EECE695M CM#4a

□ Review

- Two view points
  (i) a collection of r.v.'s

$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)$

$\tilde{C}_{XX}(t_1,t_2) = \tilde{R}_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$

$X(t)$ $\qquad (X(t) : t\in\mathbb{R})$

$(X(t))_t \qquad \{X(t)\}_t \qquad \{X(t): t\in\mathbb{R}\}$

$(X(t))_{t\in\mathbb{R}} \qquad \{X(t)\}_{t\in\mathbb{R}} \qquad \{X(t),\ t\in\mathbb{R}\}$

$(X(t),\ t\in\mathbb{R})$

→ Kolmogorov's extension theorem

$\forall N\in\mathbb{N}, \forall \underline{t}\in\mathbb{R}^N, F_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1,x_2,\cdots,x_N)$

full characterization / complete specification } of a r.p.

$\forall N\in\mathbb{N}, \forall \underline{t}\in\mathbb{R}^N, E\{\exp(j\underline{\omega}^T\underline{X})\}$

where $\underline{\omega}\in\mathbb{R}^N$, $\underline{X} = \begin{bmatrix} X(t_1)\\ X(t_2)\\ \vdots \\ X(t_N)\end{bmatrix}$

(ii) measure-theoretic

r. experiment $(\Omega, \mathcal{F}, P)$ $\xrightarrow{\omega \text{ (outcome)}}$ mapping rule $\omega \to X(\omega, t)$ $\to$ $X(\omega,t)$

sample function / path

- real random process $X(t)$
  real-valued

(a) SSS r.p.
 └ strict-sense

a r.p. is stationary in the strict sense
 strictly stationary

$\forall N\in\mathbb{N}, \forall\underline{t}\in\mathbb{R}^N,$

$F_{X(t_1),X(t_2),\cdots,X(t_N)}(\underline{x}) = F_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_N+\tau)}(\underline{x}),\ \forall\underline{x}, \forall\tau$

$x\in\mathbb{R}^N$ $\quad\tau$: translation variable
invariant in τ.

(b) WSS r.p. $\qquad$ Second-order stationary (X)
 └ wide-sense

a r.p. is stationary in the wide sense
 widely stationary

1st(order) moment $\qquad$ 2nd-order moment

(i) $\mu_X(t) \triangleq E\{X(t)\}$
$= \mu_X, \forall t = \mu_X(t+\tau), \forall t, \forall\tau$

(ii) $R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X(t_2)\}$
$= R_{XX}(t_1+\tau, t_2+\tau),\ \forall\underline{t}, \forall\tau$
$= R_{XX}(t_1-t_2, 0) = R_{XX}(t_1-t_2)$

difference

(ii)' $C_{XX}(t_1,t_2) = C_{XX}(t_1+\tau, t_2+\tau),\ \forall t, \forall\tau$

(~) covariance stationary

(c) SSCS r.p. w.r.p. T: joint CDFs are periodic in translation variable w/ period T.

(d) WSCS r.p.

a r.p. is cyclostationary in the wide sense
 widely cyclostationary
 wide-sense cyclostationary

(i) $\mu_X(t) = \mu_X(t+T), \forall t$

(i)' $\mu_X(t+\tau) = \mu_X(t+\tau+T), \forall t, \forall\tau$

(ii) $R_{XX}(t_1,t_2) = R_{XX}(t_1+T, t_2+T)$,

(ii)' $R_{XX}(t_1+\tau, t_2+\tau) = R_{XX}(t_1+\tau+T, t_2+\tau+T), \forall t, \forall\tau$

(ii)'' $C_{XX}(t_1,t_2) = C_{XX}(t_1+T, t_2+T), \forall t$

(ii)''' $C_{XX}(t_1+\tau, t_2+\tau) = C_{XX}(t_1+\tau+T, t_2+\tau+T), \forall \underline{t}, \forall\tau$

$F_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_N+\tau)}(\underline{x}) = F_{X(t_1+\tau+T),X(t_2+\tau+T),\cdots,X(t_N+\tau+T)}(\underline{x}),$
$\forall\underline{t}, \forall\tau.$

$C_{XX}(t_1+\tau, t_2+\tau) = R_{XX}(t_1+\tau, t_2+\tau) - \mu_X(t_1+\tau)\mu_X(t_2+\tau)$

