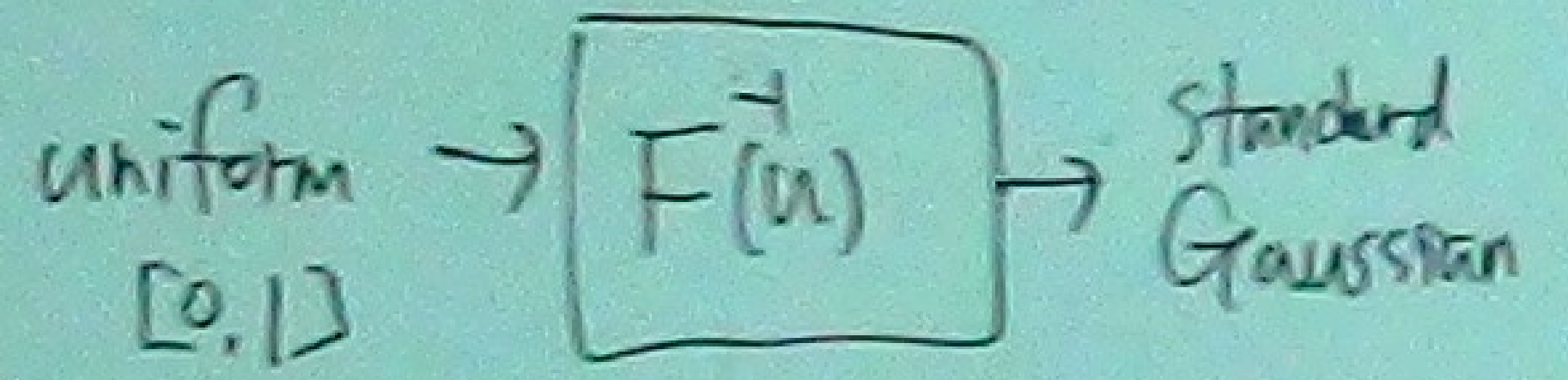
EECE695M CM#11C

☐ Complex Gaussian random vector

- $\underline{X} \sim N(\mu_{\underline{x}}, C_{\underline{x}\underline{x}} \overset{T}{=C_{xx}})$, $\boxed{\underline{X} \in \mathbb{R}^N}$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N \det C_{\underline{x}\underline{x}}}} \exp\left(-\frac{(\underline{x}-\mu_{\underline{x}})^T C_{\underline{x}\underline{x}}^{-1} (\underline{x}-\mu_{\underline{x}})}{2}\right), \underline{x} \in \mathbb{R}^N$$

$$\Phi_{\underline{X}}(\underline{\omega}) = \exp\left(j\, \underline{\omega}^T \mu_{\underline{x}} - \frac{1}{2} \underline{\omega}^T C_{\underline{x}\underline{x}} \underline{\omega}\right)$$



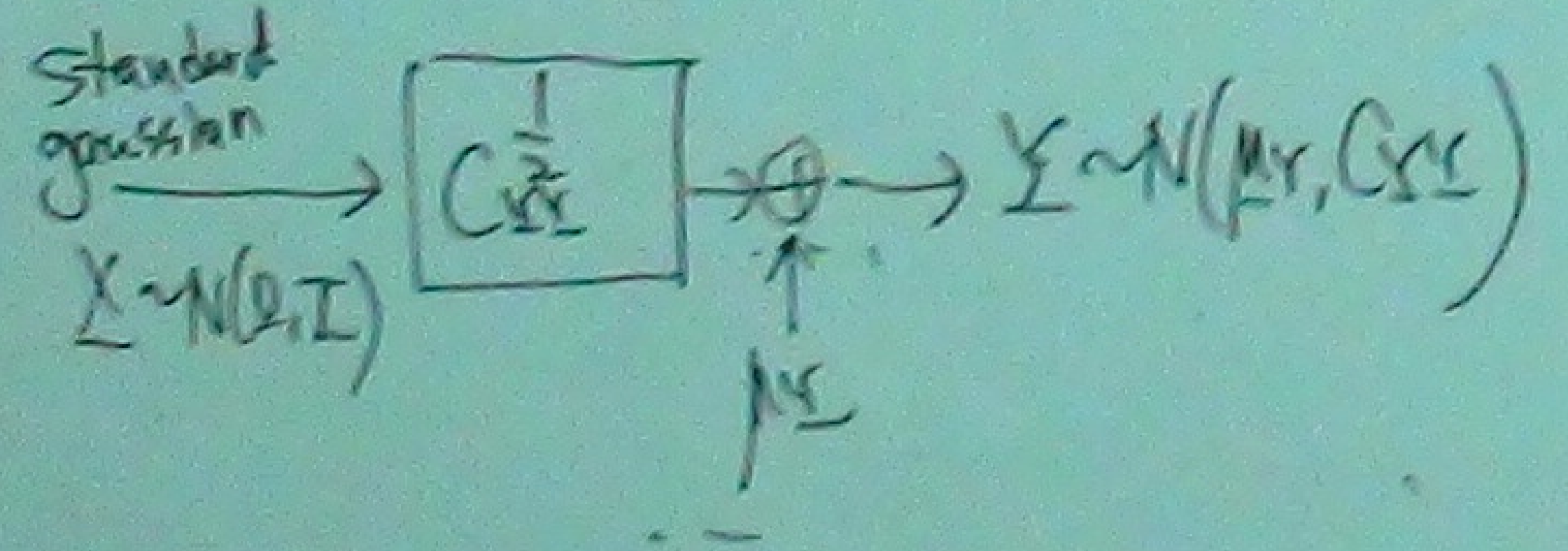
$F(x) \leftarrow$ CDF of $\cdot$.

