+\tau)}(x),\ \forall x \right) \forall \tau,$

2nd-order stationary

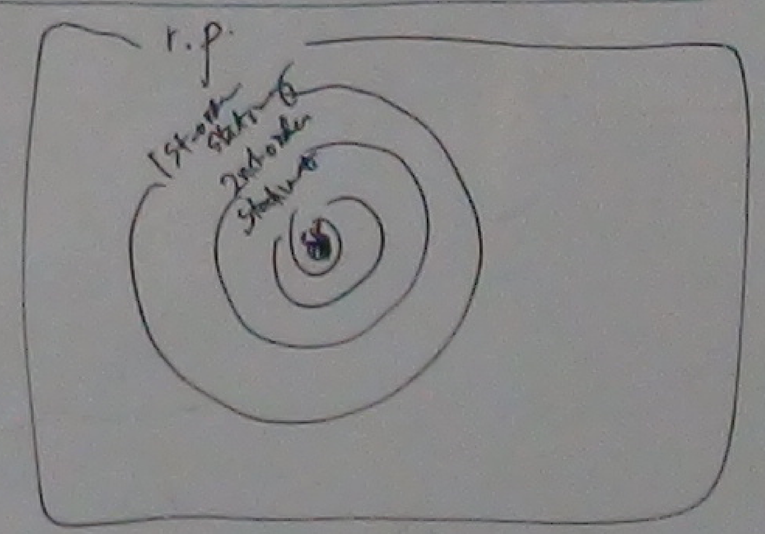
$\left(F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(t_1+\tau),X(t_2+\tau)}(x_1,x_2),\ \forall x_1,x_2 \right) \forall \tau$

Strict-sense stationary

$\forall N \in \mathbb{N}$, $\forall \underline{t} \in \mathbb{R}^N$ $\left(F_{X(t_1),X(t_2),\dots,X(t_N)}(x_1,\dots,x_N) = F_{X(t_1+\tau),\dots,X(t_N+\tau)}(x_1,\dots,x_N)\ \forall x_1 \dots x_N \right) \forall \tau$

