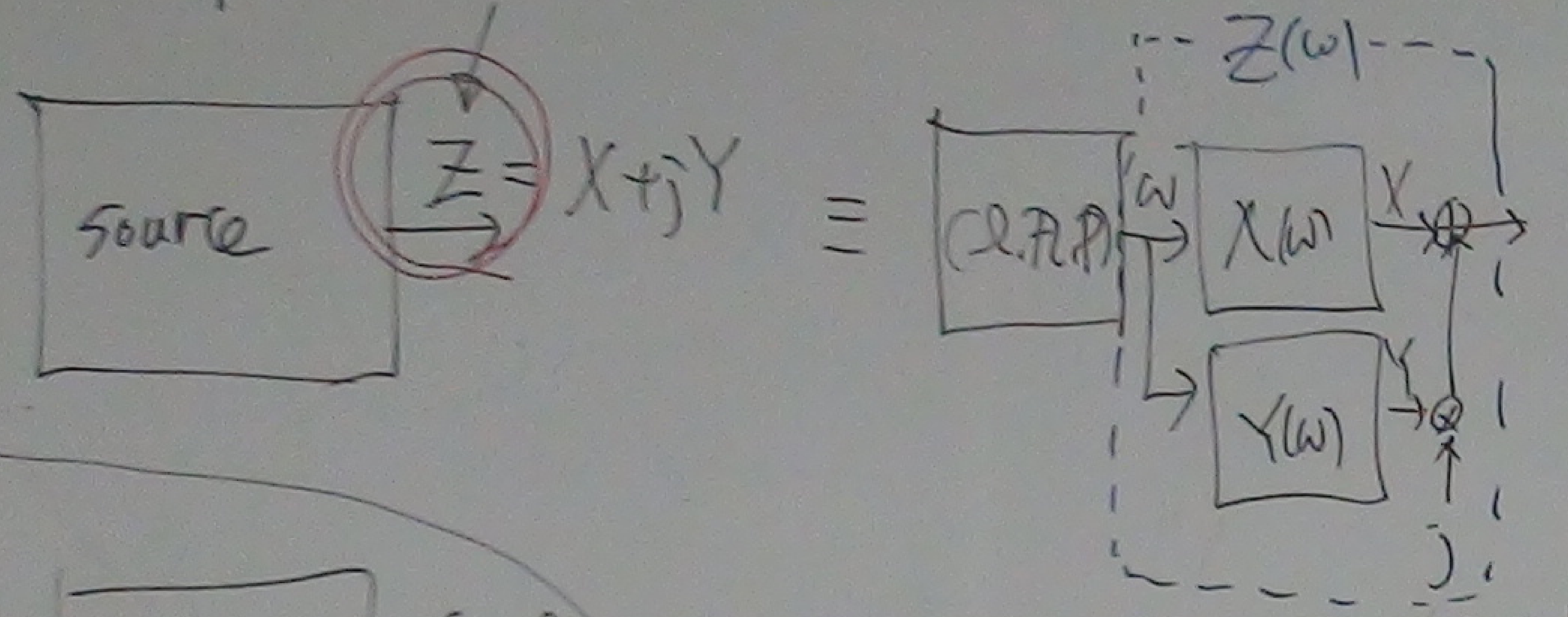
EECE695M CM#8a

□ A complex-valued random variable

$Z: \Omega \to \mathbb{C}^1$



jointly distributed.

cf) source $(\Omega, \mathcal{F}, P)$, $X(\omega)$ $\to X$ real w/ uncertainty

$X: \Omega \to \mathbb{R}$

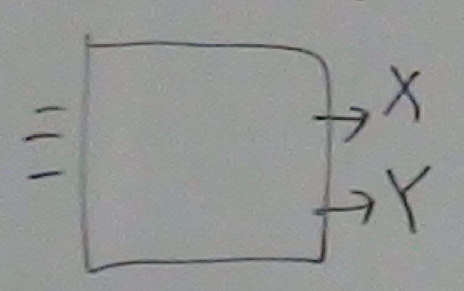
$(\mathbb{R}, \mathcal{B}, P_X)$

$F_X(x)$

$f_X(x)$

$\Phi_X(\omega) \triangleq E[e^{j\omega x}]$.