$\underline{Y} = A\underline{x} + \underline{b}$ $\quad \underline{X} \sim N(\mu_{\underline{x}}, C_{\underline{x}\underline{x}})$

$$\Phi_{\underline{Y}}(\underline{\omega}) = \exp(j\underline{\omega}^T\underline{b})\, \Phi_{\underline{X}}(A^T\underline{\omega})$$

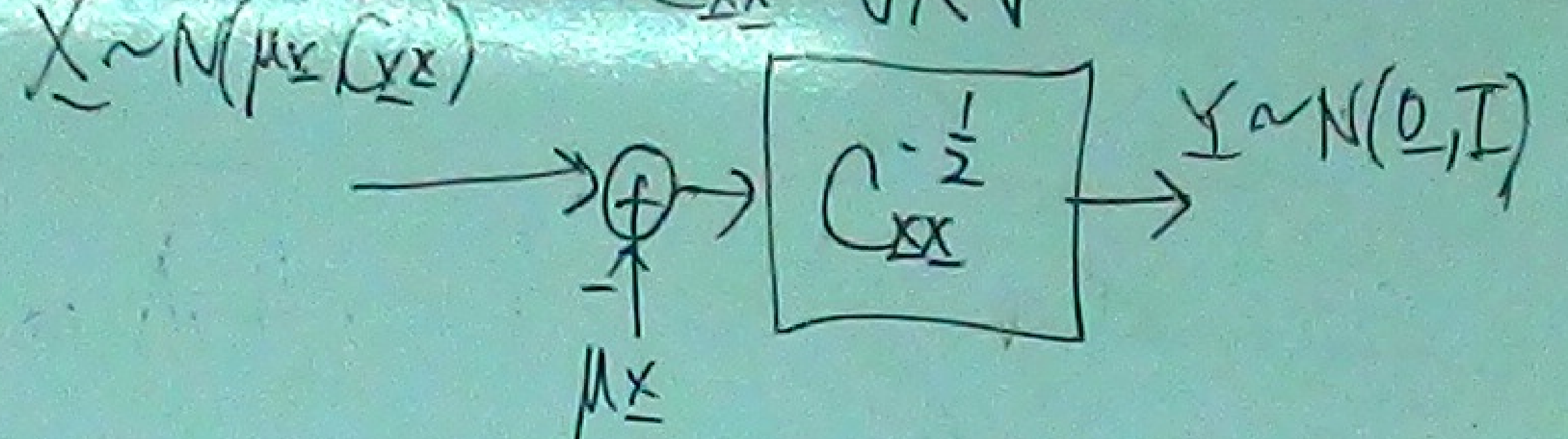
$\mu_{\underline{Y}} = A\mu_{\underline{x}} + \underline{b}$

$C_{\underline{Y}\underline{Y}} = A C_{\underline{x}\underline{x}} A^T$



$C_{\underline{x}\underline{x}} = V \Lambda V^T$

$C_{\underline{x}\underline{x}}^{-\frac{1}{2}} = V \Lambda^{-\frac{1}{2}} V^T$



$$\mu_{\underline{X}|\underline{Y}}(\underline{y}) = \mu_{\underline{x}} + C_{\underline{x}\underline{Y}} C_{\underline{Y}\underline{Y}}^{-1} (\underline{y} - \mu_{\underline{Y}})$$

$$C_{\underline{X}|\underline{Y}(\underline{y})} = C_{\underline{x}\underline{x}} - C_{\underline{x}\underline{Y}} C_{\underline{Y}\underline{Y}}^{-1} C_{\underline{Y}\underline{x}}$$

