EECE695M CM#10d

☐ SL$\underbrace{\text{TI}}_{\text{translation invariant}}$ filtering of $X(t)$

$$\text{WQ.}\ \begin{bmatrix}\underline{x}_c\\ \underline{x}_s\end{bmatrix}^T\begin{bmatrix}H_{cc} & H_{cs}\\ H_{sc} & H_{ss}\end{bmatrix}\begin{bmatrix}\underline{x}_c\\ \underline{x}_s\end{bmatrix}$$

$H_{cc}=H_{cc}^T$

$H_{ss}=H_{ss}^T$

$H_{cs}^T=H_{sc}$

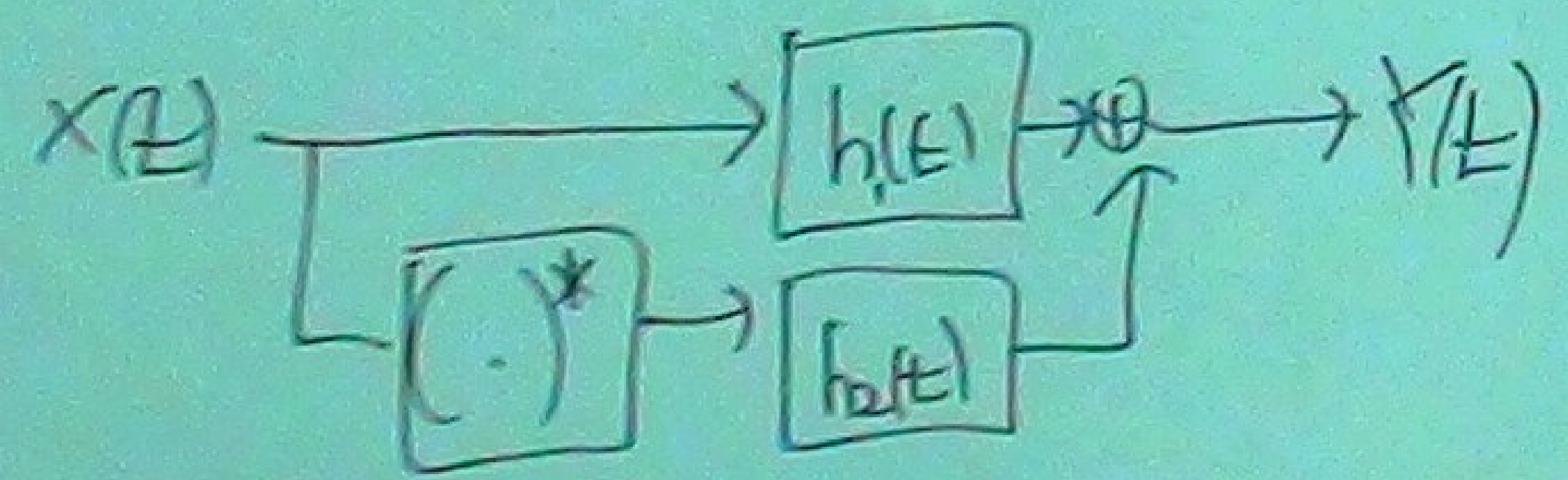
$$\begin{bmatrix}\underline{x}\\ \underline{x}^*\end{bmatrix}^H\begin{bmatrix}H_1 & H_2\\ H_2^* & H_1^*\end{bmatrix}\begin{bmatrix}\underline{x}\\ \underline{x}^*\end{bmatrix} = \underline{x}^H H_1 \underline{x} + \mathrm{Re}\{\underline{x}^H H_2 \underline{x}^*\}$$

$$Y(t)=\int_{-\infty}^{\infty} h(t-\tau)X(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$$

☐ WL-LTI filtering of $X(t)$

$X^H(t)$

$$Y(t)= h_1(t)*X(t) + h_2(t)*X^*(t)$$



☐ LTV filtering of a real bp process

$$X(t) = \mathrm{Re}\{\underbrace{(X_c(t)+jX_s(t))}_{=X_\ell(t)}e^{j2\pi f_c t}\}$$

$\mathbb{R}\nearrow$

real bp r.p

$\overset{\nearrow\ \text{real b.p}}{h(t,\tau)}$ $h(t)=\mathrm{Re}\{h_\ell(t)e^{j2\pi f_c t}\}$

$$Y(t)=\int_{-\infty}^{\infty}\underbrace{h(t,\tau)}_{\text{real}}X(\tau)d\tau$$

$$=\mathrm{Re}\{Y_\ell(t)e^{j2\pi f_c t}\}$$

$$=\mathrm{Re}\{(Y_c(t)+jY_s(t))e^{j2\pi f_c t}\}$$

$Y_\ell(t)\overset{?}{=}$ in terms of $X_\ell(t)$

$\overset{LTI}{=}\frac{1}{2}h_\ell(t)*X_\ell(t)$ (When $h(t)$ is strictly bandlimited)

LTI (specular)

$= h_\ell(t)*X_\ell(t)$ when $h(t) = \sum_{i=1}^{L}\alpha_\ell\delta(t-\tau_\ell)$