$\equiv$



$\to X$, $\to Y$ $\quad \begin{bmatrix} X \\ Y \end{bmatrix} \in \mathbb{R}^2$ real composite vector

$\begin{bmatrix} Z \\ Z^* \end{bmatrix} \in \mathbb{C}^2$ (complex) augmented vector

$\begin{bmatrix} Z \\ Z^* \end{bmatrix} = \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

redundant representation of Z.

$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2j} & -\frac{1}{2j} \end{bmatrix} \begin{bmatrix} Z \\ Z^* \end{bmatrix}$

□ complex Gaussian r.v. $Z = X + jY$.

$\Leftrightarrow$ X & Y are jointly Gaussian real r.v.s.

$\Leftrightarrow$ $\begin{bmatrix} X \\ Y \end{bmatrix}$ is a real Gaussian r. vector

$\Leftrightarrow$ Any affine combination of X & Y is a real Gaussian r. variable

$\begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X \\ Y \end{bmatrix} + c \qquad a, b, c \in \mathbb{R}$

$\Phi_W(\omega) = E\left[e^{j\omega(aX+bY)}\right] = \Phi_{X,Y}(\omega, \omega) =$

$\Leftrightarrow$ Jointly characteristic ft is in the form of $E\left[e^{j(\omega_1 X + \omega_2 Y)}\right]$

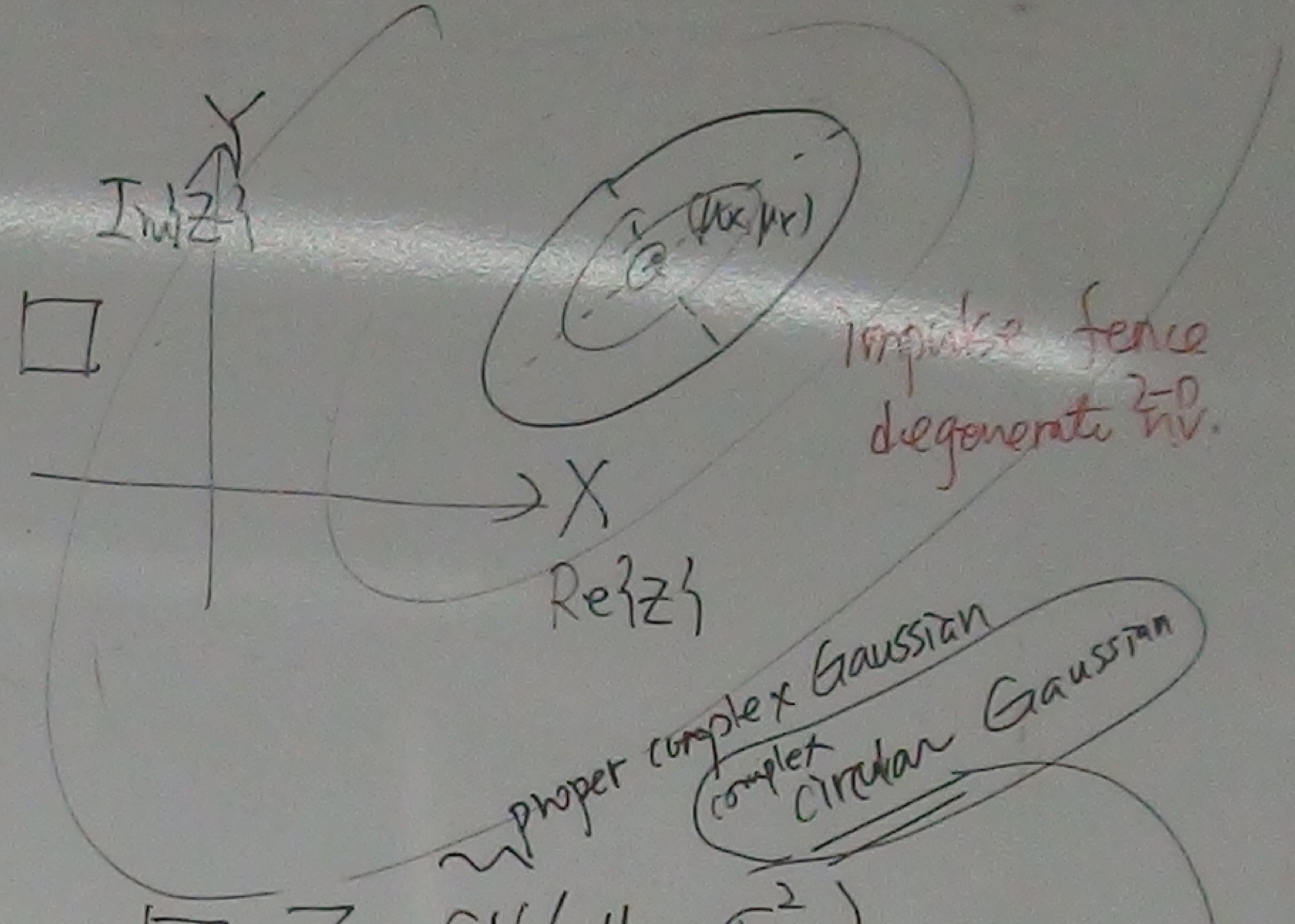
$\Phi_{X,Y}(\underline{\omega}) = \exp\{j(\omega_1 \mu_X + \omega_2 \mu_Y)\} \times \exp\left(-\frac{1}{2}\underline{\omega}^T C \underline{\omega}\right) = \exp(j\underline{\omega}^T \underline{\mu}) \times \exp\left(-\frac{1}{2}\underline{\omega}^T C \underline{\omega}\right)$