$\{\underline{Y} = \underline{y}\}$

- $\underline{X} \sim CN(\mu_{\underline{x}}, C_{\underline{x}\underline{x}} \overset{H}{=C_{xx}})$, $\underline{X} \in \mathbb{C}^N$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\pi^N \det C_{\underline{x}\underline{x}}} \exp\left(-(\underline{x}-\mu_{\underline{x}})^H C_{\underline{x}\underline{x}}^{-1} (\underline{x}-\mu_{\underline{x}})\right)$$

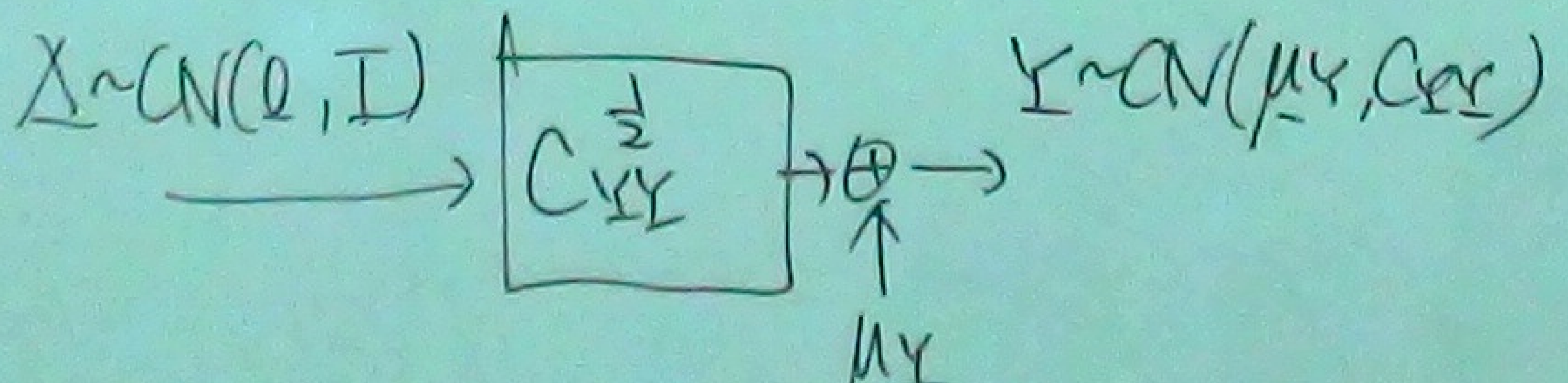
CF?

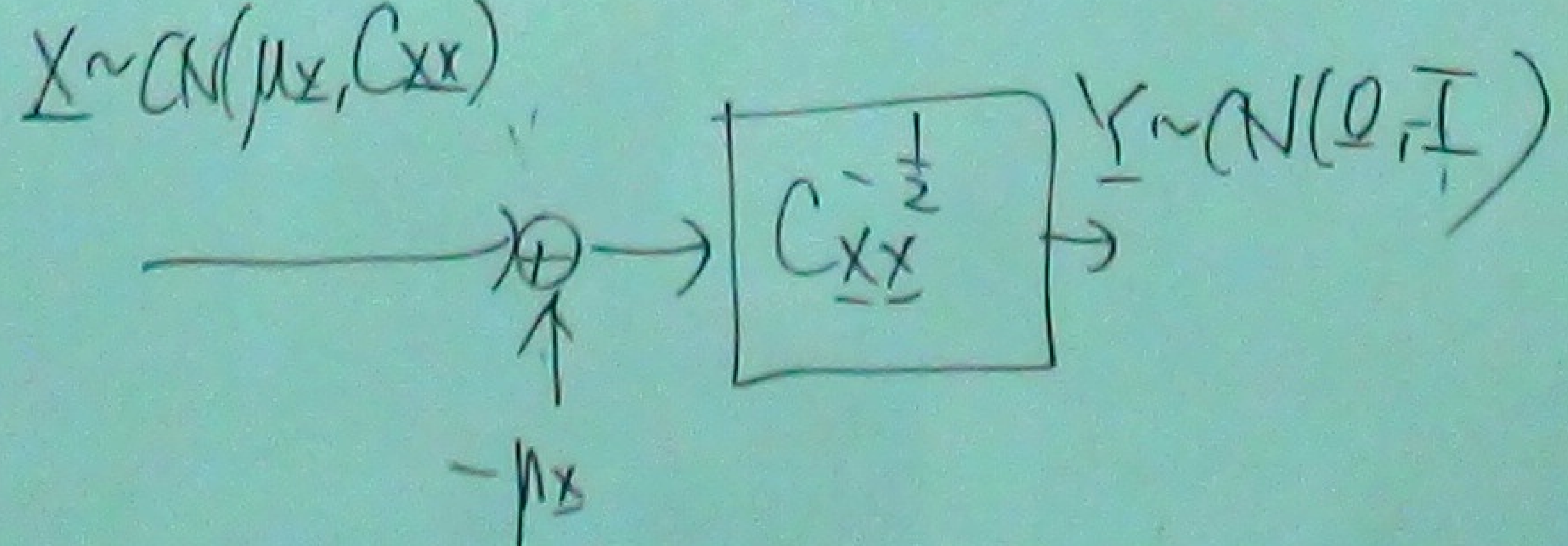
$\underline{Y} = A\underline{X} + \underline{b} \sim CN(\mu_{\underline{Y}}, C_{\underline{Y}\underline{Y}})$

