EECE695M CM#9c

□ linear processing of $\begin{bmatrix} X_c \\ X_s \end{bmatrix} \in \mathbb{R}^{2N}$

$\equiv$ linear processing of $\begin{bmatrix} X \\ X^* \end{bmatrix} \in \mathbb{C}^{2N}$

$\equiv$ WL/LCL " "

$$\begin{bmatrix} X \\ X^* \end{bmatrix} = \begin{bmatrix} I & jI \\ I & -jI \end{bmatrix}\begin{bmatrix} X_c \\ X_s \end{bmatrix} = T\begin{bmatrix} X_c \\ X_s \end{bmatrix}$$

T

$$\begin{bmatrix} X_c \\ X_s \end{bmatrix} = T^{-1}\begin{bmatrix} X \\ X^* \end{bmatrix} = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2j}I & -\frac{I}{2j} \end{bmatrix}\begin{bmatrix} X \\ X^* \end{bmatrix}$$

Schreier Scharf

$$T^H T = T T^H = 2 I_{2N}$$

Q. Given $\begin{bmatrix} Y_c \\ Y_s \end{bmatrix} = \begin{bmatrix} H_{cc} & H_{cs} \\ H_{sc} & H_{ss} \end{bmatrix}\begin{bmatrix} X_c \\ X_s \end{bmatrix}$

find $\begin{bmatrix} Y \\ Y^* \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix}\begin{bmatrix} X \\ X^* \end{bmatrix}$

A. $T\begin{bmatrix} Y_c \\ Y_s \end{bmatrix} = \left(T\begin{bmatrix} & \end{bmatrix}T^{-1}\right)T\begin{bmatrix} X_c \\ X_s \end{bmatrix}$

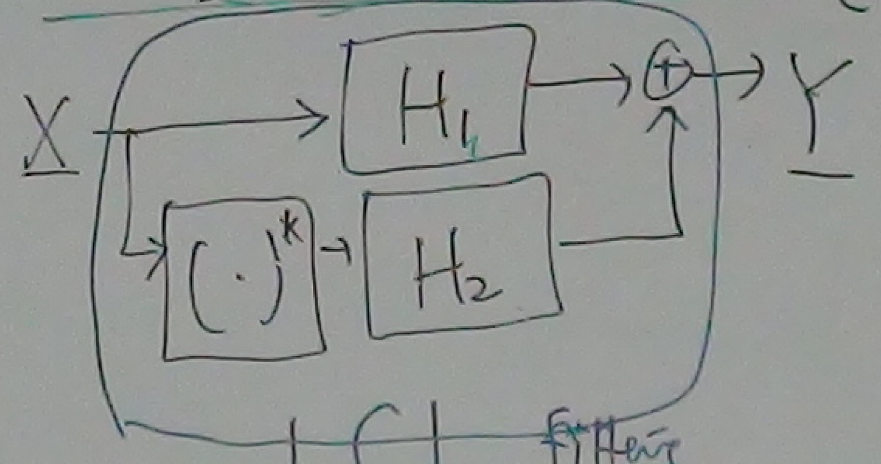
$$\begin{bmatrix} Y \\ Y^* \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_2^* & H_1^* \end{bmatrix}\begin{bmatrix} X \\ X^* \end{bmatrix}$$

...

$$H_1 = \frac{1}{2}(H_{cc}+H_{ss}) + \frac{j}{2}(H_{sc}-H_{cs}) \in \mathbb{C}^{N\times N}$$

$$H_2 = \frac{1}{2}(H_{cc}-H_{ss}) + \frac{j}{2}(H_{sc}+H_{cs}) \in \mathbb{C}^{N\times N}$$

a+b
a−b
free variables



LCL filtering

Linear Conjugate-Linear filtering

Widely Linear (WL)

