EECE695M CM#9a

□ Review

1) $X = X_c + jX_s \in \mathbb{C}^1$ complex r.v.

$\begin{bmatrix} X_c \\ X_s \end{bmatrix} \in \mathbb{R}^2$ real composite vector

$\begin{bmatrix} X \\ X^* \end{bmatrix} \in \mathbb{C}^2$ complex augmented vec. (← redundant)

2) Circular

$Y = e^{j\theta}(X-\mu_x)+\mu_x$

$f_Y(y) = f_X(y),\ \forall y \in \mathbb{C}^1,\ \forall \theta \in [0, 2\pi)$

3) Proper

$\tilde{C}_{xx} \triangleq E\{(X-\mu_x)^2\} = 0$

$(\Leftrightarrow \sigma_{x_c}^2 = \sigma_{x_s}^2,\ \sigma_{x_s x_c} = 0\ (\Leftrightarrow \rho_{x_c x_s} = 0))$

4) Circular ⇒ proper

$\tilde{C}_{xx} = \tilde{C}_{YY} = E\{(e^{j\theta}(X-\mu_x))^2\}$

$= e^{j2\theta}\tilde{C}_{xx},\ \forall\theta$

$= 0$

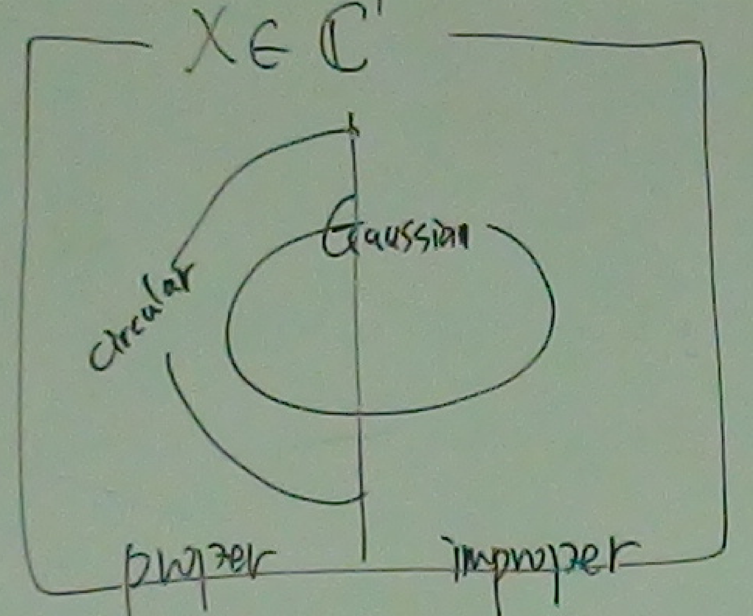
5) Gaussian $X \sim CN(\mu_x, \sigma_x^2)$

"proper ⇔ circular" when Gaussian

$f_X(x) = \frac{1}{\pi\sigma_x^2}\exp\left(-\frac{|x-\mu_x|^2}{\sigma_x^2}\right)$

$= f_Y(x),\ \forall\theta$

∴



1)' $\underline{X} = \underline{X}_c + j\underline{X}_s \in \mathbb{C}^N$

$\begin{bmatrix} \underline{X}_c \\ \underline{X}_s \end{bmatrix} \in \mathbb{R}^{2N}$

$\begin{bmatrix} \underline{X} \\ \underline{X}^* \end{bmatrix} \in \mathbb{C}^{2N}$

2)' Circular

$\underline{Y} = e^{j\theta}(\underline{X}-\underline{\mu}_x)+\underline{\mu}_x$

$f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}(\underline{y}),\ \forall \underline{y} \in \mathbb{C}^N,\ \forall\theta$

3)' Proper

$\tilde{C}_{\underline{XX}} = E\{(\underline{X}-\underline{\mu}_x)(\underline{X}-\underline{\mu}_x)^T\} = 0_{N\times N}$

$\Leftrightarrow C_{\underline{X}_c\underline{X}_c} = C_{\underline{X}_s\underline{X}_s}$, symmetry

$C_{\underline{X}_s\underline{X}_c} = -C_{\underline{X}_s\underline{X}_c}^T$ skew symmetry