where $\mu_{\underline{Y}} = A\mu_{\underline{x}} + \underline{b}$

$C_{\underline{Y}\underline{Y}} = A C_{\underline{x}\underline{x}} A^H$

$\tilde{C}_{\underline{Y}\underline{Y}} = 0$.





$$\mu_{\underline{x}|\underline{Y}}(\underline{y}) = \mu_{\underline{x}} + C_{\underline{x}\underline{Y}} C_{\underline{Y}\underline{Y}}^{-1} (\underline{y} - \mu_{\underline{Y}})$$

$$C_{\underline{X}|\underline{Y}(\underline{y})} = C_{\underline{x}\underline{x}} - C_{\underline{x}\underline{Y}} C_{\underline{Y}\underline{Y}}^{-1} C_{\underline{Y}\underline{x}}$$

$\{\underline{Y} = \underline{y}\}$

- $\underline{X} \in \mathbb{C}^N$, $\mu_{\underline{x}}, C_{\underline{x}\underline{x}}, \tilde{C}_{\underline{x}\underline{x}} \overset{H}{=C_{xx}} \neq 0$

PDF $f_{\underline{X}}(\underline{x}) =$ real composite

$=$ complex augmented

?

$$\overline{\underline{X}} = \begin{bmatrix} \underline{X} \\ \underline{X}^* \end{bmatrix}, \overline{\mu}_{\underline{x}} \begin{bmatrix} \mu_{\underline{x}} \\ \mu_{\underline{x}}^* \end{bmatrix}, \overline{C}_{\underline{x}\underline{x}} = \begin{bmatrix} C_{\underline{x}\underline{x}} & \tilde{C}_{\underline{x}\underline{x}} \\ \tilde{C}_{\underline{x}\underline{x}}^* & C_{\underline{x}\underline{x}}^* \end{bmatrix}, \cdots$$

$$f_{\underline{X}}(\underline{x}) = f_{\overline{\underline{X}}}(\overline{\underline{x}}) = \frac{1}{\pi^N \sqrt{\det \overline{C}_{\underline{x}\underline{x}}}}$$

$$\exp\left(-\frac{1}{2}(\overline{\underline{x}} - \overline{\mu}_{\underline{x}})^H \overline{C}_{\underline{x}\underline{x}}^{-1} (\overline{\underline{x}} - \overline{\mu}_{\underline{x}})\right)$$

$$\det \overline{C}_{\underline{x}\underline{x}} = 2^{2N} \det \mathrm{Cov}\left(\begin{bmatrix} \underline{X}_r \\ \underline{X}_s \end{bmatrix}, \begin{bmatrix} \underline{X}_r \\ \underline{X}_s \end{bmatrix}\right),$$

?, ?, $\boxed{S}$ $\boxed{X \in \mathbb{C}^1}$

X Gaussian

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□ Complex Gaussian random variable

$X = X_c + jX_s \in \mathbb{C}^1$

pdf.

$$f_{X_cX_s}(x_c,x_s) = \frac{1}{2\pi\sigma_c\sigma_s\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma_c^2\sigma_s^2(1-\rho^2)} \begin{bmatrix} x_c-\mu_c \\ x_s-\mu_s \end{bmatrix}^T \begin{bmatrix} \sigma_s^2 & -\rho\sigma_c\sigma_s \\ \rho\sigma_c\sigma_s & \sigma_c^2 \end{bmatrix} \begin{bmatrix} x_c-\mu_c \\ x_s-\mu_s \end{bmatrix}\right)$$

$$= \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left(-\frac{1}{2} \underbrace{\begin{bmatrix} x_c-\mu_c \\ x_s-\mu_s \end{bmatrix}^T C^{-1} \begin{bmatrix} x_c-\mu_c \\ x_s-\mu_s \end{bmatrix}}_{=C \Rightarrow \text{ellipse}}\right)$$

where $C \triangleq Cov(X_c, X_s) = \begin{bmatrix} \sigma_c^2 & \rho\sigma_c\sigma_s \\ \rho\sigma_c\sigma_s & \sigma_s^2 \end{bmatrix}$

$$\forall a \in \mathbb{R},\ 0 \le E\left[\left\{a\left(\frac{X_c-\mu_c}{\sigma_c}\right) - \left(\frac{X_s-\mu_s}{\sigma_s}\right)\right\}^2\right]$$