$h_\ell(t)=\sum_{i=1}^{L}\alpha_\ell e^{-j2\pi f_c\tau_\ell}\delta(t-\tau_\ell)$

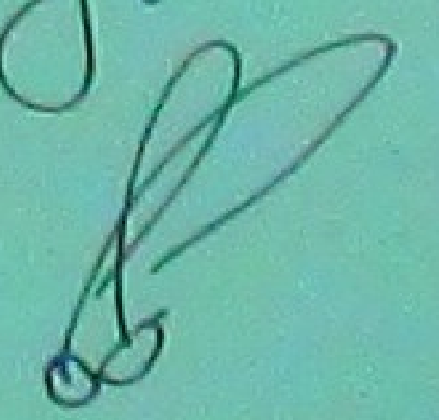
$$\overset{LTV}{=}\int_{-\infty}^{\infty}h_\ell(t,\tau)X_\ell(\tau)d\tau$$

where

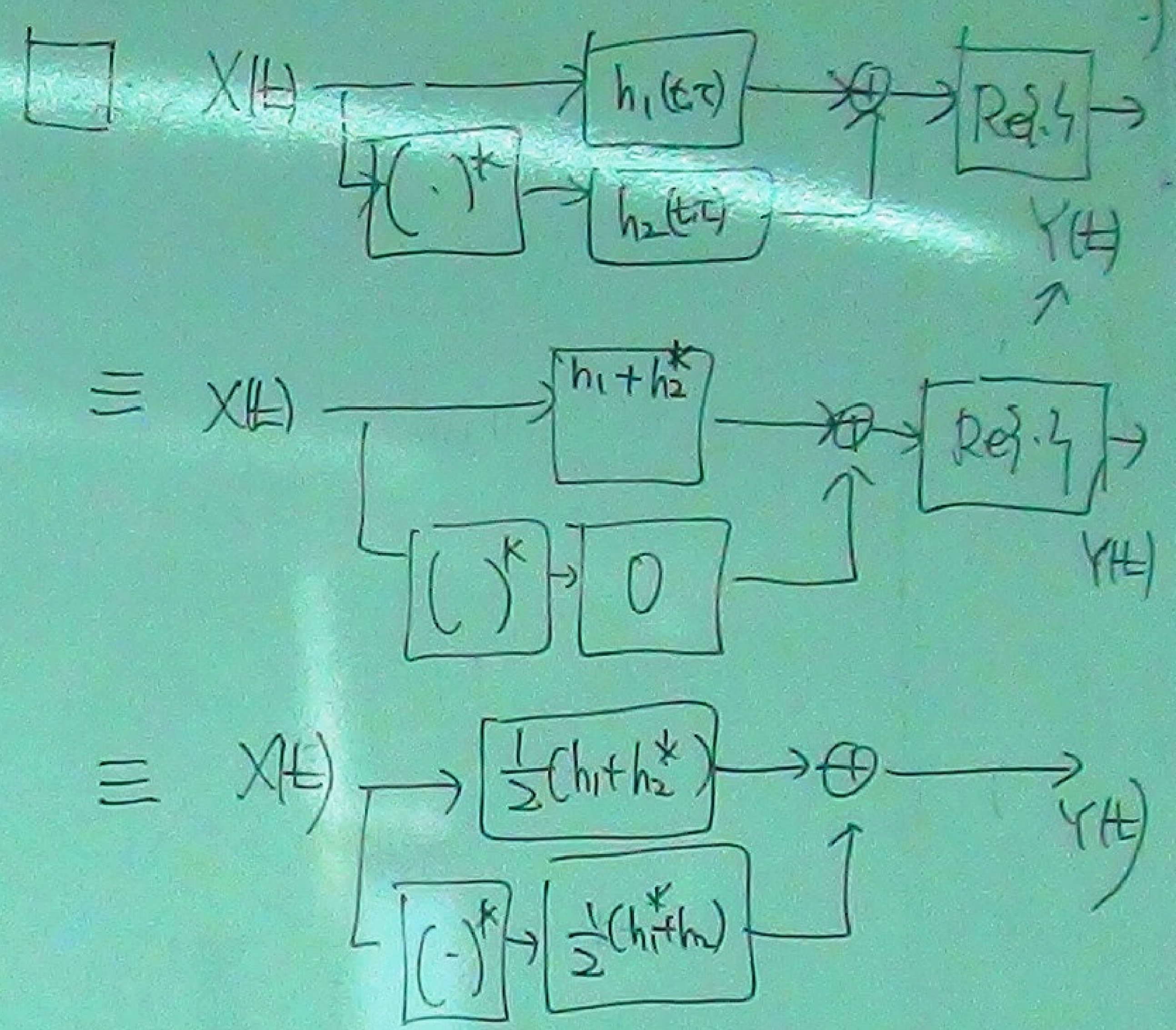
$h(t,\tau)=\mathrm{Re}\{h_\ell(t,\tau)e^{j2\pi f_c t}\}$

NOT WL$^{TV}_{TI}$ filtering!

BUT SL$^{TV}_{TI}$ ..



☐



$$Y(t)=\mathrm{Re}\{\int_{-\infty}^{\infty}(\underbrace{h_1(t,\tau)+h_2^*(t,\tau)}_{\triangleq h(t,\tau)})X(\tau)d\tau\}$$

$$\overset{LTI}{=}\mathrm{Re}\{\int_{-\infty}^{\infty}h(t-\tau)X(\tau)d\tau\}$$

$\underline{Y}=\mathrm{Re}\{H\underline{X}\}$

$Y=\mathrm{Re}\{\underline{h}^H\underline{X}\}$ = ... WL...