4)' circular ⇒ proper

$\tilde{C}_{\underline{XX}} = \tilde{C}_{\underline{YY}}$

$= E\{e^{j\theta}(\underline{X}-\underline{\mu}_x)(e^{j\theta}(\underline{X}-\underline{\mu}_x))^T\}$

$= e^{j2\theta}\tilde{C}_{\underline{XX}},\ \forall\theta$

$= 0_{N\times N}$

5)' Gaussian $\underline{X} \sim CN(\underline{\mu}_x, C_{\underline{XX}})$, $C_{\underline{XX}} \in \mathbb{C}^{N\times N}$

$f_{\underline{X}}(\underline{x}) = \frac{1}{\pi^N \det C_{\underline{XX}}}\exp\left(-(\underline{x}-\underline{\mu}_x)^H C_{\underline{XX}}^{-1}(\underline{x}-\underline{\mu}_x)\right)$

$= f_{\underline{Y}}(\underline{x}),\ \forall\theta$.

"proper ⇔ circular" when Gaussian

6)' Spherical

$\underline{V} \triangleq \begin{bmatrix} \underline{X}_c \\ \underline{X}_s \end{bmatrix}$

$\underline{W} = \underline{U}(\underline{V}-\underline{\mu}_w)+\underline{\mu}_w$

orthogonal Mx.

$f_{\underline{V}}(\underline{v}) = f_{\underline{W}}(\underline{v}),\ \forall \underline{v} \in \mathbb{R}^{2N}$

∀ orthogonal U

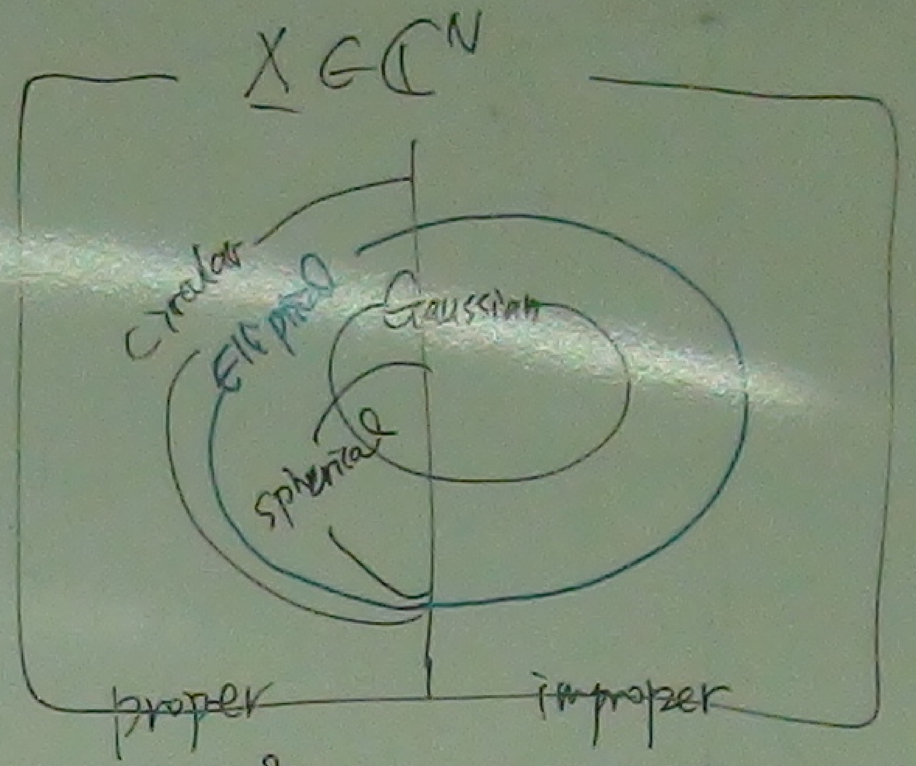
⇒ circular ⇒ proper

$\underline{Y} = U(\underline{X}-\underline{\mu}_x)+\underline{\mu}_x$,

unitary

$f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}(\underline{y}),\ \forall U$ unitary

$U = e^{j\theta}I$



7)' Elliptical, proper &

def $\begin{cases} \underline{X} \sim \mathbb{C}^N \\ f_{\underline{X}}(\underline{x}) = g\left((\underline{x}-\underline{\mu}_x)^H C_{\underline{XX}}^{-1}(\underline{x}-\underline{\mu}_x)\right) \end{cases}$

if $C_{\underline{XX}} \sim I$ then spherically distributed proper

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□ Q. $\boxed{\text{linear processing of } \underline{X} \in \mathbb{C}^N}$