□ Re{linear processing of X}

Strictly Linear

/ex/ $y = Hb + n$, $n \in \mathcal{CN}(0, \sigma^2 I)$

$b_i \in \{1, -1\}$

$$\hat{b}_{ML} = \mathrm{Re}\{H^H y\}$$

Q. Re{LCL processing of X}

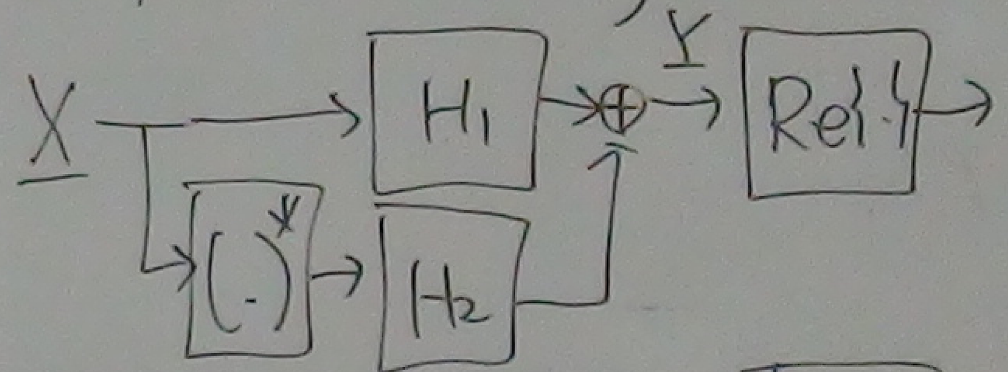
$\overset{?}{=}$ Re{linear " " X}

A. $Z \triangleq \mathrm{Re}\{H_1 X + H_2 X^*\}$, $Y = [H_1 \ H_2]\begin{bmatrix} X \\ X^* \end{bmatrix}$

$$= \frac{H_1X + H_2X^* + H_1^*X^* + H_2^*X}{2}$$

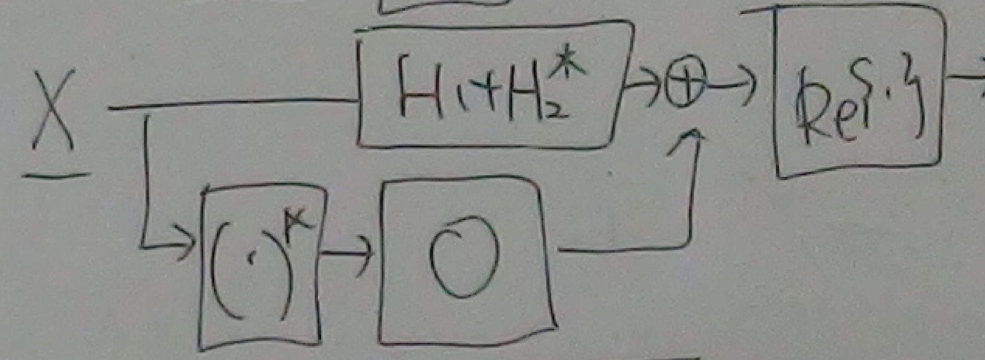
$$= \frac{(H_1+H_2^*)X + (H_1^*+H_2)X^*}{2}$$

$$= \mathrm{Re}\{(H_1+H_2^*)X\}$$



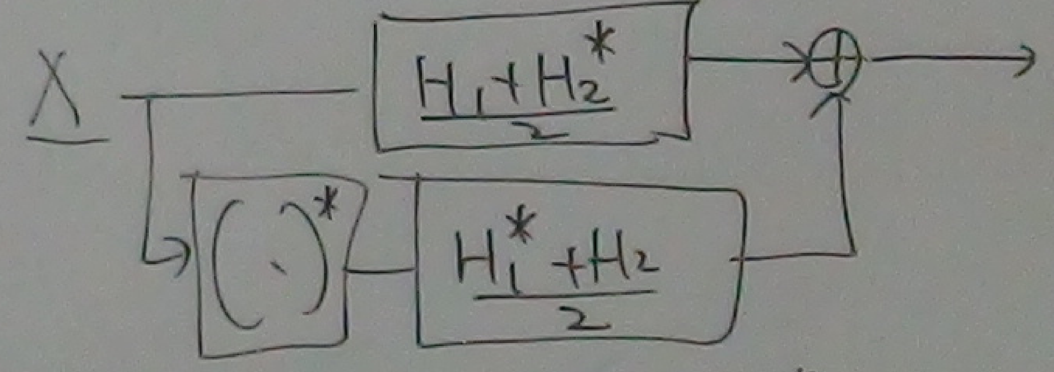
$Z = \mathrm{Re}\{H_1X + H_2X^*\}$

$\equiv$



$Z = \mathrm{Re}\{(H_1+H_2^*)X\}$

$\equiv$



$$Z = \frac{H_1+H_2^*}{2}X + \frac{H_1^*+H_2}{2}X^*$$

$$Y = \begin{bmatrix} \frac{(H_1+H_2^*)}{2} & \left(\frac{H_1+H_2^*}{2}\right)^* \end{bmatrix}\begin{bmatrix} X \\ X^* \end{bmatrix}$$

$\frac{1}{2}T^H T = \cancel{2}I$

$= \frac{1}{2} T T^H$

$T = \begin{bmatrix} I & jI \\ I & -jI \end{bmatrix}$

$\begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix} = T \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}$

EECE695M CM#10a

□ Inner product

Q. Given $\underline{x} = \underline{x}_c + j\underline{x}_s \in \mathbb{C}^N$, define an inner product!

A. $y = \begin{bmatrix} \underline{h}_c \\ \underline{h}_s \end{bmatrix}^T \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} = \underline{h}_c^T \underline{x}_c + \underline{h}_s^T \underline{x}_s \in \mathbb{R}^1$

$= \begin{bmatrix} \underline{h}_c \\ \underline{h}_s \end{bmatrix}^T \left(I \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \right) = \begin{bmatrix} \underline{h}_c \\ \underline{h}_s \end{bmatrix}^T \left(\frac{1}{2} T^H \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix} \right)$