$= a^2 - 2a\rho + 1$ $\quad \rho \triangleq E\left[\left(\frac{X_c-\mu_c}{\sigma_c}\right)\left(\frac{X_s-\mu_s}{\sigma_s}\right)\right]$

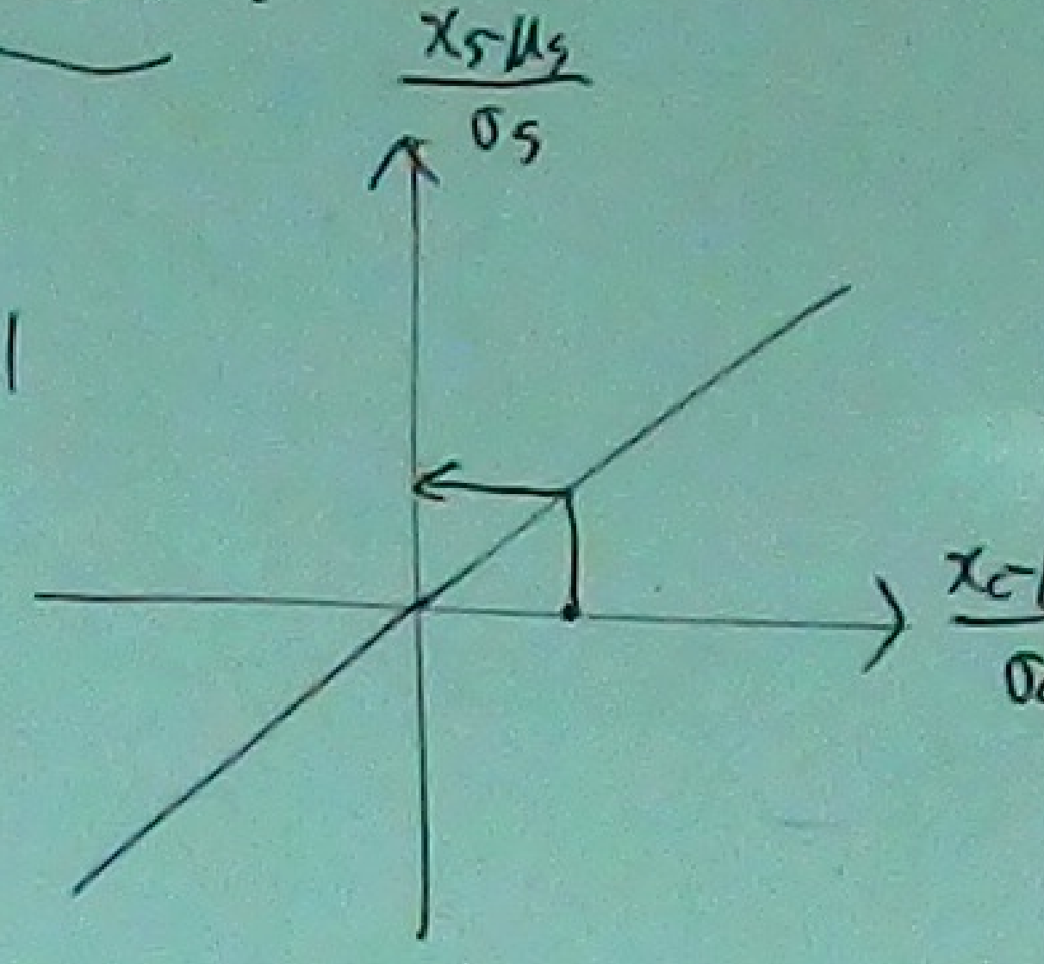
$D/4 = \rho^2 - 1 \le 0$ $\quad \rho\sigma_c\sigma_s = E[(X_c-\mu_c)(X_s-\mu_s)]$

$\Rightarrow \underset{\sim\sim\sim}{-1 \le \rho \le 1}$ $\quad = Cov(X_c, X_s)$

□ $\rho = 1$, $a = 1$

$$E\left[\underbrace{\left\{\left(\frac{X_c-\mu_c}{\sigma_c}\right) - \left(\frac{X_s-\mu_s}{\sigma_s}\right)\right\}}^2\right] = 0$$