☐ Inner Product

$$\underset{\uparrow\ \text{complex}}{Y}=\int_{-\infty}^{\infty}h^*(t)\underset{\uparrow\ \text{complex}}{X(t)}dt$$

$$\underset{\text{real}}{Y}=\mathrm{Re}\{\int_{-\infty}^{\infty}h^*(t)X(t)dt\}$$

= .. + ...

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☐ Inner product

$X(t)=X_c(t)+jX_s(t)$

$$\int_{-\infty}^{\infty}\begin{bmatrix}h_c(t)\\h_s(t)\end{bmatrix}^T\begin{bmatrix}X_c(t)\\X_s(t)\end{bmatrix}dt = \underset{\sim\sim}{Y\in R^1}$$

$$=\int_{-\infty}^{\infty}h_c(t)X_c(t)+h_s(t)X_s(t)\,dt$$

$$\begin{bmatrix}h_c\\h_s\end{bmatrix}^T\begin{bmatrix}X_c\\X_s\end{bmatrix}=Y$$

$$\int_{-\infty}^{\infty}h(t)\,X(t)\,dt$$

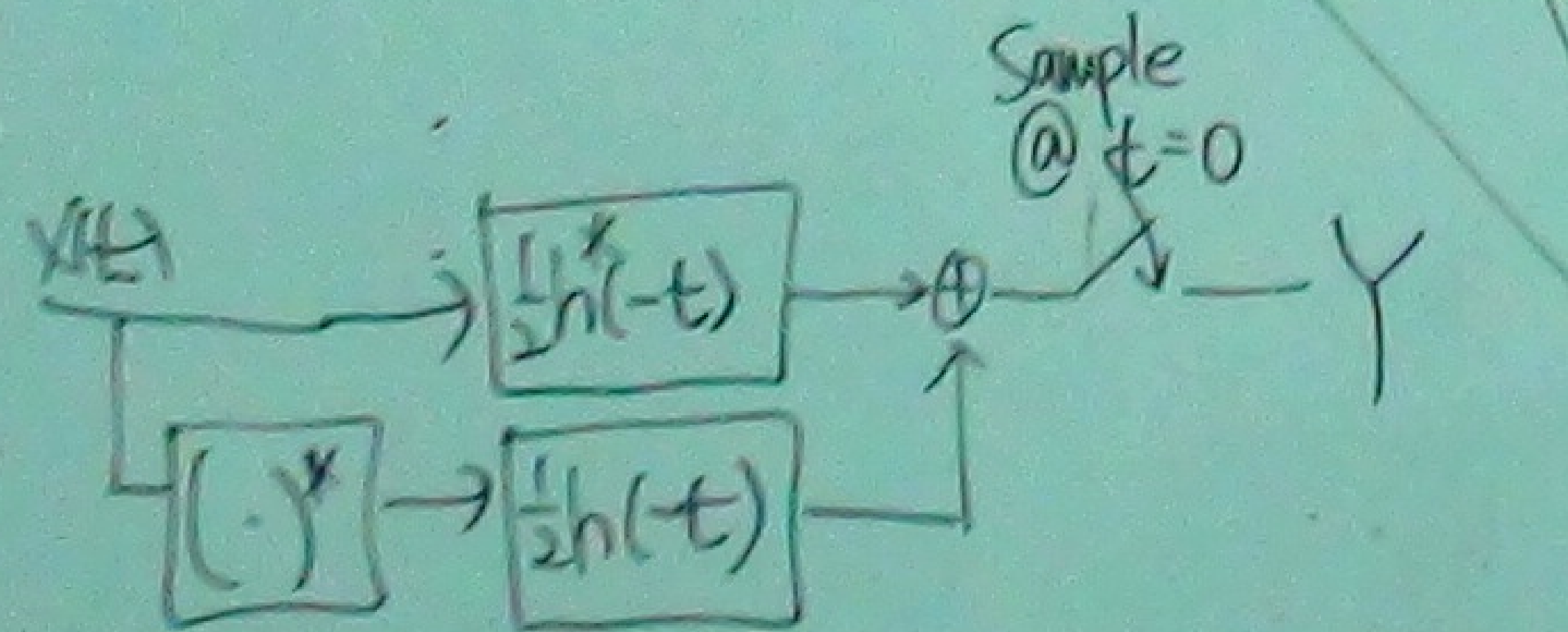
$$\rightarrow T=\begin{bmatrix}I & jI\\ I & -jI\end{bmatrix}$$

Since $\boxed{\begin{bmatrix}X\\X^*\end{bmatrix}=T\begin{bmatrix}X_c\\X_s\end{bmatrix}}$ & $TT^H=T^HT=2I$,