$= \cdots$

$= \frac{1}{2} \begin{bmatrix} \underline{h} \\ \underline{h}^* \end{bmatrix}^H \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix}$ where $\underline{h} = \underline{h}_c + j\underline{h}_s$

$(ab)^* = a^* b^*$

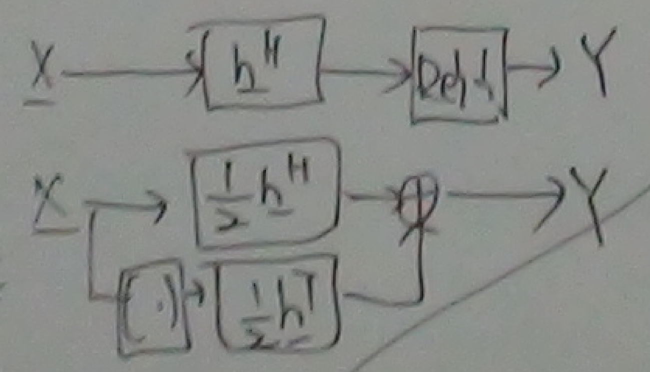
$(\underline{a}^H \underline{b})^* = \underline{a}^T \underline{b}^*$

$= (\underline{a}^H \underline{b})^H$

$= \underline{b}^H \underline{a}$

$= \frac{1}{2} \begin{bmatrix} \underline{h}^H & \underline{h}^T \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix} = \frac{1}{2}(\underline{h}^H \underline{x} + \underline{h}^T \underline{x}^*)$

$= \frac{1}{2}(\underline{h}^H \underline{x} + (\underline{h}^H \underline{x})^*) = Re(\underline{h}^H \underline{x})$



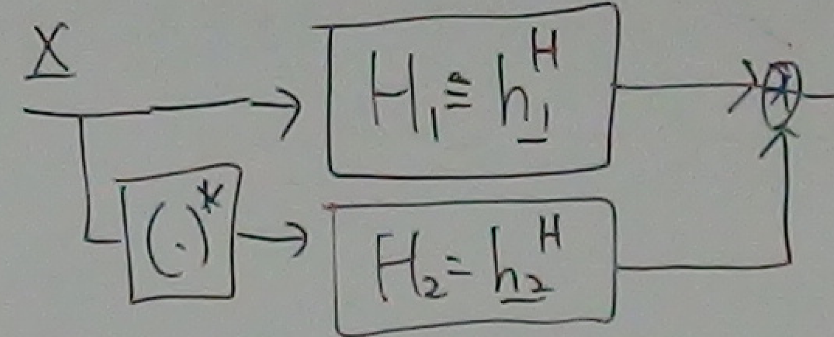
□ Inner Product ?

$\left(\begin{matrix} y \in \mathbb{C}^1 \\ \underline{x} \in \mathbb{C}^N \end{matrix} \right.$

Q. $\begin{bmatrix} y_c \\ y_s \end{bmatrix} = \begin{bmatrix} \underline{h}_{cc}^T & \underline{h}_{cs}^T \\ \underline{h}_{sc}^T & \underline{h}_{ss}^T \end{bmatrix} \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}$

is a special case of

$\begin{bmatrix} \underline{y}_c \\ \underline{y}_s \end{bmatrix} = \begin{bmatrix} H_{cc} & H_{cs} \\ H_{sc} & H_{ss} \end{bmatrix} \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \Leftrightarrow \begin{bmatrix} \underline{y} \\ \underline{y}^* \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_2^* & H_1^* \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix}$



$y = \underline{h}_1^H \underline{x} + \underline{h}_2^H \underline{x}^*$

where $H_1 = \frac{1}{2}(H_{cc} + H_{ss}) + \frac{j}{2}(H_{sc} - H_{cs})$

$H_2 = \frac{1}{2}(H_{cc} - H_{ss}) + \frac{j}{2}(H_{sc} + H_{cs})$

(X)

□ Quadratic Form

Q. Given $\underline{x} = \underline{x}_c + j\underline{x}_s \in \mathbb{C}^N$ define a quadratic form.

A. $\boxed{y = \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}^T \begin{bmatrix} H_{cc} & H_{cs} \\ H_{sc} & H_{ss} \end{bmatrix} \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}}$

$= \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}^H \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}$

$I = \frac{1}{2} T^H T$

$= \left(I \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \right)^H \begin{bmatrix} H_{cc} & H_{cs} \\ & \end{bmatrix} \left(I \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \right)$

$= \left(\frac{1}{2} T^H \left(T \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \right) \right)^H \cdots$

$= \frac{1}{2} \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix}^H \left(T \begin{bmatrix} H_{cc} & H_{cs} \\ & \end{bmatrix} T^H \right) \frac{1}{2} \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix}$