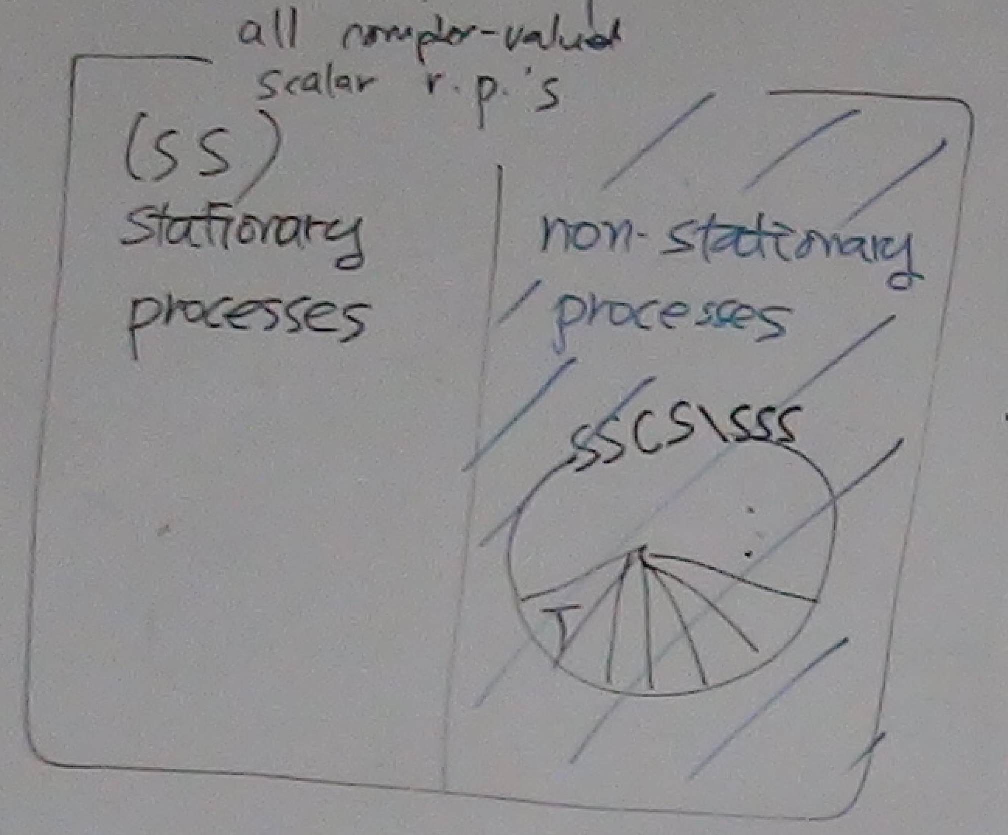
(ii) cyclostationary process

$\forall \tau \to T(\cdot)$

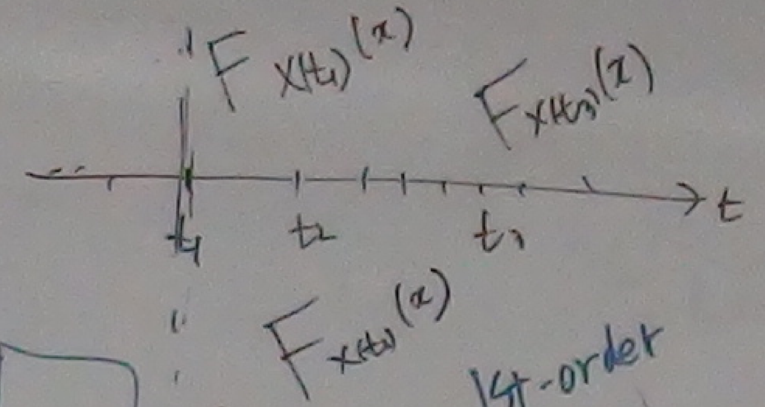


EECE695M CM#3a

☐ Classification of r.p.'s



strict-sense
(ii) cyclostationary process
$\forall \tau \to T(\cdot)$



1st-order

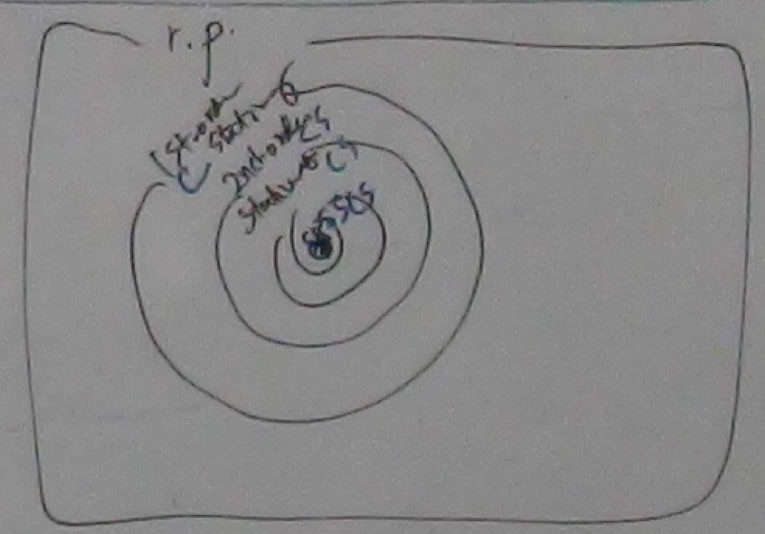
$N=1.\ \boxed{\forall t_1},\ \left(F_{X(t_1)}(x) = F_{X(t_1+T)}(x),\ \forall x \right) \cancel{\forall \tau},$

2nd-order

$N=2,\ \boxed{\forall \underline{t}} \in \mathbb{R}^2 \left(F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(t_1+T),X(t_2+T)}(x_1,x_2),\ \forall x_1,x_2 \right) \cancel{\forall \tau}$