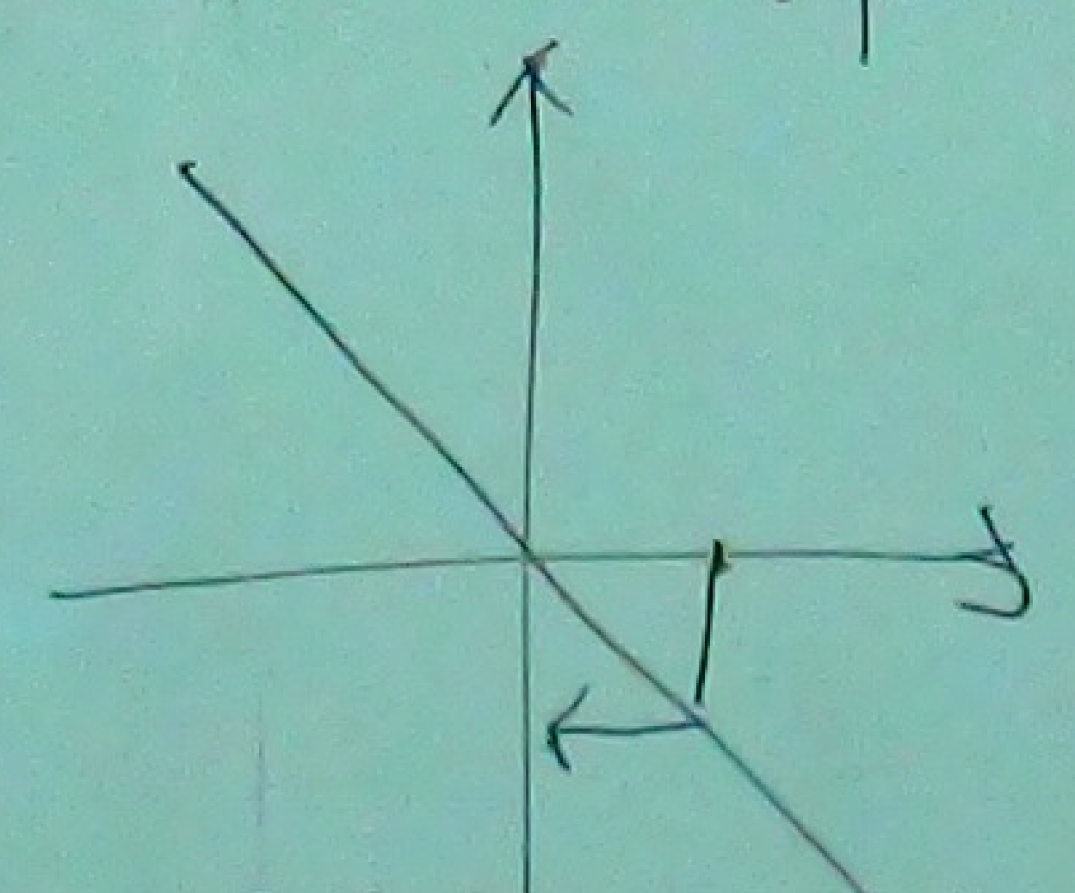
$= 0$, a.s.
w.p.1



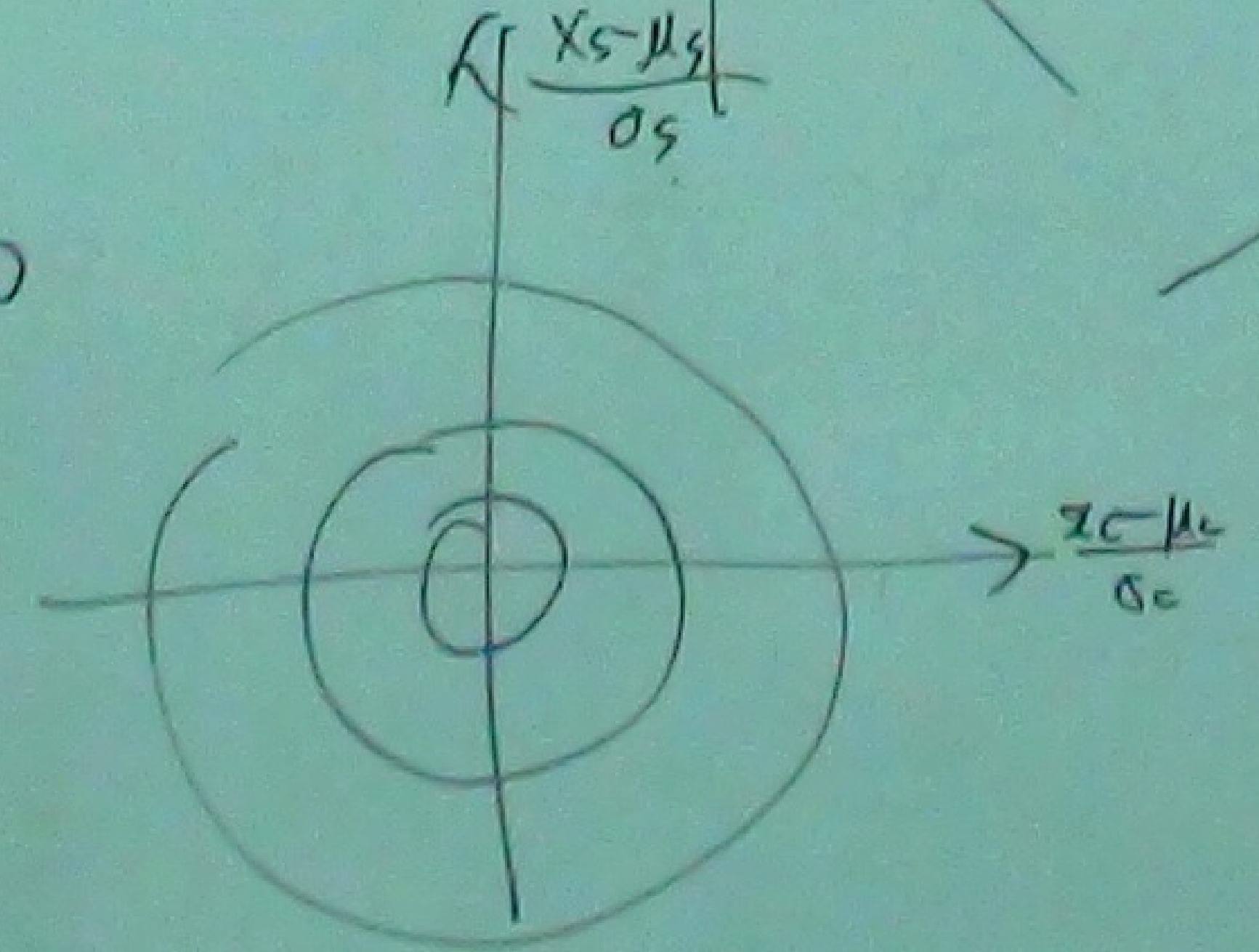
□ $\rho = -1$, $a = -1$

$$E\left[\underbrace{\left\{\left(\frac{X_c-\mu_c}{\sigma_c}\right) + \left(\frac{X_s-\mu_s}{\sigma_s}\right)\right\}}^2\right] = 0$$