$C_{XX}(t_1+\tau+T, t_2+\tau+T)$
$= R_{XX}(t_1+\tau+T, t_2+\tau+T) - \mu_X(t_1+\tau+T)\mu_X(t_2+\tau+T)$

EECE695M CM#4b

□ Review

- Two view points

(1)

$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)$

$\tilde{C}_{XX}(t_1,t_2) = \tilde{R}_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$

→ Kolmogorov's extension theorem

$\forall N \in \mathbb{N}, \forall \underline{t} \in \mathbb{R}^N, F_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1,x_2,\cdots,x_N)$

full characterization / complete specification } of a r.p.

$\forall N \in \mathbb{N}, \forall \underline{t} \in \mathbb{R}^N, E\{\exp(j\underline{\omega}^T\underline{X})\}$

where $\underline{\omega} \in \mathbb{R}^N$, $\underline{X} = \begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_N) \end{bmatrix}$

- real random process $X(t)$
real-valued

(a) SSS r.p.
└ strict-sense

a r.p. is stationary in the strict sense.
strictly stationary.

$\forall N \in \mathbb{N}, \forall \underline{t} \in \mathbb{R}^N,$

$F_{X(t_1),X(t_2),\cdots,X(t_N)}(\underline{x}) = F_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_N+\tau)}(\underline{x}), \forall \underline{x}, \forall \tau$

$x \in \mathbb{R}^N$ τ: translation variable
invariant in τ.

(b) WSS r.p. Second-order stationary (X)
└ wide-sense

a r.p. is stationary in the wide sense
widely stationary

1st (order) moment 2nd-order moment

(i) $\mu_X(t) \triangleq E\{X(t)\}$
$= \mu_X, \forall t = \mu_X(t+\tau), \forall t, \forall \tau$

(ii) $R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X(t_2)\}$
$= R_{XX}(t_1+\tau, t_2+\tau), \forall \underline{t}, \forall \tau$
$= R_{XX}(t_1-t_2, 0) = R_{XX}(t_1-t_2)$

difference

(ii)' $C_{XX}(t_1,t_2) = C_{XX}(t_1+\tau, t_2+\tau), \forall t, \forall \tau$

( ) covariance stationary

(c) SSCS r.p. w/ r.p. T: joint CDFs are periodic in translation variable w/ period T.

(d) WSCS r.p.

a r.p. is cyclostationary in the wide sense
widely cyclostationary
wide-sense cyclostationary.

(i) $\mu_X(t) = \mu_X(t+T), \forall t.$

(i)' $\mu_X(t+\tau) = \mu_X(t+\tau+T), \forall t, \forall \tau$

(ii) $R_{XX}(t_1,t_2) = R_{XX}(t_1+T, t_2+T),$

(ii)' $R_{XX}(t_1+\tau, t_2+\tau) = R_{XX}(t_1+\tau+T, t_2+\tau+T), \forall t, \forall \tau$

(ii)'' $C_{XX}(t_1,t_2) = C_{XX}(t_1+T, t_2+T), \forall \underline{t}$

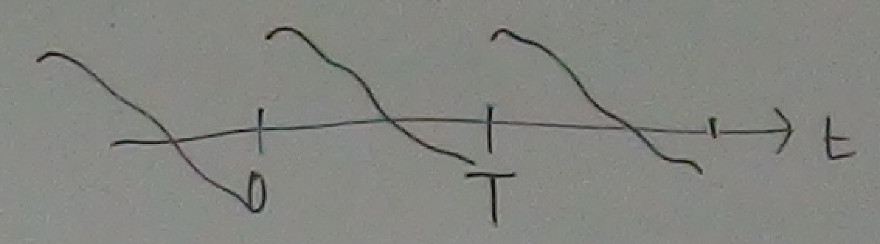
(ii)''' $C_{XX}(t_1+\tau, t_2+\tau) = C_{XX}(t_1+\tau+T, t_2+\tau+T), \forall \underline{t}, \forall \tau$

Def. wide-sense periodic r.p. $X(t)$

$Pr(\{X(t) = X(t+T)\}) = 1, \forall t \iff \mu_X(t) = \mu_X(t+T), \forall t$

$\iff R_{XX}(0) = R_{XX}(T)$

$\iff R_{XX}(t_1,t_2)$ ...