$\overset{?}{\equiv}$ linear processing of $\begin{bmatrix} \underline{X}_c \\ \underline{X}_s \end{bmatrix} \in \mathbb{R}^{2N}$

$\underline{x} \in \mathbb{C}^N \longrightarrow \boxed{\mathcal{M}} \longrightarrow \underline{y} \in \mathbb{C}^M$

$\underline{y} = \mathcal{M}(\underline{x})$

is linear iff there exists $H \in \mathbb{C}^{M \times N}$ s.t.

$\boxed{\underline{y} = H\underline{x}.}$

A. $\underline{h}_n \triangleq \mathcal{M}(\underline{e}_n), \quad n = 1, 2, \cdots, N$

$\in \mathbb{C}^M$

$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \sum_{n=1}^{N} x_n \underline{e}_n$

$$\begin{aligned} \underline{y} = \mathcal{M}(\underline{x}) &= \mathcal{M}\left(\sum_{n=1}^{N} x_n \underline{e}_n\right) \\ &= \sum_{n=1}^{N} x_n \mathcal{M}(\underline{e}_n) \\ &= \sum_{n=1}^{N} x_n \underline{h}_n = [\underline{h}_1 \cdots \underline{h}_N]\underline{x} \end{aligned}$$

$H \in \mathbb{C}^{M \times N}$

A (cont.) $\underline{y} = \underline{y}_c + j\underline{y}_s$

$\begin{bmatrix} \underline{y}_c \\ \underline{y}_s \end{bmatrix} \in \mathbb{R}^{2M}$

$\begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \longrightarrow \boxed{\mathcal{M}} \longrightarrow \begin{bmatrix} \underline{y}_c \\ \underline{y}_s \end{bmatrix}$

$$\begin{bmatrix} \underline{y}_c \\ \underline{y}_s \end{bmatrix} = \begin{bmatrix} H_{cc} & H_{cs} \\ H_{sc} & H_{ss} \end{bmatrix} \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}$$

$H_{\cdot\cdot} \in \mathbb{R}^{M \times N}$

$$\underline{y} = H_{cc}\underline{x}_c + H_{cs}\underline{x}_s + j\left(H_{sc}\underline{x}_c + H_{ss}\underline{x}_s\right) = \left(H_{cc} + jH_{sc}\right)\underline{x}_c + j\left(H_{ss} - jH_{cs}\right)\underline{x}_s$$

not in general

$$\begin{aligned} &\overset{?}{\neq} H\underline{x} \\ &= H(\underline{x}_c + j\underline{x}_s) \\ &= H\underline{x}_c + jH\underline{x}_s \end{aligned}$$

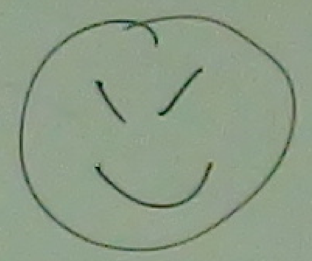
Q.

□ When equivalent?

A. $H_{cc} = H_{ss}$

$H_{sc} = -H_{cs}$

$$\begin{bmatrix} H_{cc} & -H_{sc} \\ H_{sc} & H_{cc} \end{bmatrix}$$

□ 

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

real bandpass signals

$x(t) = \mathrm{Re}\{x_e(t)e^{j2\pi f_c t}\}$

$y(t) = \mathrm{Re}\{y_e(t) \quad " \quad \}$

$h(t) = \mathrm{Re}\{h_e(t) \quad " \quad \}$

$$\begin{aligned} y_e(t) &= \boxed{\tfrac{1}{2}h_e(t)} * x_e(t) \\ &= \tilde{h}_e(t) * x_e(t) \end{aligned}$$

$$\begin{aligned} y_c(t) &= \left(\tilde{h}_c(t) + j\tilde{h}_s(t)\right) * \left(x_c(t) + jx_s(t)\right) \\ &= \left(\tilde{h}_c(t) + j\tilde{h}_s(t)\right) * x_c(t) \\ &+ \left(\tilde{h}_c(t) + j\tilde{h}_s(t)\right) * jx_s(t) \end{aligned}$$