where $E\left\{\begin{bmatrix} X \\ Y \end{bmatrix}\right\} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$

$C \triangleq \mathrm{Cov}\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right)$

$\Rightarrow$ Any affine transform of $\begin{bmatrix} X \\ Y \end{bmatrix}$ is Gaussian $A\begin{bmatrix} X \\ Y \end{bmatrix} + \underline{b}$

□



~ proper complex Gaussian

complex circular Gaussian

□ $Z \sim \mathcal{CN}(\mu_Z, \sigma_Z^2)$

$\mu_Z \triangleq E\{Z\} = E\{X + jY\} = E\{X\} + jE\{Y\}$

$\sigma_Z^2 \triangleq E\{|Z - \mu_Z|^2\} = 2\sigma_X^2$

$\tilde{\sigma}_Z^2 \triangleq E\{(Z - \mu_Z)^2\} = 0. \Rightarrow \sigma_X^2 = \sigma_Y^2$, $\sigma_{XY} = 0$

when C^{-1} exists

$\Rightarrow f_{X,Y}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \det(C)}} \exp\left(-\frac{\begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}^T C^{-1} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}}{2}\right)$

$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X^2} \exp\left(-\frac{(x-\mu_X)^2 + (y-\mu_Y)^2}{2\sigma_X^2}\right)$

$= |z - \mu_Z|^2$

$f_Z(z) = \frac{1}{\pi\sigma_Z^2} \exp\left(-\frac{|z-\mu_Z|^2}{\sigma_Z^2}\right)$

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☐ "$Z \in \mathbb{C}^1$ is proper."

$E\{Z\} = \mu_Z = \mu_X + j\mu_Y$

$Z = X + jY$

$\Leftrightarrow \quad E\{(Z-\mu_Z)^2\} = 0$

$E\{|Z-\mu_Z|^2\} = 2\sigma_X^2 = 2\sigma_Y^2 = \sigma_Z^2$

$\Rightarrow \quad \sigma_X^2 + 2j\,\sigma_{XY} - \sigma_Y^2 = 0$

$\Leftrightarrow \quad \sigma_X^2 = \sigma_Y^2, \quad \sigma_{XY} = 0$

$\sigma_{XY} = \rho_{XY}\,\sigma_X\sigma_Y$ (unitless)

$-1 \le \rho_{XY} = E\left\{\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right\} \le 1$

☐ "A complex Gaussian Z is proper."