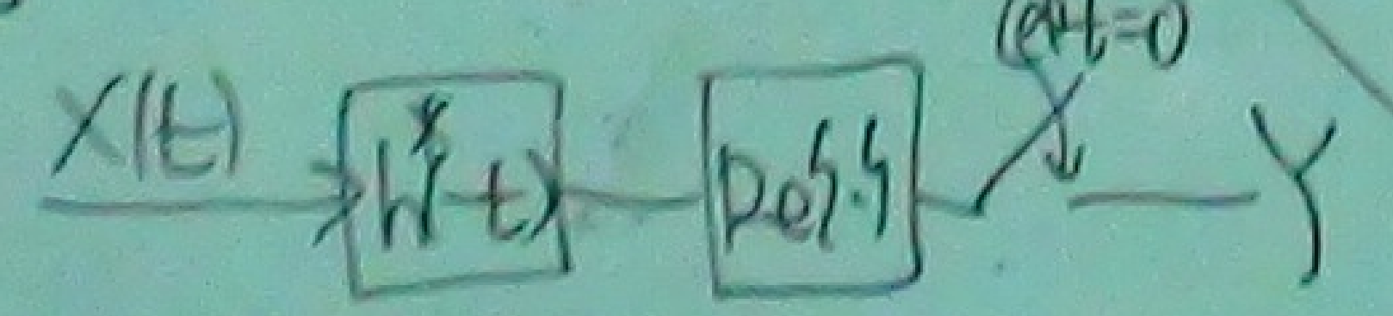
$$Y=\int_{\infty}^{\infty}\begin{bmatrix}h_c\\h_s\end{bmatrix}^H\frac{T^H}{2}T\begin{bmatrix}X_c\\X_s\end{bmatrix}dt$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\left(T\begin{bmatrix}h_c\\h_s\end{bmatrix}\right)^H\left(T\begin{bmatrix}X_c\\X_s\end{bmatrix}\right)dt$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\begin{bmatrix}h(t)\\h^*(t)\end{bmatrix}^H\begin{bmatrix}X(t)\\X^*(t)\end{bmatrix}dt =$$



$$=\frac{1}{2}\int_{-\infty}^{\infty}h^*(t)X(t)\,dt+\frac{1}{2}\int_{-\infty}^{\infty}h(t)X^*(t)\,dt=Re\left\{\int_{-\infty}^{\infty}h^*(t)X(t)\,dt\right\}=$$



WQ. $\begin{bmatrix}\underline{x_c}\\\underline{x_s}\end{bmatrix}^T\begin{bmatrix}H_{cc} & H_{cs}\\ H_{sc} & H_{ss}\end{bmatrix}\begin{bmatrix}\underline{x_c}\\\underline{x_s}\end{bmatrix}$

$H_{cc}=H_{cc}^T$

$H_{ss}=H_{ss}^T$

$H_{cs}^T=H_{sc}$

$$\begin{bmatrix}\underline{x}\\\underline{x}^*\end{bmatrix}^H\begin{bmatrix}H_1 & H_2\\ H_2^* & H_1^*\end{bmatrix}\begin{bmatrix}\underline{x}\\\underline{x}^*\end{bmatrix}$$

$$=\underline{x}^H H_1\underline{x}+Re\{\underline{x}^H H_2\underline{x}^*\}$$

☐ Quadratic Form

$X(t)=X_c(t)+jX_s(t)$

$$\begin{bmatrix}X_c(t)\\X_s(t)\end{bmatrix}^T\begin{bmatrix}h_{cc} & h_{cs}\\ h_{sc} & h_{ss}\end{bmatrix}\begin{bmatrix}X_c(t)\\X_s(t)\end{bmatrix}\quad (X)$$

$$=\cdots$$

$$=\begin{bmatrix}X(t)\\X^*(t)\end{bmatrix}^H\begin{bmatrix}h_1 & h_2\\ h_2^* & h_1^*\end{bmatrix}\begin{bmatrix}X(t)\\X^*(t)\end{bmatrix}$$

$$Y\triangleq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underset{\frac{T^HT}{2}}{\begin{bmatrix}X_c(t_1)\\X_s(t_1)\end{bmatrix}^T}\begin{bmatrix}h_{cc}(t_1,t_2) & h_{cs}(t_1,t_2)\\ h_{sc}(t_1,t_2) & h_{ss}(t_1,t_2)\end{bmatrix}\underset{\frac{T^HT}{2}}{\begin{bmatrix}X_c(t_2)\\X_s(t_2)\end{bmatrix}}dt_1dt_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\ dt_1dt_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X^*(t_1)h_1(t_1,t_2)X(t_2)\,dt_1dt_2$$

$$+Re\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X^*(t_1)h_2(t_1,t_2)X^*(t_2)\,dt_1dt_2\right\}.$$

↳ Widely Quadratic Form of X(t).

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X^*(t_1)\,h(t_1,t_2)\,X(t_2)\,dt_1dt_2 \longrightarrow$$ Strictly " " " X(t)

Where $h(t_1,t_2)=h(t_2,t_1)^*$

$$\int_{-\infty}^{\infty}R(t_1,t_2)\,v_i(t_2)\,dt_2=\lambda_i v_i(t_1).$$

where $R(t_1,t_2)=R^*(t_2,t_1)$.

$\boxed{R\underline{v}_i=\lambda_i\underline{v}_i.}$ $R=R^H\geq 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v_i^*(t_1)\,R(t_1,t_2)\,v_i(t_2)\,dt_2dt_1$$

$$=\int_{-\infty}^{\infty}v_i^*(t_1)\,\lambda_i\,v_i(t_1)\,dt_1$$

$$=\lambda_i \qquad \boxed{\underline{v}_i^H R\underline{v}_i=\lambda_i}$$

☐ Inner Product

$$\underset{\substack{\uparrow\\ \text{complex}}}{Y}=\int_{-\infty}^{\infty}h^*(t)\underset{\substack{\uparrow\\ \text{complex}}}{X(t)}\,dt$$

$$\underset{\text{real}}{Y}=Re\left\{\int_{-\infty}^{\infty}h^*(t)X(t)\,dt\right\}$$