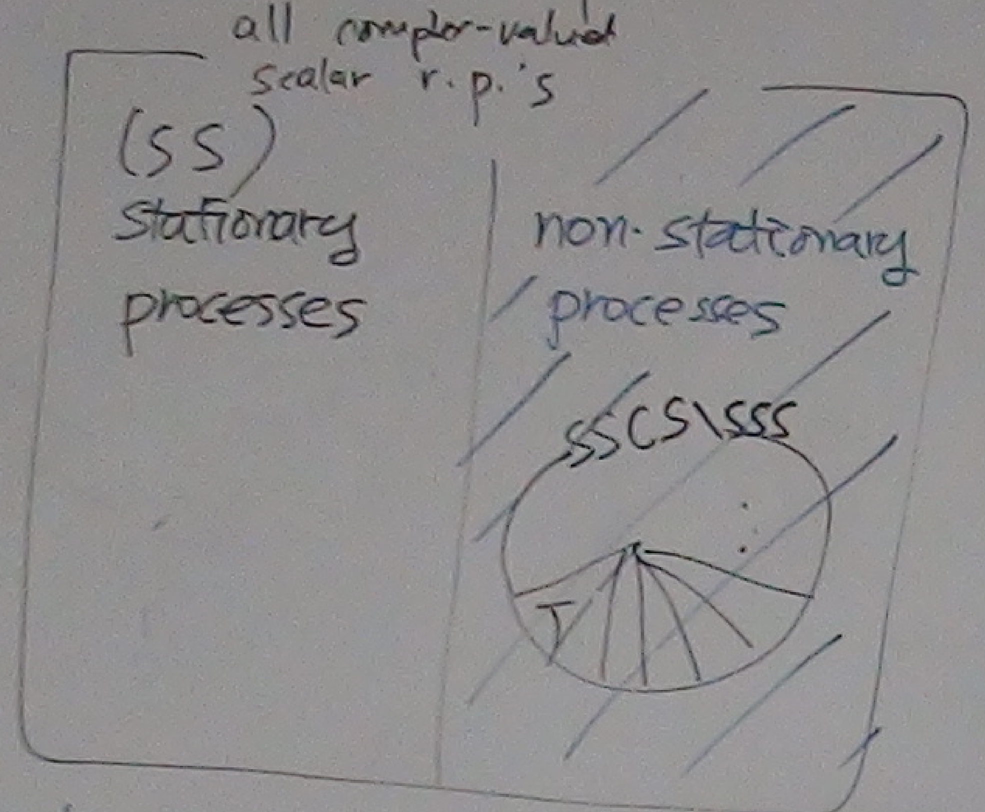
Strict-sense stationary

$\forall N \in \mathbb{N}$
$\forall \underline{t} \in \mathbb{R}^N \left(F_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1,\cdots,x_N) = F_{X(t_1+T),\cdots,X(t_N+T)}(x_1,\cdots,x_N)\ \forall x_1 \cdots x_N \right) \cancel{\forall \tau}$



EECE695M CM#3b

☐ Classification of r.p.'s



Kolmogorov's extension theorem

CT
- real Gaussian WSS
- Poisson r.p

DT Markov process

☐ real-valued r.p. $X(t)$ WSS (wide-sense)

Def. " " " is " if

(i) $E\{X(t)\} = \mu_X, \forall t$

(ii) $E\{X(t_1)X(t_2)\} = R_{XX}(t_1,t_2)$
$= R_{XX}(t_1+\tau, t_2+\tau), \forall \tau$
$= R_{XX}(t_1-t_2), \forall t \quad = R_{XX}(t_1-t_2, 0)$

$\equiv$

(i)
(ii)' $E[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))]$
$= C_{XX}(t_1,t_2) = C_{XX}(t_1+\tau, t_2+\tau)$
$= C_{XX}(t_1-t_2, 0) = C_{XX}(t_1-t_2)$

$R_{XX}: \mathbb{R}^2 \to \mathbb{R}$
$R_{XX}: \mathbb{R} \to \mathbb{R}$

Properties

Q. $C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$

if real WSS, then?

A. $C_{XX}(t_1,t_2) = C_{XX}(t_1-t_2, 0) = C_{XX}(t_1-t_2)$

Def " " " ~~WS~~ Covariance Stationary (CS)

if (ii)' " "