$= \boxed{\frac{1}{2} \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix}^H \begin{bmatrix} H_1 & H_2 \\ H_2^* & H_1^* \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix}}$

$(\frac{1}{2} \underline{x}^H H_1 \underline{x})^*$

$(\frac{1}{2} \underline{x}^H H_1^* \underline{x}^*)^*$

$+\frac{1}{2} \underline{x}^T H_1^* \underline{x}^*$

$= \frac{1}{2} \underline{x}^H H_1 \underline{x} + \frac{1}{2} \underline{x}^H H_2 \underline{x}^* + \frac{1}{2} \underline{x}^T H_2^* \underline{x}$

$= \frac{1}{2} \underline{x}^H H_1 \underline{x} + \frac{1}{2} \underline{x}^H H_2 \underline{x}^* + (\frac{1}{2} \underline{x}^H H_2 \underline{x}^*)^* + (\frac{1}{2} \underline{x}^H H_1 \underline{x})^*$

$= Re(\underline{x}^H H_1 \underline{x}) + Re(\underline{x}^H H_2 \underline{x}^*)$

$= \boxed{\underline{x}^H H_1 \underline{x} + Re(\underline{x}^H H_2 \underline{x}^*)}$

cf) $\underline{x}^T A \underline{x}$, $\underline{x} \in \mathbb{R}^N$

$= \underline{x}^T \frac{A + A^T}{2} \underline{x}$

$\underline{x}^H H_1 \underline{x}$

$= \underline{x}^H \frac{H_1 + H_1^H}{2} \underline{x}$

$=$ real

"Widely Quadratic Form"

□ Quadratic Form?

$\begin{bmatrix} y_c \\ y_s \end{bmatrix} = \cdots\cdots$

(X)

$\frac{1}{2}T^H T = I$

$= \frac{1}{2} T T^H$

$T = \begin{bmatrix} I & jI \\ I & -jI \end{bmatrix}$

$\begin{bmatrix} x \\ x^* \end{bmatrix} = T \begin{bmatrix} x_c \\ x_s \end{bmatrix}$

EECE695M CM#10b

□ Widely Linear vs. Linear Conjugate-Linear

$\underline{x} \rightarrow \boxed{H_1} \rightarrow \oplus \rightarrow \underline{y}$

$\underline{x} \rightarrow \boxed{(\cdot)^*} \rightarrow \boxed{H_2} \rightarrow \oplus$

↕

Strictly Linear

$\underline{y} = H\underline{x}$

Widely Quadratic vs. $\underline{x}^H H_1 \underline{x} + Re(\underline{x}^H H_2 \underline{x}^*)$

quadratic ?

↕

Strictly quadratic

$\underline{x}^H H \underline{x}$

□ WQ revisited

$\underline{x} = \underline{x}_c + j\underline{x}_s$

$y = \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}^T \begin{bmatrix} H_{cc} & H_{cs} \\ H_{sc} & H_{ss} \end{bmatrix} \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}$ real composite의 quadratic form

real

$= \frac{1}{2} \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix}^H \begin{bmatrix} H_1 & H_2 \\ H_2^* & H_1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix}$ complex augmented의 quadratic form (w/ structure)

$= \underline{x}^H H_1 \underline{x} + Re(\underline{x}^H H_2 \underline{x}^*)$ $\underline{x}$의 widely quadratic form.

— —

complex random vector → CT complex r.p

□ Given a CT complex-valued r.p $X(t)$

$X(t) \rightarrow \boxed{\mathcal{M}} \rightarrow Y(t)$

Q. Linear mapping?

$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$ (sift property)

$x(\tau)\delta(t-\tau)$

τ

t

$\delta(t-\tau) \rightarrow \boxed{\mathcal{L}} \rightarrow h_\tau(t)$

$h(t,\tau)$: response of the system to the impulse applied at time τ

: impulse response from any τ

t

τ

$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \rightarrow \boxed{\mathcal{L}} \rightarrow \int_{-\infty}^{\infty} h(t,\tau)x(\tau)\,d\tau$

$= x(t)$ LTV $= y(t)$

$\therefore y(t) = \int_{-\infty}^{\infty} h(t,\tau)x(\tau)\,d\tau$

$\Rightarrow Y(t) = \int_{-\infty}^{\infty} h(t,\tau)X(\tau)\,d\tau$ Strictly Linear processing filtering of $X(t)$.