$= \cdot\cdot + \cdot\cdot\cdot$

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☐ $\begin{bmatrix} X_c(t) \\ X_s(t) \end{bmatrix}$ inner product

$Y \in \mathbb{R}^1$

$Y_c = \mathrm{Re}\{\int_{-\infty}^{\infty} h_1(t)^* X(t)\,dt\}$

$Y_s = \mathrm{Re}\{\int_{-\infty}^{\infty} h_2(t)^* X(t)\,dt\}$

$$Y_c + jY_s = \int_{-\infty}^{\infty} \tilde{h}_1^*(t) X(t)\,dt + \int_{-\infty}^{\infty} \tilde{h}_2^*(t) X^*(t)\,dt$$

☐ $Y = \iint x^*(t_1) h_1(t_1-t_2) x(t_2)\,dt_1 dt_2 + \mathrm{Re}\{\iint x^*(t_1) h_2(t_1-t_2) x^*(t_2)\,dt_1 dt_2\} + \iint$

$= \iint \cdots$

$+ \iint \cdots$

☐ Correlation Analysis of a complex vector

$\underline{X} \in \mathbb{C}^N \rightarrow \boxed{L, WL} \rightarrow \underline{Y} \in \mathbb{C}^M$

- $\mu_{\underline{X}}$
- $C_{\underline{X}\underline{X}}$
- $\tilde{C}_{\underline{X}\underline{X}}$

- $\mu_{\underline{Y}}$
- $C_{\underline{Y}\underline{Y}}$
- $\tilde{C}_{\underline{Y}\underline{Y}}$

- $\underline{Y} = H\underline{X}$ $\quad \underline{X} \rightarrow \boxed{H} \rightarrow \underline{Y}$

$\mu_{\underline{Y}} = H\mu_{\underline{X}}$

$C_{\underline{Y}\underline{Y}} = E\{(\underline{Y}-\mu_{\underline{Y}})(\underline{Y}-\mu_{\underline{Y}})^H\}$

$= E\{H(\underline{X}-\mu_{\underline{X}})(\underline{X}-\mu_{\underline{X}})^H H^H\}$

$= HC_{\underline{X}\underline{X}}H^H$

$\tilde{C}_{\underline{Y}\underline{Y}} = \cdots$

$= H\tilde{C}_{\underline{X}\underline{X}}H^T$

1) SL processing of a proper-cmplx $\underline{X}$

⇒ a proper-complex $\underline{Y}$.

2) SL " " " improper " $\underline{X}$

⇒ improper $\underline{Y}$ in general.

- $\underline{X} \rightarrow \boxed{H_1} \rightarrow \oplus \rightarrow \underline{Y} = H_1\underline{X} + H_2\underline{X}^*$, $\quad \underline{X} \rightarrow \boxed{(\cdot)^*} \rightarrow \boxed{H_2} \rightarrow \oplus$

$\mu_{\underline{Y}} = H_1\mu_{\underline{X}} + H_2\mu_{\underline{X}}^*$

$C_{\underline{Y}\underline{Y}} = H_1 C_{\underline{X}\underline{X}} H_1^H + H_1 \tilde{C}_{\underline{X}\underline{X}} H_2^H + H_2 \tilde{C}_{\underline{X}\underline{X}}^* H_1^H + H_2 C_{\underline{X}\underline{X}}^* H_2^H$

$\tilde{C}_{\underline{Y}\underline{Y}} \overset{?}{=} H_1 \tilde{C}_{\underline{X}\underline{X}} H_1^T + H_1 C_{\underline{X}\underline{X}} H_2^T + H_2 \tilde{C}_{\underline{X}\underline{X}}^* H_1^T + H_2 C_{\underline{X}\underline{X}}^* H_2^T$

3) WL processing of a proper-complex $\underline{X}$

⇒ improper $\underline{Y}$ in general.

4) WL " improper " $\underline{X}$

⇒ improper " $\underline{Y}$ in general.

For special improper-complex $\underline{X}$, (asymptotic) ~~[illegible]~~
there exists an LCL / a WL filter called a properizer
invertible
that makes $\underline{Y}$ proper (asymptotically.)

SOCS $X(t)$ period T

$X_d[n] = X(\frac{nT}{M})$, $n \in \mathbb{Z}$

$\begin{bmatrix} X_d[0] \\ \vdots \\ X_d[N-1] \end{bmatrix} \rightarrow \boxed{\begin{matrix}\text{invertible} \\ LCL\end{matrix}} \rightarrow \begin{bmatrix} Y[0] \\ \vdots \\ Y[N-1] \end{bmatrix}$ ← almost proper