/ex/ $E\{X(t)\} = 0, \forall t$
WSS. $\to$ CS r.p

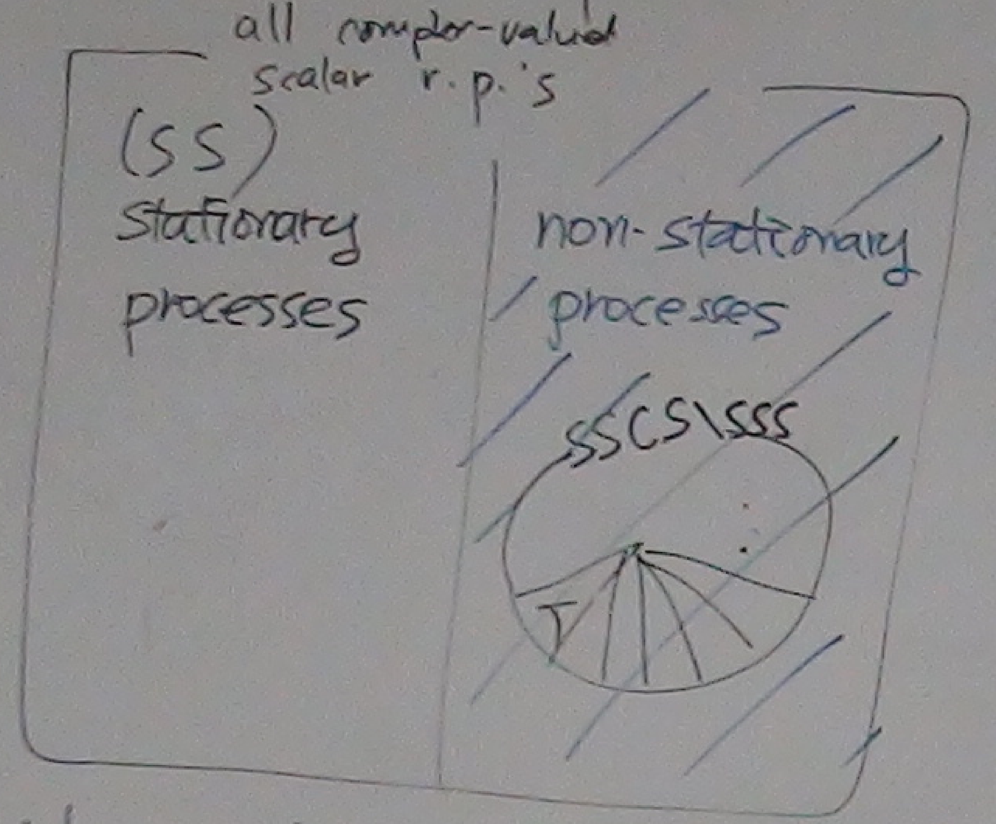
$Y(t) = x(t) + X(t)$

non-constant ft

$C_{YY}(t_1,t_2) = R_{XX}(t_1,t_2) = R_{XX}(t_1-t_2, 0)$

EECE695M CM#3C

☐ Classification of r.p.'s



Kolmogorov's extension theorem

CT — real Gaussian WSS
— Poisson r.p

DT Markov process

☐ ~~real~~ complex -valued r.p. $X(t)$ WSS (wide-sense)

Def. " " " is " if

(i) $E\{X(t)\} = \mu_X, \forall t$

(ii) $E\{X(t_1)X^*(t_2)\} = R_{XX}(t_1,t_2)$

$= R_{XX}(t_1+\tau, t_2+\tau), \forall \tau$

$= R_{XX}(t_1-t_2), \forall t \quad = R_{XX}(t_1-t_2, 0)$

(iii) $\tilde{R}_{XX}(t_1,t_2) = $ constant. $\forall t$

$R_{XX}: \mathbb{R} \to \mathbb{K}$

$R_{XX}: \mathbb{R} \to \mathbb{R}$

$=$

(i)

(ii)′ $E\left[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))^*\right]$

$= C_{XX}(t_1,t_2) = C_{XX}(t_1+\tau, t_2+\tau)$

$= C_{XX}(t_1-t_2, 0) = C_{XX}(t_1-t_2)$

(iii)′ $E\{(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))\} = 0, \forall t$

$\tilde{C}_{XX}$

☐ complex-valued r.p $X(t)$ is proper & WSS

proper-complex r.p. is WSS

improper
impropriety

/ex/ $E\{X(t)\} = 0, \forall t$

WSS. → CS r.p

$Y(t) = x(t) + X(t)$

non-constant ft

$C_{YY}(t_1,t_2) = R_{XX}(t_1,t_2) = R_{XX}(t_1-t_2, 0)$.

Properties

Q. $\tilde{C}_{XX}(t_1,t_2) = \tilde{R}_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$

if real WSS, then?