□ $X(t) = X_c(t) + jX_s(t)$ $Y(t) = Y_c(t) + jY_s(t)$

$\begin{bmatrix} X_c(t) \\ X_s(t) \end{bmatrix} \rightarrow \boxed{\mathcal{L}} \rightarrow \begin{bmatrix} Y_c(t) \\ Y_s(t) \end{bmatrix}$

$\vdots$

$\begin{bmatrix} Y_c(t) \\ Y_s(t) \end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix} h_{cc}(t,\tau) & h_{cs}(t,\tau) \\ h_{sc}(t,\tau) & h_{ss}(t,\tau) \end{bmatrix} \begin{bmatrix} X_c(\tau) \\ X_s(\tau) \end{bmatrix} d\tau$

$\begin{bmatrix} Y(t) \\ Y^*(t) \end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix} h_1(t,\tau) & h_2(t,\tau) \\ h_2^*(t,\tau) & h_1^*(t,\tau) \end{bmatrix} \begin{bmatrix} X(\tau) \\ X^*(\tau) \end{bmatrix} d\tau$

where $h_1(t,\tau) = \frac{1}{2}(h_{cc}(t,\tau) + h_{ss}(t,\tau)) + \frac{j}{2}(\ -\ )$

$h_2(t,\tau) = \frac{1}{2}(h_{cc}(t,\tau) - h_{ss}(t,\tau)) + \frac{j}{2}(\ +\ )$

$Y(t) = \int_{-\infty}^{\infty} h_1(t,\tau)X(\tau)\,d\tau + \int_{-\infty}^{\infty} h_2(t,\tau)X^*(\tau)\,d\tau$

~~WL-TV~~

$X(t) \rightarrow \boxed{h_1(t,\tau)} \rightarrow \oplus \rightarrow Y(t)$

$X(t) \rightarrow \boxed{(\cdot)^*} \rightarrow \boxed{h_2(t,\tau)} \rightarrow \oplus$

W-LTV

LCL-TV

EECE695M CM#10c

□ SLTI filtering of $X(t)$

$\underbrace{\text{TI}}_{\text{translation invariant}}$

$$h(t,\tau) = h(t-\tau,0) \triangleq h(t-\tau)$$

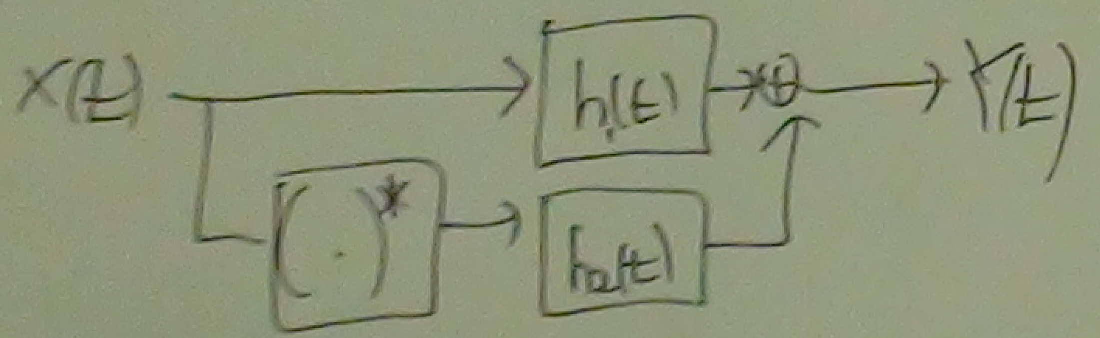
$$Y(t) = \int_{-\infty}^{\infty} h(t-\tau) X(\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau$$