$F_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_N+\tau)}(\underline{x}) = F_{X(t_1+\tau+T),X(t_2+\tau+T),\cdots,X(t_N+\tau+T)}(\underline{x})$ (2),
$\forall \underline{t}, \forall \tau.$

$C_{XX}(t_1+\tau, t_2+\tau) = R_{XX}(t_1+\tau, t_2+\tau) - \mu_X(t_1+\tau)\mu_X(t_2+\tau)$

$C_{XX}(t_1+\tau+T, t_2+\tau+T)$
$= R_{XX}(t_1+\tau+T, t_2+\tau+T) - \mu_X(t_1+\tau+\cancel{T})\mu_X(t_2+\tau+\cancel{T})$

EECE695M CM#4C

Def. A complex-valued r.p. $X(t)$ is proper if $\tilde{C}_{XX}(t_1,t_2)=0,\ \forall \underline{t}$ ⟷ improper

$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)$

$\tilde{C}_{XX}(t_1,t_2) = \tilde{R}_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$

- $X(t)$ is a proper-complex r.p.
- a proper-complex r.p. $X(t)$
- a proper r.p. $X(t)$

→ Kolmogorov's extension theorem

$\forall N\in\mathbb{N}, \forall \underline{t}\in\mathbb{R}^N, F_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1,x_2,\cdots,x_N)$

full characterization / complete specification } of a r.p.

proper-complex ~~real~~ random process $X(t)$ ~~real-valued~~

(a) SSS r.p.
└ strict-sense

a r.p. is stationary in the strict sense.
strictly stationary.

$\forall N\in\mathbb{N}, \forall \underline{t}\in\mathbb{R}^N,$

$F_{X(t_1),X(t_2),\cdots,X(t_N)}(\underline{x}) = F_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_N+\tau)}(\underline{x}),\ \forall \underline{x}, \forall\tau$ ~~$\in\mathbb{R}^N$~~ or $\mathbb{C}^N$

$\underline{x}\in\mathbb{R}^N$ τ: translation variable
invariant in τ.

proper-complex
(b) WSS r.p.
└ wide-sense

Second-order stationary (X)

a r.p. is stationary in the wide sense
widely stationary

1st-order moment 2nd-order moment

(i) $\mu_X(t) \triangleq E\{X(t)\}$
$= \mu_X, \forall t = \mu_X(t+\tau), \forall t, \forall\tau$

(ii) $R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X^*(t_2)\}$
$= R_{XX}(t_1+\tau, t_2+\tau), \forall \underline{t}, \forall\tau$
$= R_{XX}(t_1-t_2, 0) = R_{XX}(t_1-t_2)$

(ii)' $C_{XX}(t_1,t_2) = C_{XX}(t_1+\tau, t_2+\tau), \forall \underline{t}, \forall\tau$

difference

(-) covariance stationary

(iii) $\tilde{C}_{XX}=0$

(c) SSCS r.p. w/ c.p. T : joint CDFs are periodic in translation variable w/ period T.

(d) WSCS r.p.
proper-complex

a r.p. is cyclostationary in the wide sense
widely cyclostationary
wide-sense cyclostationary

(i) $\mu_X(t) = \mu_X(t+T), \forall t$.
(i)' $\mu_X(t+\tau) = \mu_X(t+\tau+T), \forall t, \forall\tau$
(ii) $R_{XX}(t_1,t_2) = R_{XX}(t_1+T, t_2+T)$,
(ii)' $R_{XX}(t_1+\tau, t_2+\tau) = R_{XX}(t_1+\tau+T, t_2+\tau+T), \forall \underline{t}, \forall\tau$

(iii) $\tilde{C}_{XX}=0$

(ii)'' $C_{XX}(t_1,t_2) = C_{XX}(t_1+T, t_2+T), \forall \underline{t}$
(ii)''' $C_{XX}(t_1+\tau, t_2+\tau) = C_{XX}(t_1+\tau+T, t_2+\tau+T), \forall \underline{t}, \forall\tau$