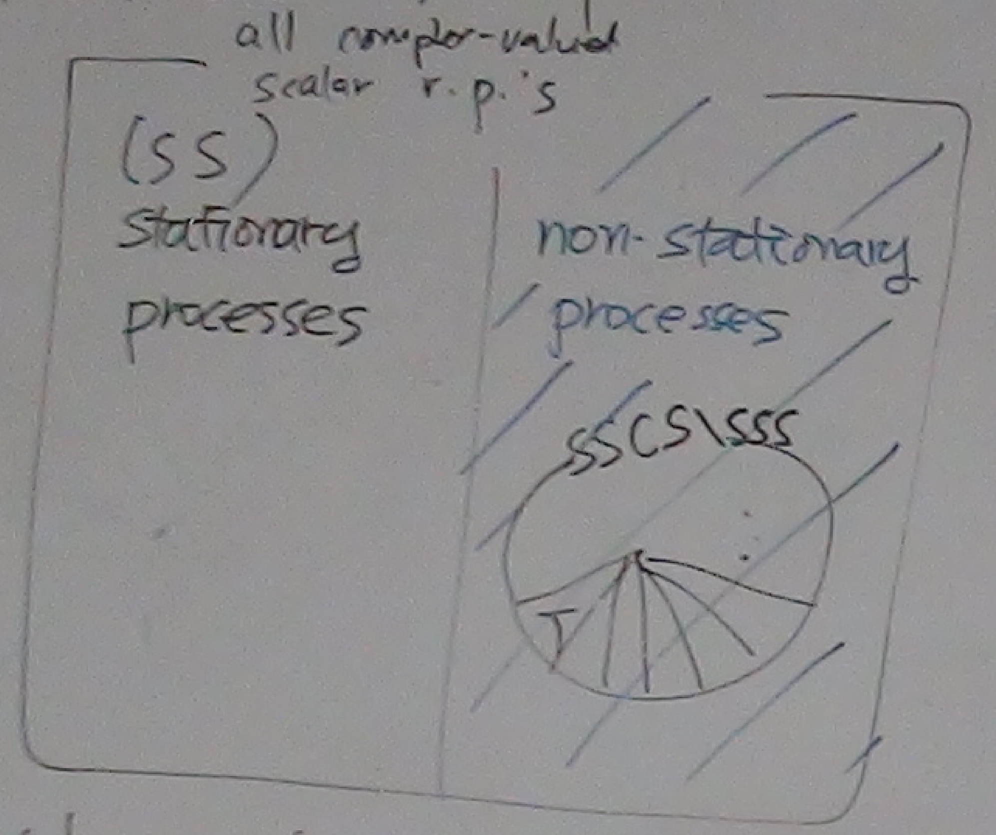
A. $C_{XX}(t_1,t_2) = C_{XX}(t_1-t_2, 0) = C_{XX}(t_1-t_2)$

Def " " " ~~WSS~~ Covariance Stationary (CS)

if (ii)′.

EECE695M CM#3d

□ Classification of r.p.'s



Kolmogorov's extension theorem

CT — real Gaussian WSS; Poisson r.p

DT Markov process

□ real-valued r.p. $X(t)$ WSCS (wide-sense, WSS, w/ period T, cycle)

Def. " " " is " if

(i) $E\{X(t)\} = \mu_X(t) = \mu_X(t+T), \forall t$

(ii) $E\{X(t_1)X(t_2)\} = R_{XX}(t_1,t_2) = R_{XX}(t_1+T, t_2+T), \forall t$

?⇔

(i) $\mu_X(t) = \mu_X(t+T), \forall t$

(ii)' $E[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))] = C_{XX}(t_1,t_2) = C_{XX}(t_1+T, t_2+T), \forall t$

$Y(t) = x(t) + X(t)$ zero-mean WSCS w/ c.p. T

where $x(t) \neq x(t+T)$.