$$= h(t) * X(t) = X(t) * h(t)$$

□ W-LTI filtering of $X(t)$

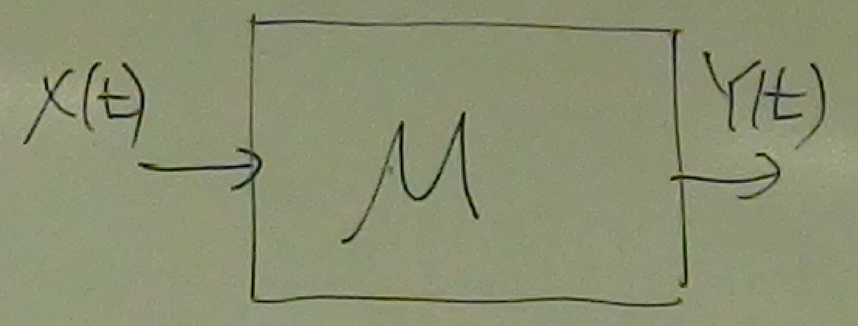
$X^H(t)$

$$Y(t) = h_1(t) * X(t) + h_2(t) * X^*(t)$$



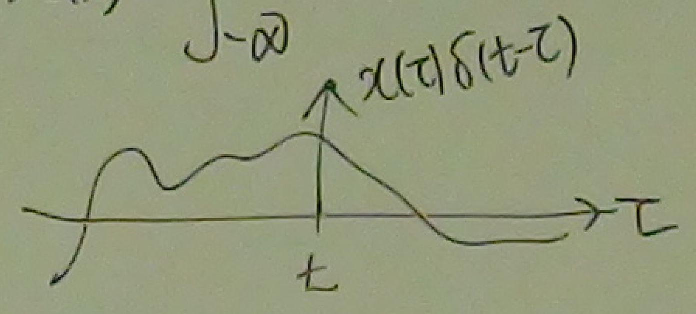
CT complex r.p

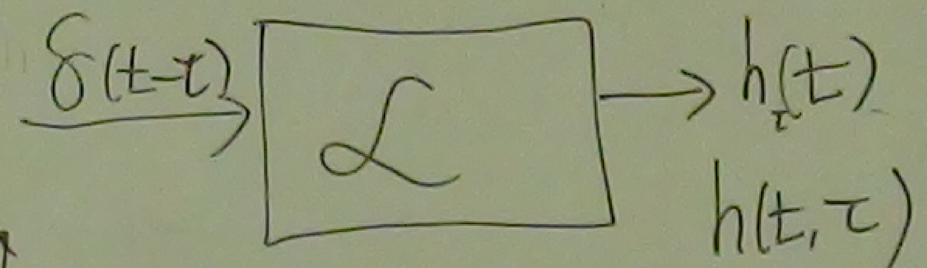
□ Given a CT r.p $X(t)$ (complex-valued)



Q. Linear mapping?

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau \quad \text{(sift property)}$$





$h(t,\tau)$: response of the system to the impulse applied at time τ : impulse response pam by τ

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau \to \boxed{\mathcal{L}} \to \int_{-\infty}^{\infty} h(t,\tau) x(\tau) d\tau$$

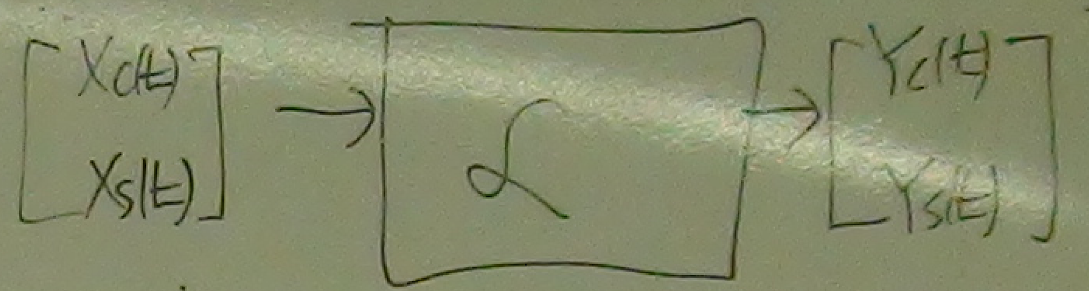
$= x(t)$ (S)LTV $= y(t)$

$$\therefore y(t) = \int_{-\infty}^{\infty} h(t,\tau) x(\tau) d\tau$$

$$Y(t) = \int_{-\infty}^{\infty} h(t,\tau) X(\tau) d\tau$$

Strictly Linear processing filtering of $X(t)$.

□ $X(t) = X_c(t) + j X_s(t)$ $\quad Y(t) = Y_c(t) + j Y_s(t)$



⋮

$$\begin{bmatrix} Y_c(t) \\ Y_s(t) \end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix} h_{cc}(t,\tau) & h_{cs}(t,\tau) \\ h_{sc}(t,\tau) & h_{ss}(t,\tau) \end{bmatrix} \begin{bmatrix} X_c(\tau) \\ X_s(\tau) \end{bmatrix} d\tau$$

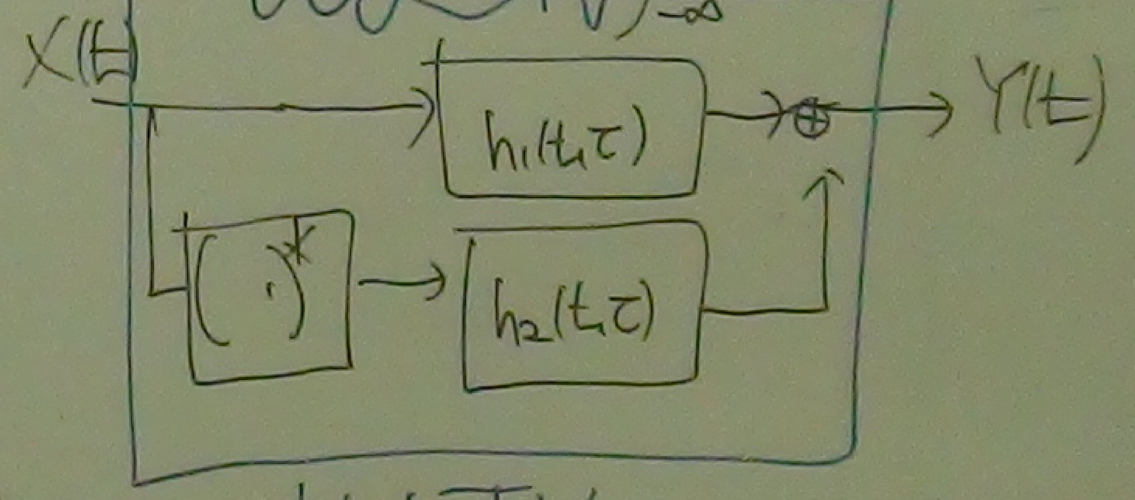
$$\begin{bmatrix} Y(t) \\ Y^*(t) \end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix} h_1(t,\tau) & h_2(t,\tau) \\ h_2^*(t,\tau) & h_1^*(t,\tau) \end{bmatrix} \begin{bmatrix} X(\tau) \\ X^*(\tau) \end{bmatrix} d\tau$$

where $h_1(t,\tau) = \frac{1}{2}(h_{cc}(t,\tau) + h_{ss}(t,\tau)) + \frac{j}{2}(-)$