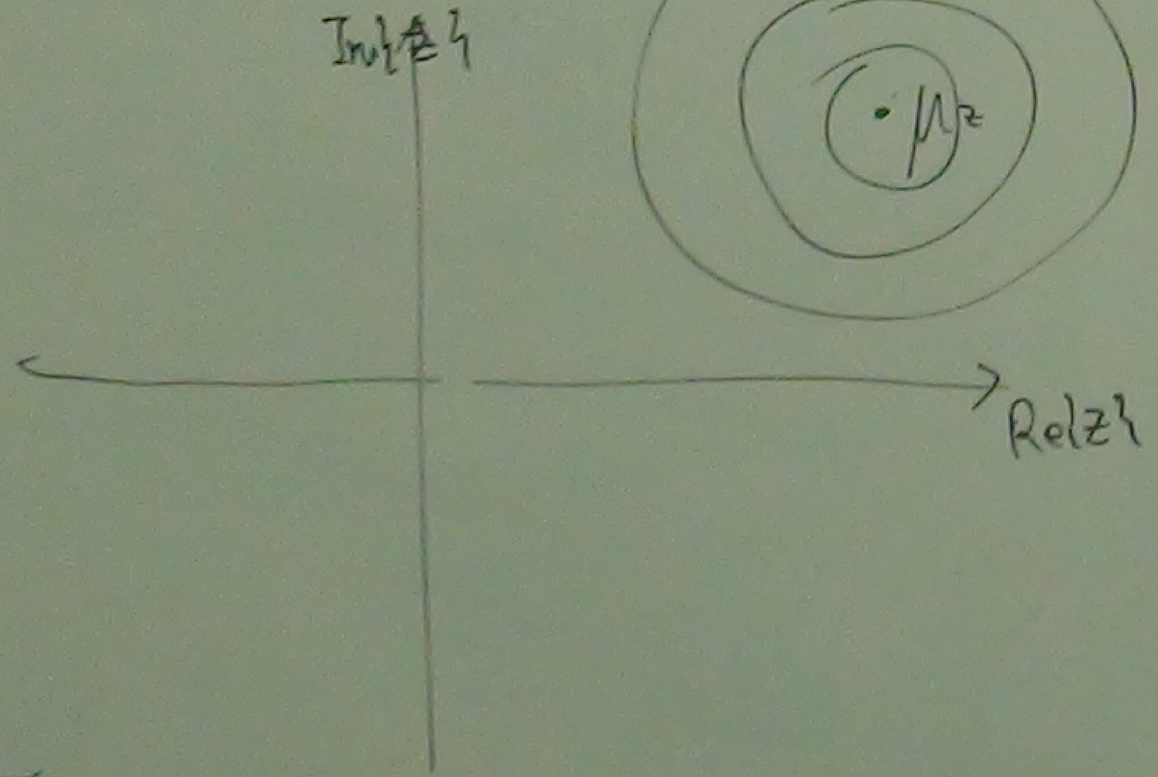
$f_{X,Y}(x,y) \overset{\sigma_{XY}=0}{=} f_X(x)f_Y(y)$

$= \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)\frac{1}{\sqrt{2\pi\sigma_Y^2}}\exp\left(-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right)$

$\sigma_X^2 = \sigma_Y^2$

$= \frac{1}{\pi\sigma_Z^2}\exp\left(-\frac{|z-\mu_Z|^2}{\sigma_Z^2}\right)$

$z = x + jy$



☐ "If $Z \sim \mathcal{CN}(\mu_Z, \sigma_Z^2)$, then $f_Z(z)$ is circularly symmetric w.r.t. μ_Z."

☐ Def. A complex r.v. Z is circular if

$W = e^{j\theta}(Z-\mu_Z) + \mu_Z$ is invariant to the choice of $\theta \in [0, 2\pi)$. $f_Z(z) = f_W(z), \forall z \in \mathbb{C}$.