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☐ Q. $\boxed{\text{linear processing of } \underline{x} \in \mathbb{C}^N}$

$\overset{?}{\equiv}$ $\boxed{\text{linear processing of } \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \in \mathbb{R}^{2N}}$

$\underline{x} \in \mathbb{C}^N \longrightarrow \boxed{\mathcal{M}} \longrightarrow \underline{y} \in \mathbb{C}^M, \quad \underline{y} = \mathcal{M}(\underline{x})$

is linear iff there exists $H \in \mathbb{C}^{M \times N}$ s.t. $\boxed{\underline{y} = H\underline{x}.}$

A. $\underline{h}_n \triangleq \mathcal{M}(\underline{e}_n), \; n = 1, 2, \cdots, N$

$\in \mathbb{C}^M$

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \sum_{n=1}^{N} x_n \underline{e}_n$$

$$\begin{aligned} \underline{y} = \mathcal{M}(\underline{x}) &= \mathcal{M}\Big(\sum_{n=1}^{N} x_n \underline{e}_n\Big) \\ &= \sum_{n=1}^{N} x_n \mathcal{M}(\underline{e}_n) \\ &= \sum_{n=1}^{N} x_n \underline{h}_n = \underbrace{[\underline{h}_1 \cdots \underline{h}_N]}_{H \in \mathbb{C}^{M\times N}} \underline{x} \end{aligned}$$

A. (cont.) $\underline{y} = \underline{y}_c + j\underline{y}_s$

$\begin{bmatrix} \underline{y}_c \\ \underline{y}_s \end{bmatrix} \in \mathbb{R}^{2M}$

$\begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \longrightarrow \boxed{\mathcal{M}} \longrightarrow \begin{bmatrix} \underline{y}_c \\ \underline{y}_s \end{bmatrix}$

$$\boxed{\begin{bmatrix} \underline{y}_c \\ \underline{y}_s \end{bmatrix} = \begin{bmatrix} H_{cc} & H_{cs} \\ H_{sc} & H_{ss} \end{bmatrix} \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}}$$

$H_{\cdot\cdot} \in \mathbb{R}^{M \times N}$

$$\begin{aligned} \underline{y} &= H_{cc}\underline{x}_c + H_{cs}\underline{x}_s + j\big(H_{sc}\underline{x}_c + H_{ss}\underline{x}_s\big) = (H_{cc} + jH_{sc})\underline{x}_c + j(H_{ss} - jH_{cs})\underline{x}_s \\ &\overset{?}{\neq} H\underline{x} \quad \text{not in general} \\ &= H(\underline{x}_c + j\underline{x}_s) \\ &= H\underline{x}_c + jH\underline{x}_s \end{aligned}$$

Q. ☐ When equivalent?

A. $H_{cc} = H_{ss}$

$H_{sc} = -H_{cs}$

$$\begin{bmatrix} H_{cc} & -H_{sc} \\ H_{sc} & H_{cc} \end{bmatrix}$$

☐ 

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

real bandpass signals

$x(t) = \mathrm{Re}\{x_\ell(t) e^{j2\pi f_c t}\}$

$y(t) = \mathrm{Re}\{y_\ell(t) \quad " \quad \}$

$h(t) = \mathrm{Re}\{h_\ell(t) \quad " \quad \}$

$$\begin{aligned} y_\ell(t) &= \boxed{\tfrac{1}{2} h_\ell(t)} * x_\ell(t) \\ &= \tilde{h}_\ell(t) * x_\ell(t) \end{aligned}$$

$$\begin{aligned} y_c(t) &= \big(\tilde{h}_c(t) + j\tilde{h}_s(t)\big) * \big(x_c(t) + jx_s(t)\big) \\ &= \big(\tilde{h}_c(t) + j\tilde{h}_s(t)\big) * x_c(t) \\ &\quad + \big(\tilde{h}_c(t) + j\tilde{h}_s(t)\big) * jx_s(t) \end{aligned}$$

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□ linear processing of $\begin{bmatrix} X_c \\ X_s \end{bmatrix} \in \mathbb{R}^{2N}$