$F_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_N+\tau)}(\underline{x}) = F_{X(t_1+\tau+T),X(t_2+\tau+T),\cdots,X(t_N+\tau+T)}(\underline{x}),$
$\forall \underline{t}, \forall\tau.$

$C_{XX}(t_1+\tau, t_2+\tau) = R_{XX}(t_1+\tau, t_2+\tau) - \mu_X(t_1+\tau)\mu_X^*(t_2+\tau)$

$C_{XX}(t_1+\tau+T, t_2+\tau+T)$
$= R_{XX}(t_1+\tau+T, t_2+\tau+T) - \mu_X(t_1+\tau+T)\mu_X^*(t_2+\tau+T)$

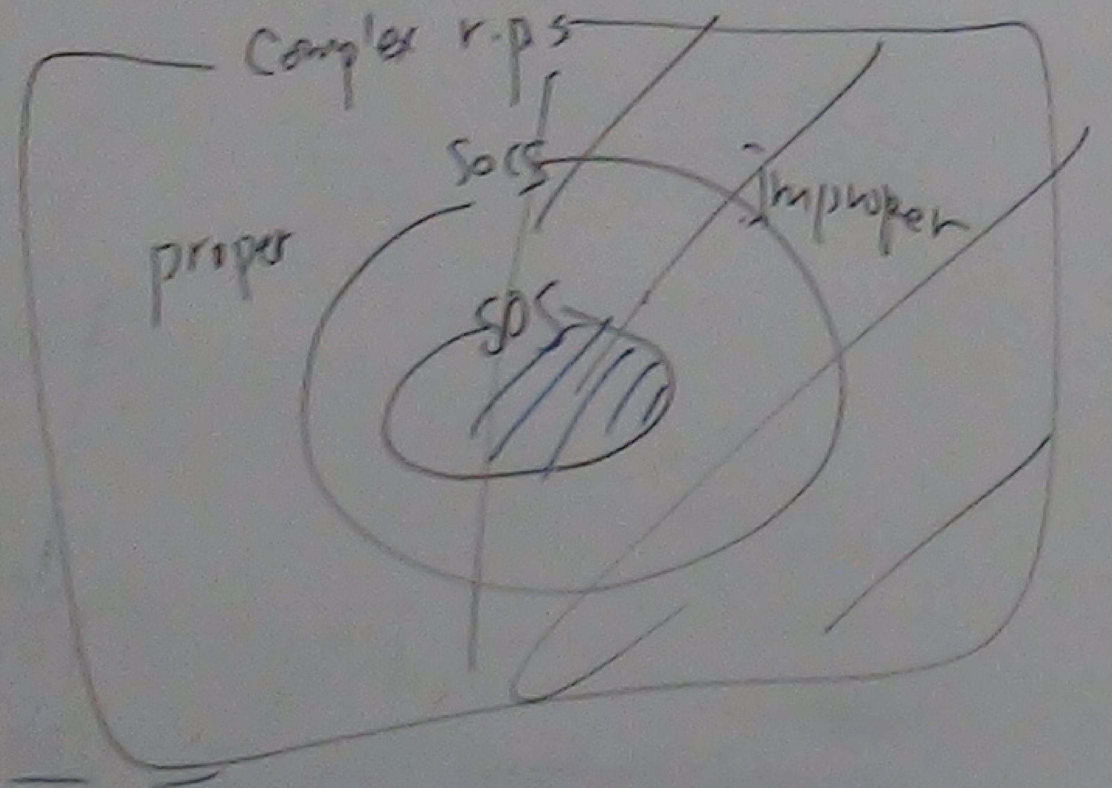
EECE695M CM#4d

Def. A complex-valued r.p. $X(t)$ is ~~proper~~ if $\longleftrightarrow$ improper

$\tilde{C}_{XX}(t_1,t_2)=0,\ \forall \underline{t}$

$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)$

$\tilde{C}_{XX}(t_1,t_2) = \tilde{R}_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$

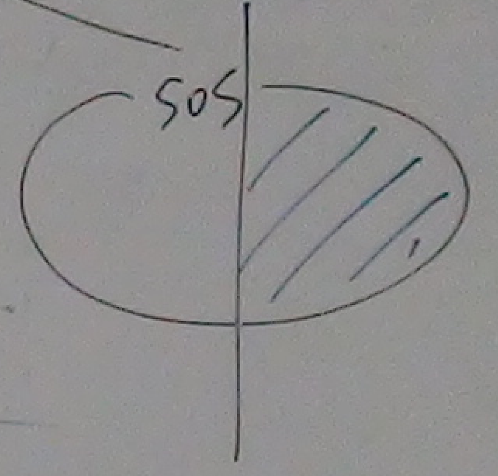


τ: translation variable

invariant in τ.