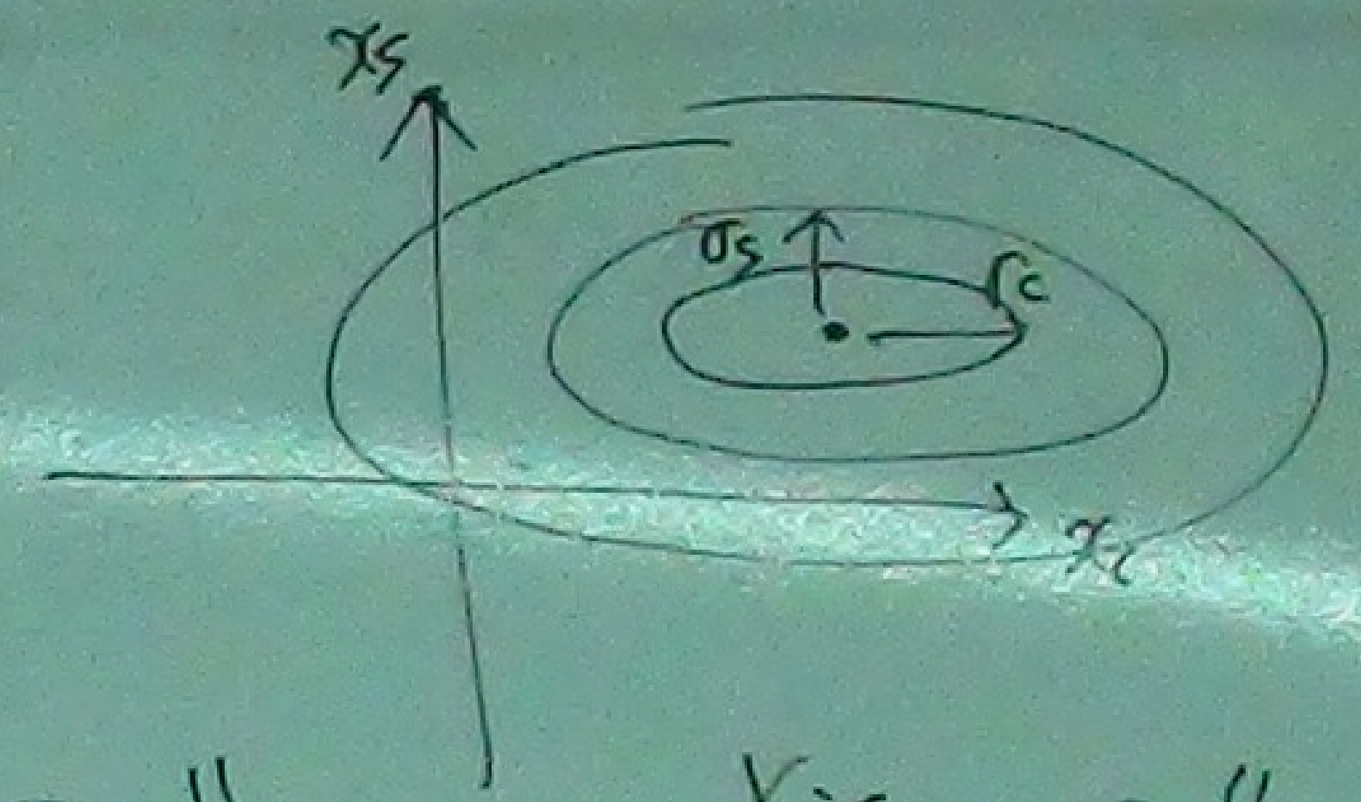
$= 0$, a.s.
w.p.1



□ $\rho = 0$



□