$\equiv$ linear processing of $\begin{bmatrix} X \\ X^* \end{bmatrix} \in \mathbb{C}^{2N}$

$$\begin{bmatrix} X \\ X^* \end{bmatrix} = \begin{bmatrix} I & jI \\ I & -jI \end{bmatrix} \begin{bmatrix} X_c \\ X_s \end{bmatrix} = T \begin{bmatrix} X_c \\ X_s \end{bmatrix}$$

$$\begin{bmatrix} X_c \\ X_s \end{bmatrix} = T^{-1} \begin{bmatrix} X \\ X^* \end{bmatrix} = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2j}I & -\frac{I}{2j} \end{bmatrix} \begin{bmatrix} X \\ X^* \end{bmatrix}$$

Q. Given

$$\begin{bmatrix} Y_c \\ Y_s \end{bmatrix} = \begin{bmatrix} H_{cc} & H_{cs} \\ H_{sc} & H_{ss} \end{bmatrix} \begin{bmatrix} X_c \\ X_s \end{bmatrix}$$

find $\begin{bmatrix} Y \\ Y^* \end{bmatrix} = \begin{bmatrix} \; ? \; \end{bmatrix} \begin{bmatrix} X \\ X^* \end{bmatrix}$

A. $T \begin{bmatrix} Y_c \\ Y_s \end{bmatrix} = \left(T \begin{bmatrix} \; \end{bmatrix} T^{-1} \right) T \begin{bmatrix} X_c \\ X_s \end{bmatrix}$

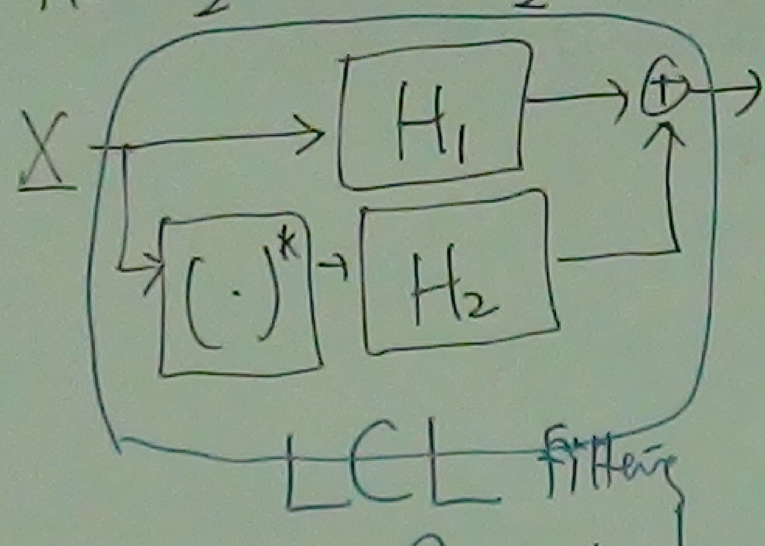
$$\begin{bmatrix} Y \\ Y^* \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_2^* & H_1^* \end{bmatrix} \begin{bmatrix} X \\ X^* \end{bmatrix}$$

...

$H_1 = \frac{1}{2}(H_{cc}+H_{ss}) + \frac{j}{2}(H_{sc}-H_{cs}) \in \mathbb{C}^{N \times N}$

$H_2 = \frac{1}{2}(H_{cc}-H_{ss}) + \frac{j}{2}(H_{sc}+H_{cs}) \in \mathbb{C}^{N \times N}$

free variables



LCL filtering

Linear Conjugate-Linear filtering

Widely Linear (WL)