$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$ ← Always true! ☺

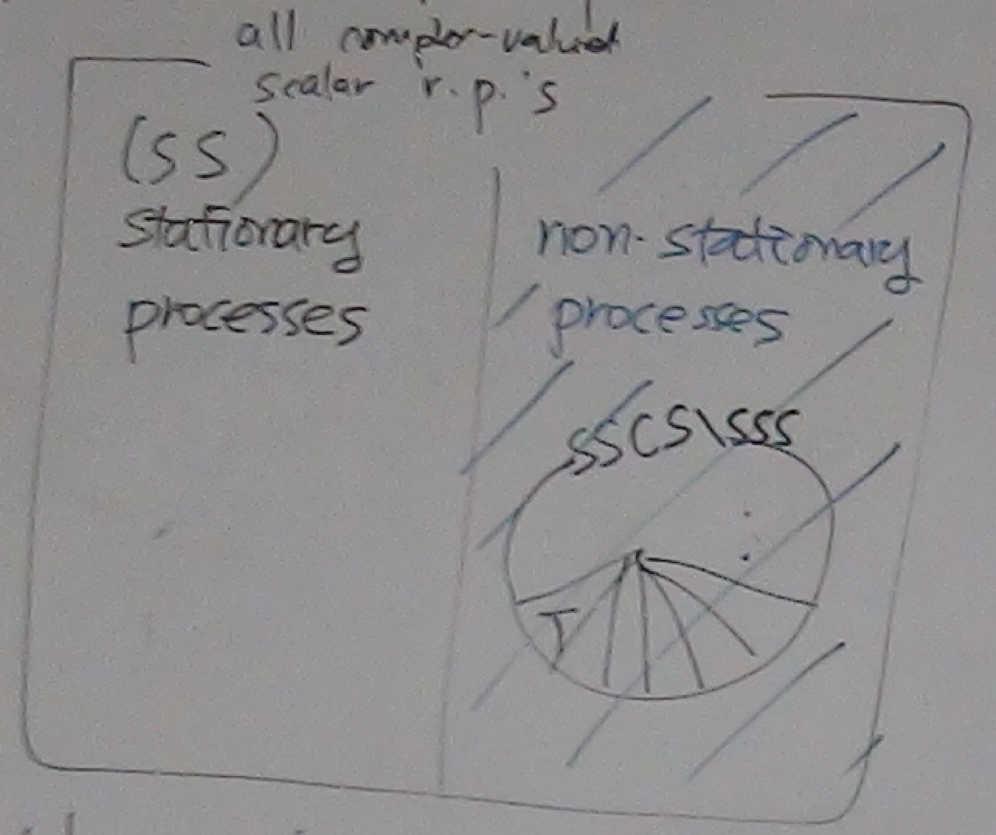
$= E[(X_1-\mu_1)(X_2-\mu_2)] = \boxed{R_{XX}(t_1,t_2) - \mu_1\mu_2} \cancel{- \mu_1\mu_2 + \mu_1\mu_2}$

Def. " " " Covariance (cyclo) Stationary (CS) (C)

if (ii)'.

EECE695M CM#3e

□ Classification of r.p.'s



Kolmogorov's extension theorem

CT: real Gaussian WSS; Poisson r.p

DT: Markov process

□ complex ~~real~~-valued r.p. $X(t)$ WSS / WSCS (wide-sense, proper-complex)

Def. " " " is " w/ period T (cycle)

if

(i) $E\{X(t)\} = \mu_X(t) = \mu_X(t+T), \forall t$

(ii) $E\{X(t_1)X(t_2)^*\} = R_{XX}(t_1,t_2)$
$= R_{XX}(t_1+T, t_2+T), \ \forall t$

(iii) $\tilde{R}_{XX}(t_1,t_2) = \mu_X(t_1)\mu_X(t_2)$

?⇔ =

(i) $\mu_X(t) = \mu_X(t+T), \forall t$

(ii)' $E[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))^*]$
$= C_{XX}(t_1,t_2) = C_{XX}(t_1+T, t_2+T), \forall t$

(iii)' $\tilde{C}_{XX}(t_1,t_2) = 0, \forall t$

$Y(t) = x(t) + X(t)$ zero-mean WSCS w/ c.p. T

where $x(t) \neq x(t+T)$.

$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$ ← Always true! ☺

$= E[(X_1-\mu_1)(X_2-\mu_2)] = \boxed{R_{XX}(t_1,t_2) - \mu_1\mu_2} \cancel{- \mu_1\mu_2 + \mu_1\mu_2}$

Def. complex " " " cyclo covariance stationary (CS) C

if (ii)' proper