$h_2(t,\tau) = \frac{1}{2}(h_{cc}(t,\tau) - h_{ss}(t,\tau)) + \frac{j}{2}(+)$

$$Y(t) = \int_{-\infty}^{\infty} h_1(t,\tau) X(\tau) d\tau + \int_{-\infty}^{\infty} h_2(t,\tau) X^*(\tau) d\tau$$

~~WL~~ TV



W-LTV

LCL-TV

EECE695M CM#10d

□ LTI filtering of X(t)

translation invariant

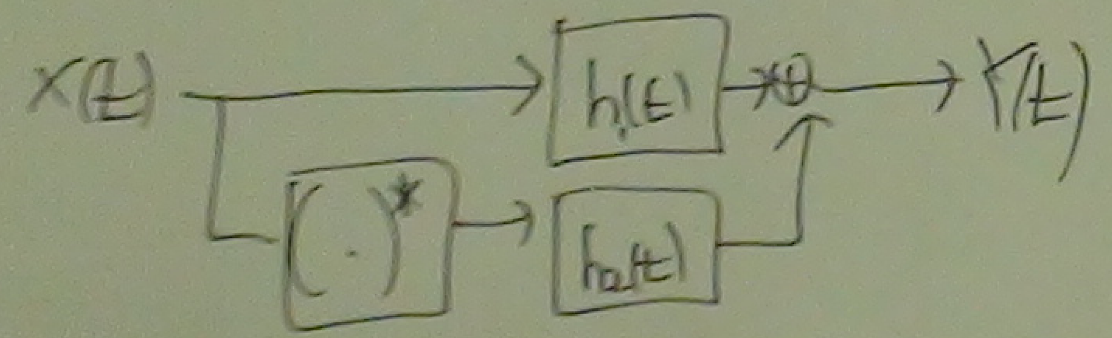
$h(t,\tau) = h(t-\tau, 0) \triangleq h(t-\tau)$

$Y(t) = \int_{-\infty}^{\infty} h(t-\tau)X(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$

$= h(t) * X(t) = X(t) * h(t)$

□ WL-LTI filtering of X(t) $\quad X^H(t)$

$Y(t) = h_1(t) * X(t) + h_2(t) * X^*(t)$



□ LTV filtering of a real bp process

$X(t) = Re\{\underbrace{(X_c(t) + jX_s(t))}_{= X_\ell(t)} e^{j2\pi f_c t}\}$

WQ. $\begin{bmatrix} x_c \\ x_s \end{bmatrix}^T \begin{bmatrix} H_{cc} & H_{cs} \\ H_{sc} & H_{ss} \end{bmatrix} \begin{bmatrix} x_c \\ x_s \end{bmatrix}$

$H_{cc} = H_{cc}^T$

$H_{ss} = H_{ss}^T$

$H_{cs}^T = H_{sc}$

$\begin{bmatrix} x \\ x^* \end{bmatrix}^H \begin{bmatrix} H_1 & H_2 \\ H_2^* & H_1^* \end{bmatrix} \begin{bmatrix} x \\ x^* \end{bmatrix} = x^H H_1 x + Re\{x^H H_2 x^*\}$

real bp r.p. $\quad$ real b.p $h(t,\tau)$ $\quad h(t) = Re\{h_\ell(t) e^{j2\pi f_c t}\}$

$Y(t) = \int_{-\infty}^{\infty} \underbrace{h(t,\tau)}_{\text{real}} X(\tau)d\tau$

$= Re\{Y_\ell(t) e^{j2\pi f_c t}\}$

$= Re\{(Y_c(t) + jY_s(t)) e^{j2\pi f_c t}\}$

$Y_\ell(t) \overset{?}{=}$ in terms of $X_\ell(t)$

$\overset{LTI}{=} \frac{1}{2} h_\ell(t) * X_\ell(t)$ (When h(t) is strictly bandlimited)

LTI (specular) $= h_\ell(t) * X_\ell(t)$

when $h(t) = \sum_{i=1}^{L} \alpha_\ell \delta(t-\tau_\ell)$

$h_\ell(t) = \sum_{i=1}^{L} \alpha_\ell e^{-j2\pi f_c \tau_\ell} \delta(t-\tau_\ell)$