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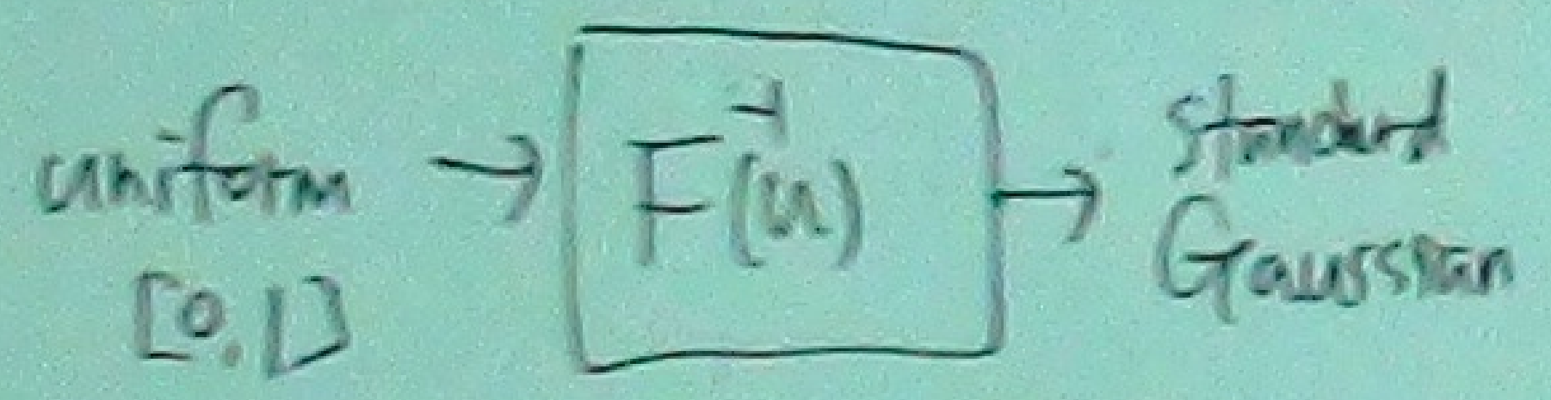
☐ Complex Gaussian random vector

- $\underline{X} \sim N(\mu_{\underline{X}}, C_{\underline{XX}} = C_{XX}^T)$, $\boxed{\underline{X}\in\mathbb{R}^N}$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N \det C_{\underline{XX}}}} \exp\left(-\frac{(\underline{x}-\mu_{\underline{X}})^T C_{\underline{XX}}^{-1}(\underline{x}-\mu_{\underline{X}})}{2}\right), \underline{x}\in\mathbb{R}^N$$

$$\Phi_{\underline{X}}(\underline{\omega}) = \exp\left(j\underline{\omega}^T\mu_{\underline{X}} - \frac{1}{2}\underline{\omega}^T C_{\underline{XX}}\underline{\omega}\right)$$



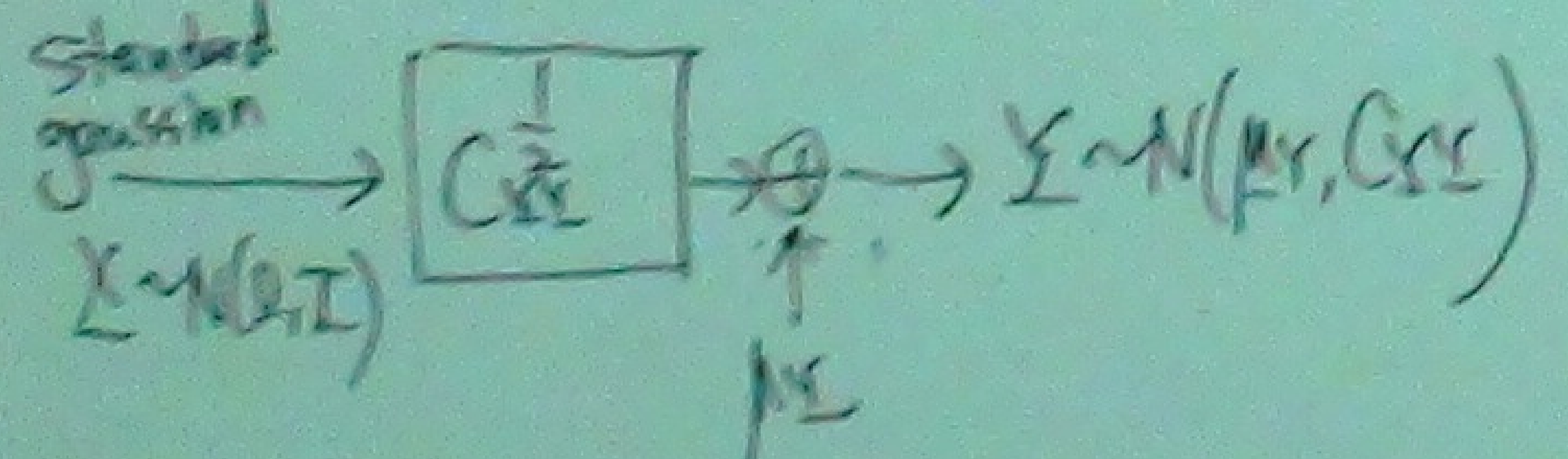
$F(x) \leftarrow$ CDF of $\cdot$.

$\underline{Y} = A\underline{X}+\underline{b}$ $\quad \underline{X}\sim N(\mu_{\underline{X}}, C_{\underline{XX}})$

$\Phi_{\underline{Y}}(\underline{\omega}) = \exp(j\underline{\omega}^T\underline{b})\,\Phi_{\underline{X}}(A^T\underline{\omega})$