☐ "For a proper-complex r.v. Z, propriety $\Leftrightarrow$ circularity"

☐ "For a general complex r.v. Z, propriety $\Leftarrow$ circularity"

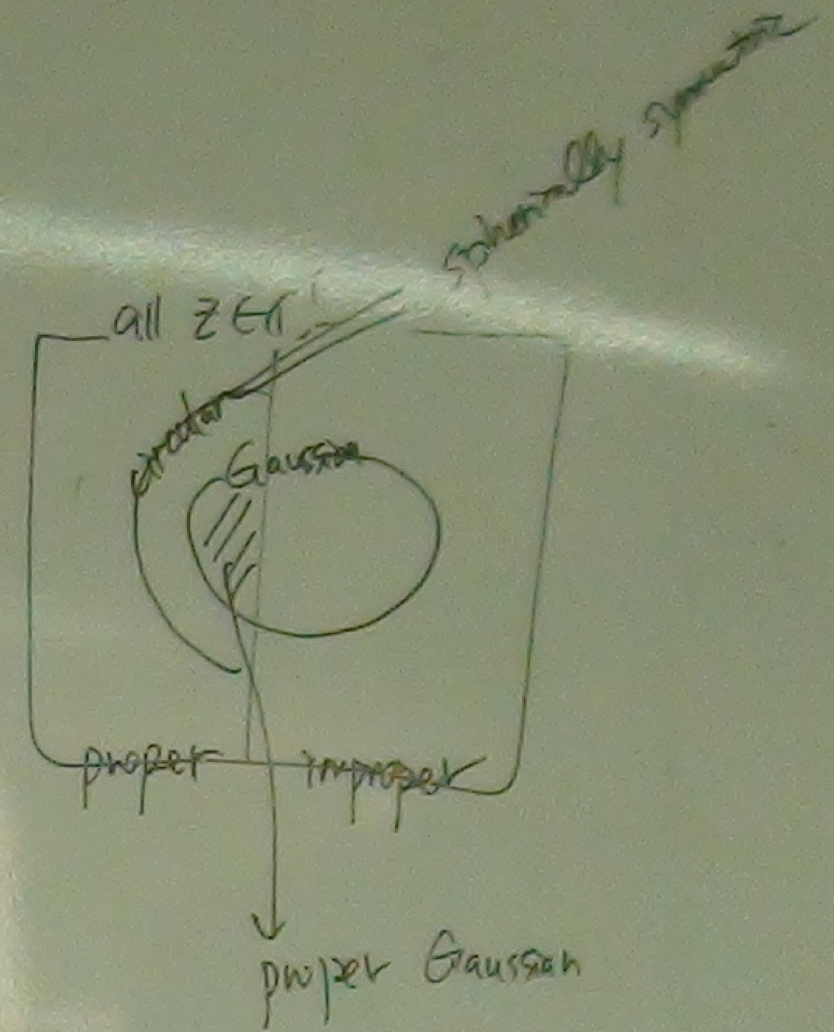
$E\{(Z-\mu_Z)^2\}$

$= E\left[\{e^{j\theta}(Z-\mu_Z)\}^2\right]$ $\because$ circular

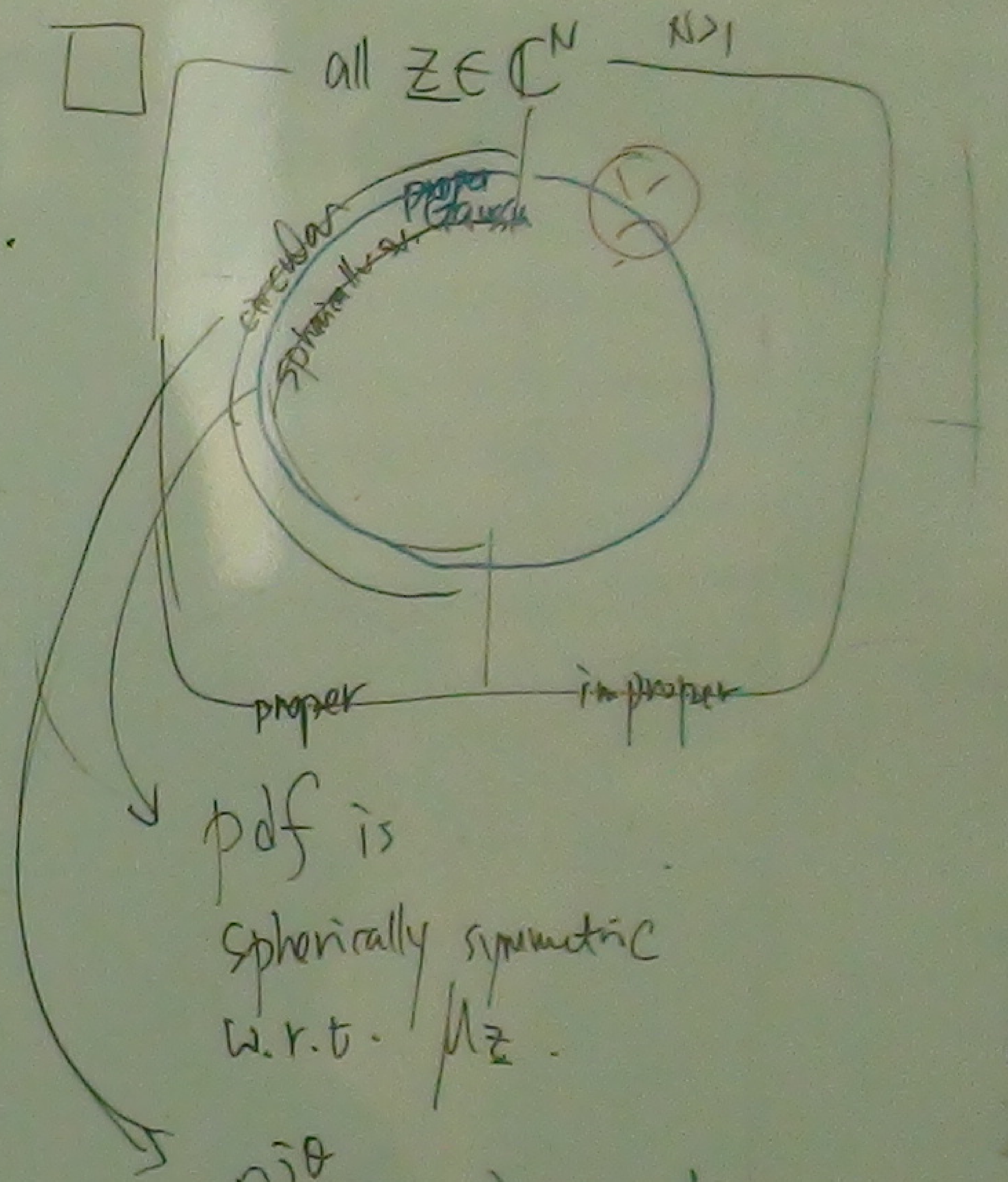
$= e^{j2\theta}E\{(Z-\mu_Z)^2\}, \ \forall\theta$

$\Rightarrow \quad E\{(Z-\mu_Z)^2\} = 0.$

☐



☐



pdf is spherically symmetric w.r.t. μ_Z.

$e^{j\theta}(Z-\mu_Z)+\mu_Z$ has invariant pdf.

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☐ "$\underline{Z} \in \mathbb{C}^N$ is proper." $E\{\underline{Z}\} = \mu_{\underline{Z}}$.

$\Leftrightarrow E\{(\underline{Z}-\mu_{\underline{Z}})(\underline{Z}-\mu_{\underline{Z}})^T\} = 0_{N\times N}$ $\quad \underline{Z} = \underline{X} + j\underline{Y}$. $\mu_{\underline{X}}, \mu_{\underline{Y}}$

$\Leftrightarrow E[\{(\underline{X}-\mu_{\underline{X}}) + j(\underline{Y}-\mu_{\underline{Y}})\}\{(\underline{X}-\mu_{\underline{X}}) + j(\underline{Y}-\mu_{\underline{Y}})\}^T] = 0$ $\quad (\underline{X}\underline{Y}^T)^T = \underline{Y}\underline{X}^T$

$\Rightarrow Cov(\underline{X}) = Cov(\underline{Y})$, $\quad Cov(\underline{X},\underline{Y}) = -Cov(\underline{Y},\underline{X}) = -\{Cov(\underline{X},\underline{Y})\}^T$. skew/odd symmetric.