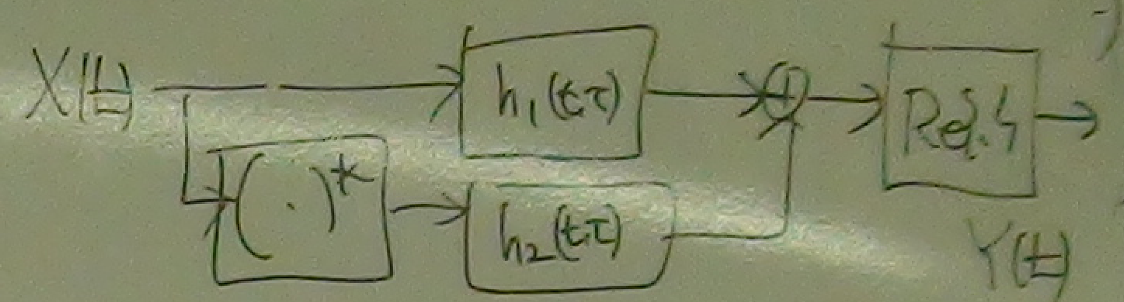
$\overset{LTV}{=} \int_{-\infty}^{\infty} h_\ell(t,\tau) X_\ell(\tau) d\tau$

where $h(t,\tau) = Re\{h_\ell(t,\tau) e^{j2\pi f_c t}\}$

NOT WL^{TV}_{TI} filtering!

BUT SL^{TV}_{TI} ..!

□



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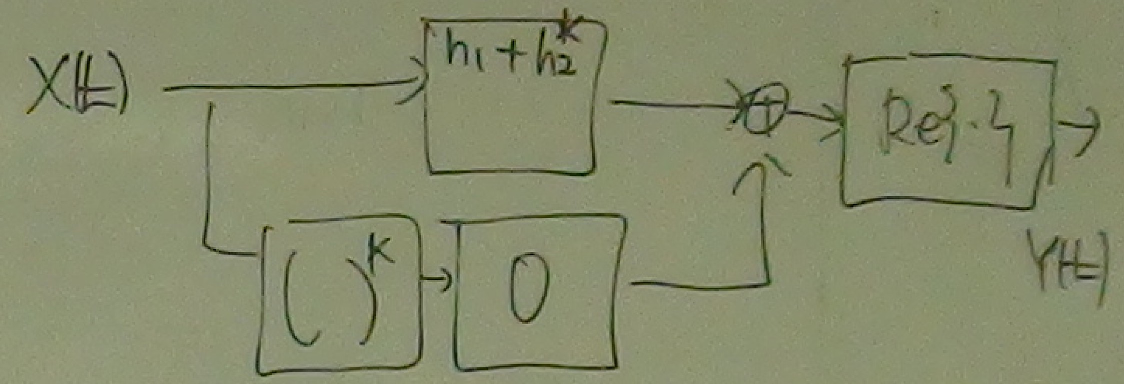

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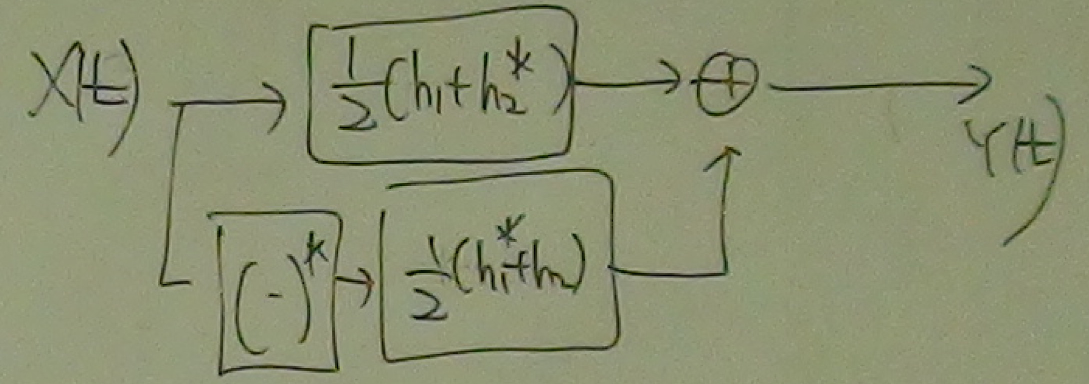

$Y(t) = Re\{\int_{-\infty}^{\infty} \underbrace{(h_1(t,\tau) + h_2^*(t,\tau))}_{\triangleq h(t,\tau)} X(\tau) d\tau\}$

$\overset{LTI}{=} Re\{\int_{-\infty}^{\infty} h(t-\tau) X(\tau) d\tau\}$

$Y = Re\{H^H X\}$

$Y = Re\{h^H x\} = $ WL...

□ Inner Product

$Y = \int_{-\infty}^{\infty} h^*(t) X(t) dt$

complex $\quad$ complex

$Y = Re\{\int_{-\infty}^{\infty} h^*(t) X(t) dt\}$

real

$= \ldots + \ldots$