(b) SOS (Second-order stationary)
- proper
- Improper-complex SOS

~~proper-complex SOS~~ = proper-complex WSS



difference

(i) $\mu_X(t) \triangleq E\{X(t)\}$

$= \mu_X,\ \forall t = \mu_X(t+\tau),\ \forall t, \forall \tau$

(ii)' $C_{XX}(t_1,t_2) = C_{XX}(t_1-t_2)$ $\equiv$ (ii) $R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X^*(t_2)\}$

$= R_{XX}(t_1+\tau, t_2+\tau),\ \forall \underline{t}, \forall \tau$

$= R_{XX}(t_1-t_2, 0) = R_{XX}(t_1-t_2)$

(iii)' $\tilde{C}_{XX}(t_1,t_2) = \tilde{C}_{XX}(t_1-t_2)$ $\equiv$ (iii) $\tilde{R}_{XX}(t_1,t_2) = \tilde{R}_{XX}(t_1-t_2)$, $\forall \underline{t}, \forall \tau$

(c) covariance stationary

(d) SOCS (Second-order cyclostationary)
- proper — proper-complex WSCS
- improper-complex SOCS

(i) $\mu_X(t) = \mu_X(t+T),\ \forall t.$

(ii) $R_{XX}(t_1,t_2) = R_{XX}(t_1+T, t_2+T)$ $\equiv$ (ii)'' $C_{XX}(t_1,t_2) = C_{XX}(t_1+T, t_2+T),\ \forall \underline{t}$

(iii) $\tilde{R}_{XX}(t_1,t_2) = \tilde{R}_{XX}(t_1+T, t_2+T),\ \forall \underline{t}$