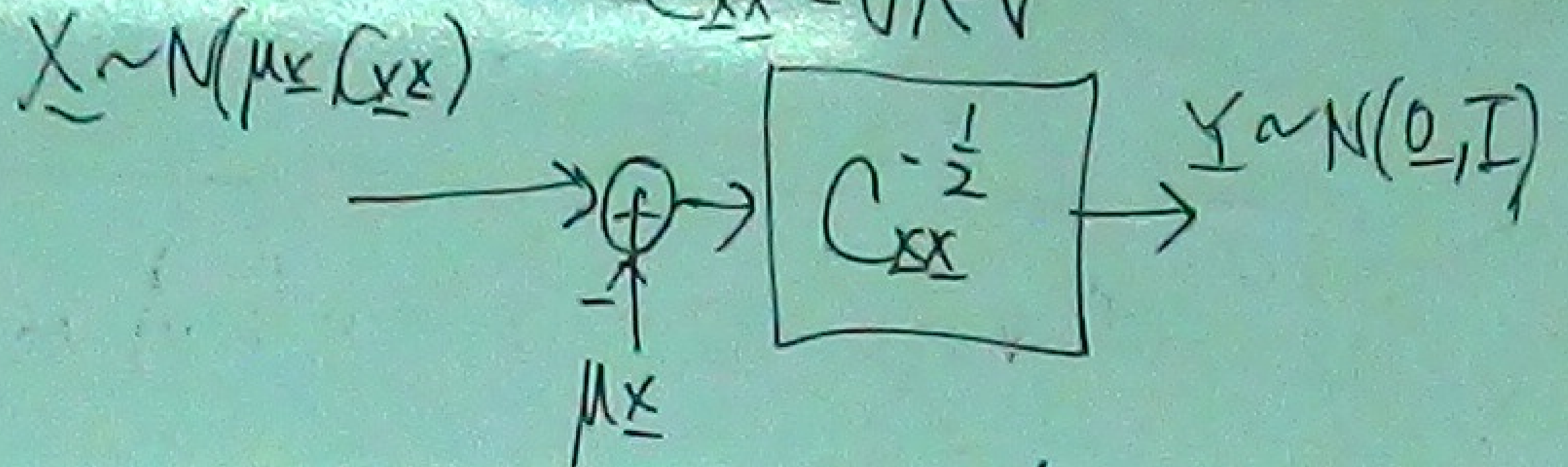
$\mu_{\underline{Y}} = A\mu_{\underline{X}}+\underline{b}$

$C_{\underline{YY}} = A C_{\underline{XX}} A^T$



$C_{\underline{XX}} = V\Lambda V^T$

$C_{\underline{XX}}^{-\frac{1}{2}} = V\Lambda^{-\frac{1}{2}}V^T$



$\mu_{\underline{X}|\underline{Y}}(\underline{y}) = \mu_{\underline{X}} + C_{\underline{XY}}C_{\underline{YY}}^{-1}(\underline{y}-\mu_{\underline{Y}})$

$C_{\underline{X}|\underline{Y}}(\underline{y}) = C_{\underline{XX}} - C_{\underline{XY}}C_{\underline{YY}}^{-1}C_{\underline{YX}}$

$\{\underline{Y}=\underline{y}\}$

- $\underline{X} \sim CN(\mu_{\underline{X}}, C_{\underline{XX}} = C_{XX}^H)$, $\underline{X}\in\mathbb{C}^N$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\pi^N \det C_{\underline{XX}}} \exp\left(-(\underline{x}-\mu_{\underline{X}})^H C_{\underline{XX}}^{-1}(\underline{x}-\mu_{\underline{X}})\right)$$

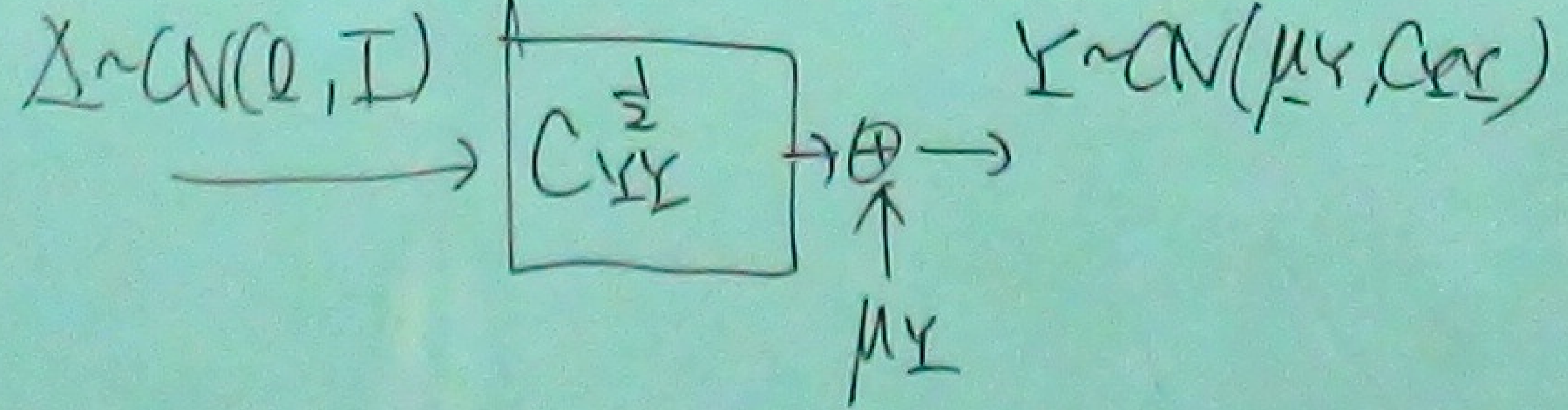
CF?