$C_{\underline{X}\underline{X}} = C_{\underline{Y}\underline{Y}}$ $\quad C_{\underline{X}\underline{Y}} = -C_{\underline{Y}\underline{X}} = -C_{\underline{X}\underline{Y}}^T$

☐ "A complex Gaussian $\underline{Z} \in \mathbb{C}^N$ is proper." $\quad \underline{Z} \sim CN(\mu_{\underline{Z}}, C_{\underline{Z}\underline{Z}})$

$$\Leftrightarrow \underbrace{\begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix}}_{\in \mathbb{R}^{2N}} \sim N\left(\begin{bmatrix} \mu_{\underline{X}} \\ \mu_{\underline{Y}} \end{bmatrix}, \begin{bmatrix} C_{\underline{X}\underline{X}} & C_{\underline{X}\underline{Y}} \\ C_{\underline{Y}\underline{X}} & C_{\underline{Y}\underline{Y}} \end{bmatrix}\right)$$

where $C_{\underline{X}\underline{X}} = C_{\underline{Y}\underline{Y}}$, $C_{\underline{X}\underline{Y}} = -C_{\underline{Y}\underline{X}}$

$$\begin{bmatrix} C_{\underline{X}\underline{X}} & -C_{\underline{Y}\underline{X}} \\ C_{\underline{Y}\underline{X}} & C_{\underline{X}\underline{X}} \end{bmatrix}$$

$$\Leftrightarrow f_{\underline{X},\underline{Y}}(\underline{x},\underline{y}) = \frac{1}{\sqrt{(2\pi)^{2N}\det C}} \exp\left(-\frac{\begin{bmatrix} \underline{x}-\mu_{\underline{X}} \\ \underline{y}-\mu_{\underline{Y}} \end{bmatrix}^T C^{-1} [\;\;]}{2}\right) \quad \& \quad C = \begin{bmatrix} C_{XX} & -C_{YX} \\ C_{YX} & C_{XX} \end{bmatrix}$$

$$\Leftrightarrow f_{\underline{Z}}(\underline{z}) = \frac{1}{\pi^N \det(C_{\underline{Z}\underline{Z}})} \exp\left(-(\underline{z}-\mu_{\underline{Z}})^H C_{\underline{Z}\underline{Z}}^{-1} (\underline{z}-\mu_{\underline{Z}})\right), \quad C_{\underline{Z}\underline{Z}} = 2C_{\underline{X}\underline{X}}$$

☐ Def. $\underline{Z}$ is circular if

$$e^{j\theta}(\underline{Z}-\mu_{\underline{Z}}) + \mu_{\underline{Z}} = \begin{bmatrix} e^{j\theta} & & \\ & e^{j\theta} & \\ & & \ddots & \\ & & & e^{j\theta} \end{bmatrix}(\underline{Z}-\mu_{\underline{Z}}) + \mu_{\underline{Z}}$$

does not affect the pdf.

☐ $\underline{Z}$ is spherically symmetric if

all rotation reflection $(\underline{Z}-\underline{\mu}) + \underline{\mu}$

does not affect the PDF.

ex) $\underline{Z} \sim CN(\underline{0}, \underline{I})$

☐ "If $\underline{Z}$ is circular, then $\underline{Z}$ is proper."

$E\{(\underline{Z}-\mu_{\underline{Z}})(\underline{Z}-\mu_{\underline{Z}})^T\} = 0$

$\times e^{j2\theta}$ $\quad e^{j2\theta}$

$$\boxed{\begin{aligned} &E\{e^{j\theta}(\underline{Z}-\mu_{\underline{Z}})\, e^{j\theta}(\underline{Z}-\mu_{\underline{Z}})\} \\ &= e^{j2\theta}\tilde{C}_{\underline{Z}\underline{Z}}, \quad \forall\theta \end{aligned}}$$

$\theta = 0, \frac{\pi}{2}$ $\quad \tilde{C}_{\underline{Z}\underline{Z}} = 0$.