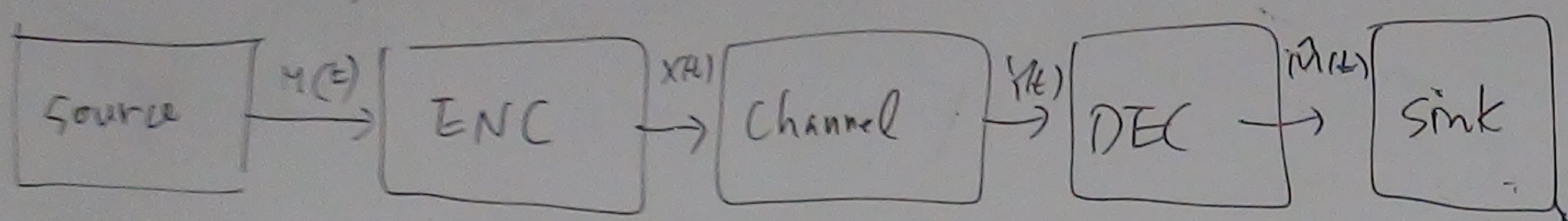
$\equiv$

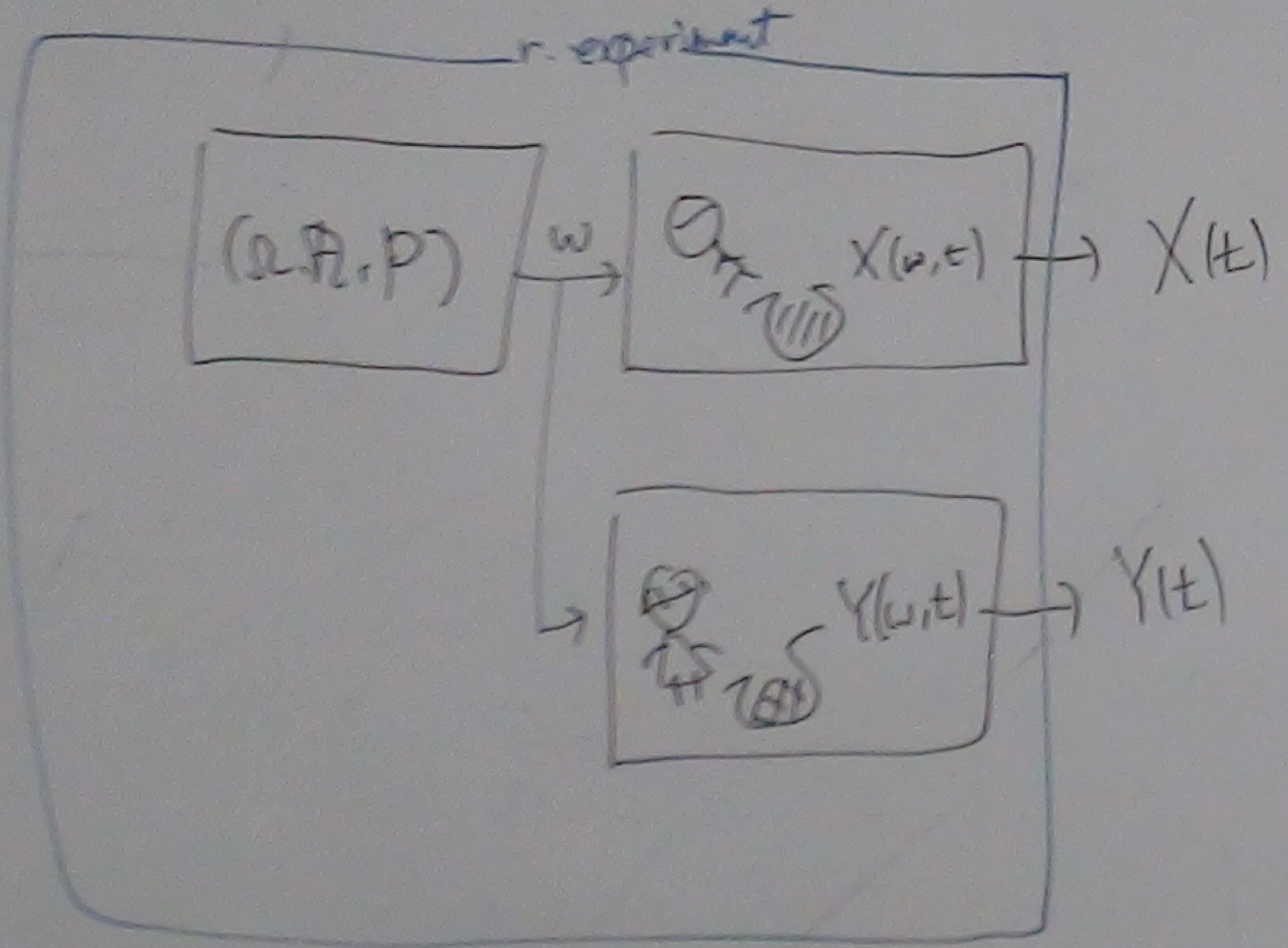
(iii)'' $\tilde{C}_{XX}(t_1,t_2) = C_{XX}(t_1+T, t_2+T),\ \forall \underline{t}$

EECE695M CM#4e

□ Two jointly distributed r.p.'s

jointly random processes.





□ Two r.p.'s $X(t)$ & $Y(t)$ are independent.

Def. $\forall N \in \mathbb{N}$, $\forall \underline{t} \in \mathbb{R}^N$, $\forall \underline{s} \in \mathbb{R}^N$

$$F_{X(t_1),X(t_2),\dots,X(t_N),Y(s_1),Y(s_2),\dots,Y(s_N)}(\underline{x},\underline{y}) = F_{\underline{X}}(\underline{x})F_{\underline{Y}}(\underline{y}), \quad \forall \underline{x},\underline{y} \in \mathbb{R}^N \quad \forall \underline{x},\underline{y} \in \mathbb{C}^N$$

☑ Two real r.p.'s are uncorrelated / orthogonal

□ Two jointly proper-complex r.p.'s are uncorrelated / orthogonal

real-valued

☑ Jointly WSS r.p.'s

□ jointly proper-complex WSS r.p.'s

□ jointly " " WSCS ".

□ ? improper