□ Re{linear processing of X}

Strictly Linear

/ex/ $y = Hb + n$, $n \in \mathcal{CN}(0, \sigma^2 I)$, $b_i \in \{1, -1\}$

$\hat{b}_{ML} = \mathrm{Re}\{H^H y\}$

Q. Re{LCL processing of X}

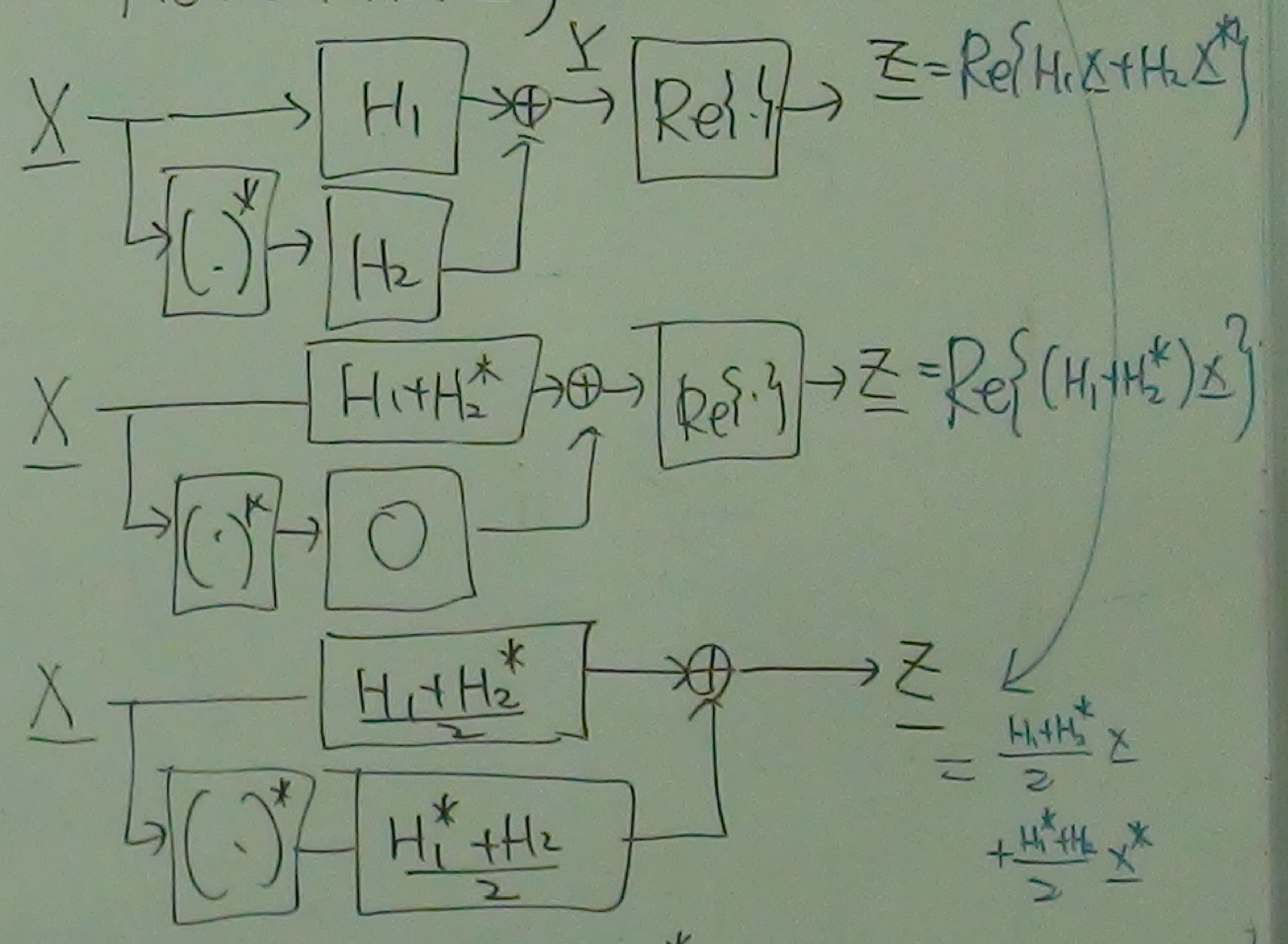
$\overset{?}{=}$ Re{linear processing of X}

A. $Z \triangleq \mathrm{Re}\{H_1 X + H_2 X^*\}$, $Y = [H_1 \; H_2] \begin{bmatrix} X \\ X^* \end{bmatrix}$

$$= \frac{H_1 X + H_2 X^* + H_1^* X^* + H_2^* X}{2}$$

$$= \frac{(H_1 + H_2^*)X + (H_1^* + H_2)X^*}{2}$$

$$= \mathrm{Re}\{(H_1 + H_2^*)X\}$$



$$Y = \begin{bmatrix} \frac{(H_1+H_2^*)}{2} & \frac{(H_1+H_2^*)^*}{2} \end{bmatrix} \begin{bmatrix} X \\ X^* \end{bmatrix}$$