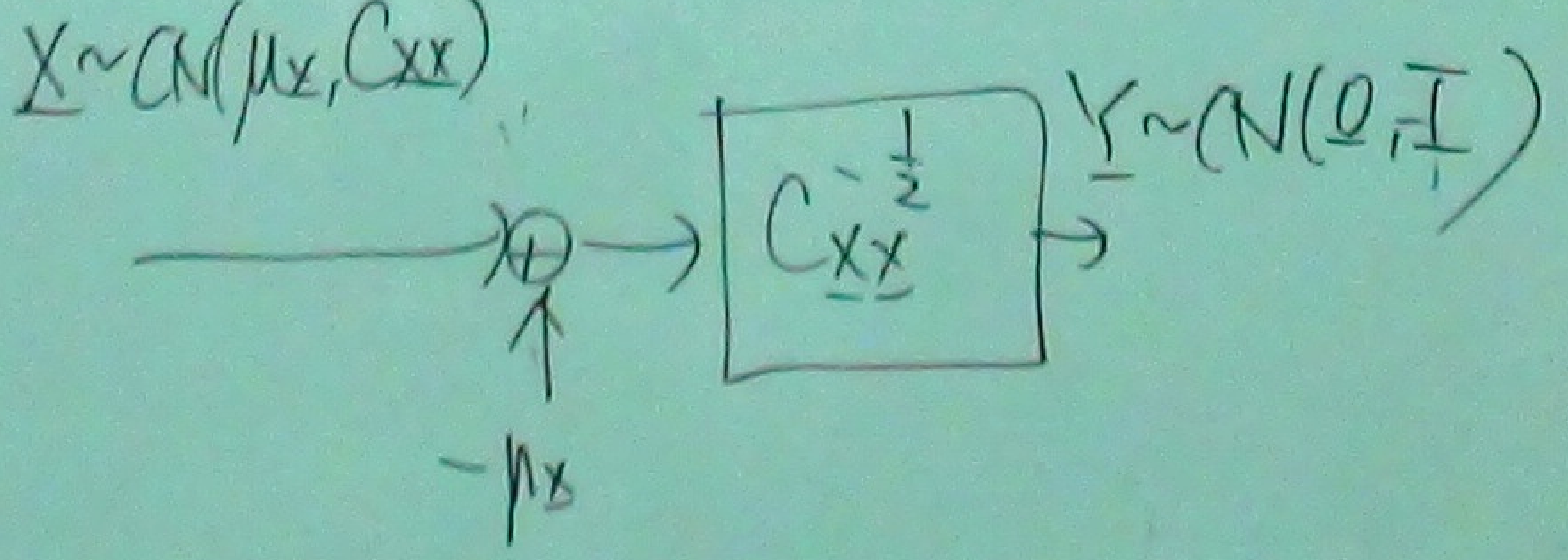
$\underline{Y} = A\underline{X}+\underline{b} \sim CN(\mu_{\underline{Y}}, C_{\underline{YY}})$

where $\mu_{\underline{Y}} = A\mu_{\underline{X}}+\underline{b}$

$C_{\underline{YY}} = A C_{\underline{XX}} A^H$

$\tilde{C}_{\underline{YY}} = 0$.





$\mu_{\underline{X}|\underline{Y}}(\underline{y}) = \mu_{\underline{X}} + C_{\underline{XY}}C_{\underline{YY}}^{-1}(\underline{y}-\mu_{\underline{Y}})$

$C_{\underline{X}|\underline{Y}}(\underline{y}) = C_{\underline{XX}} - C_{\underline{XY}}C_{\underline{YY}}^{-1}C_{\underline{YX}}$

$\{\underline{Y}=\underline{y}\}$

- $\underline{X}\in\mathbb{C}^N$, $\mu_{\underline{X}}$, $C_{\underline{XX}} = C_{XX}^H$, $\tilde{C}_{\underline{XX}} \neq 0$

PDF $f_{\underline{X}}(\underline{x}) =$ real composite

$=$ complex augmented

$\overline{\underline{X}} = \begin{bmatrix}\underline{X}\\ \underline{X}^*\end{bmatrix}$, $\overline{\mu}_{\underline{X}} = \begin{bmatrix}\mu_{\underline{X}}\\ \mu_{\underline{X}}^*\end{bmatrix}$, $\overline{C}_{\underline{XX}} = \begin{bmatrix}C_{\underline{XX}} & \tilde{C}_{\underline{XX}}\\ \tilde{C}_{\underline{XX}}^* & C_{\underline{XX}}^*\end{bmatrix}$, $\cdots$

$$f_{\underline{X}}(\underline{x}) = f_{\overline{\underline{X}}}(\overline{\underline{x}}) = \frac{1}{\pi^N\sqrt{\det \overline{C}_{\underline{XX}}}} \exp\left(-\frac{1}{2}(\overline{\underline{x}}-\overline{\mu}_{\underline{X}})^H\overline{C}_{\underline{XX}}^{-1}(\overline{\underline{x}}-\overline{\mu}_{\underline{X}})\right)$$

$\det\overline{C}_{\underline{XX}} = 2^{2N}\det \mathrm{Cov}\left(\begin{bmatrix}\underline{X}_r\\ \underline{X}_s\end{bmatrix}, \begin{bmatrix}\underline{X}_r\\ \underline{X}_s\end{bmatrix}\right)$,

?, ?, $\boxed{S}$ $\quad \boxed{X\in\mathbb{C}^1}$

X Gaussian