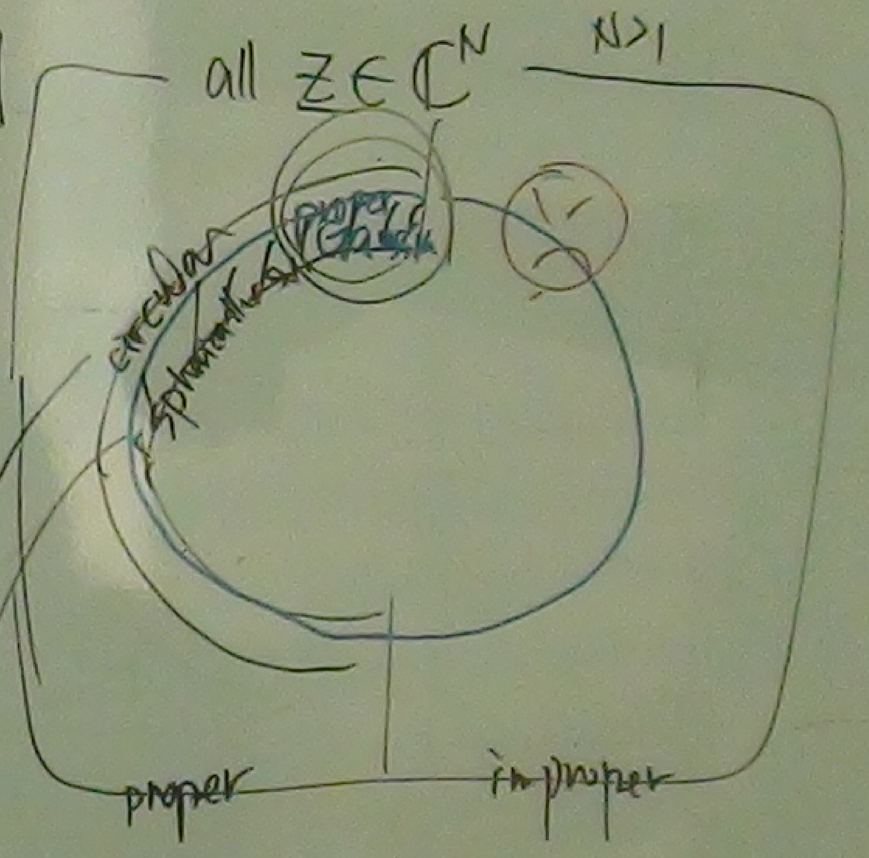
☐ If $\underline{Z}$ is Gaussian, then

propriety $\Rightarrow$ circularity

$E\{\cancel{e^{j\theta}}(\underline{Z}-\mu_{\underline{Z}})\,(e^{j\theta}(\underline{Z}-\mu_{\underline{Z}}))^H\}$

$= C_{\underline{Z}\underline{Z}}, \forall\theta$

☐



pdf is spherically symmetric w.r.t. $\mu_{\underline{Z}}$.

$e^{j\theta}(\underline{Z}-\mu_{\underline{Z}}) + \mu_{\underline{Z}}$ has invariant pdf.

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□ $\underline{z} \in \mathbb{C}^N$ $\qquad \underbrace{\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix}}_{\in \mathbb{R}^{2N}} \cdot \underbrace{\begin{bmatrix} \underline{z} \\ \underline{z}^* \end{bmatrix}}_{\in \mathbb{C}^{2N}}$

□ Linear Transform of $\underline{z}$

$$\underset{\in \mathbb{C}^N}{\underline{z}} \rightarrow \boxed{\mathcal{L}} \rightarrow \underline{w} \in \mathbb{C}^M$$

There exists $H \in \mathbb{C}^{M \times N}$ s.t.

$$\underbrace{\underline{w} = H \underline{z}.}$$

□ Q. This covers every linear transform of $\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix}$?

$$\underset{\in \mathbb{R}^{2N}}{\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix}} \rightarrow \boxed{\mathcal{L}} \rightarrow \underset{\in \mathbb{R}^{2M}}{\begin{bmatrix} \underline{w}_r \\ \underline{w}_i \end{bmatrix}}$$

A. NO!