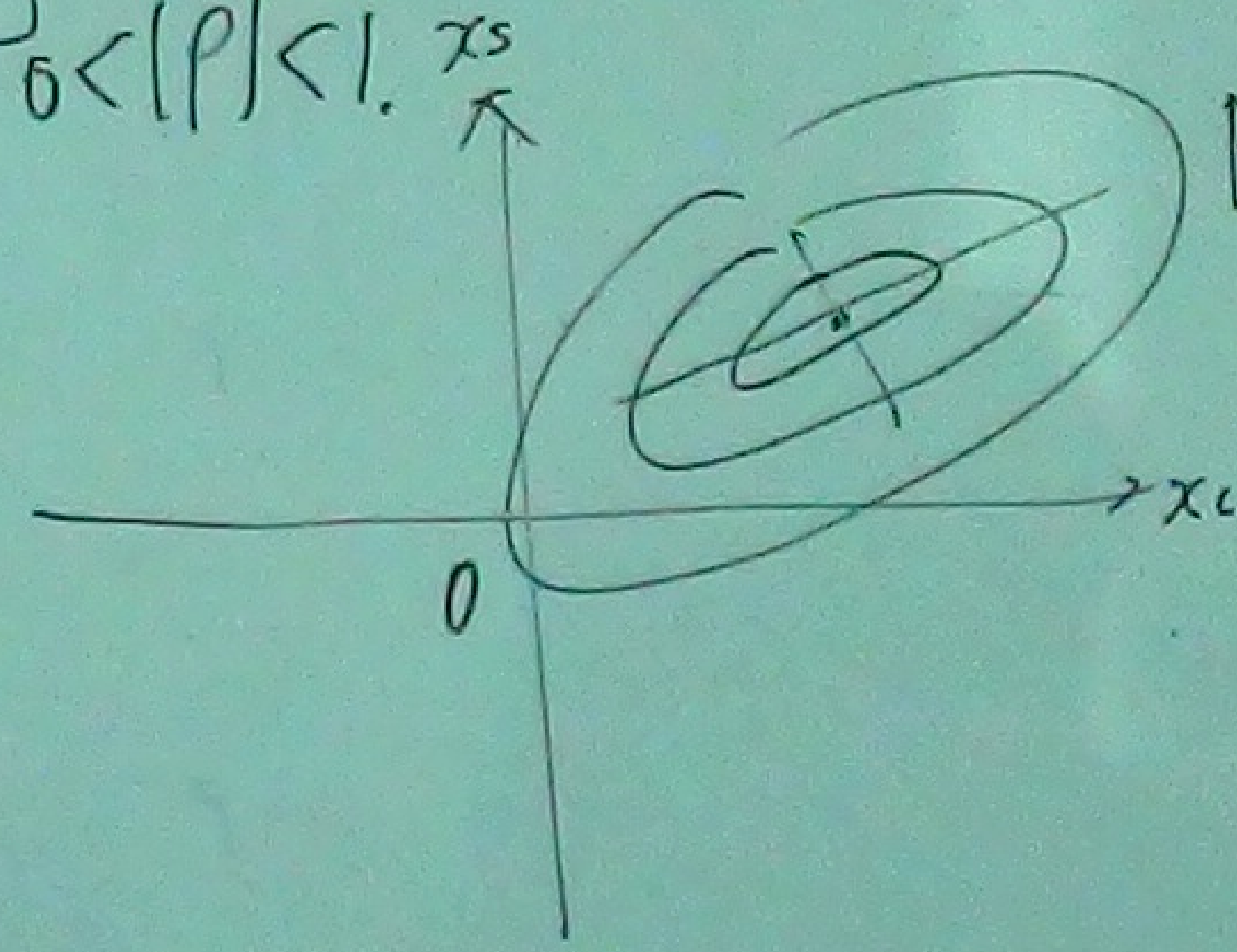
Q. "$\rho = 0 \Rightarrow$ X is proper ?" ← WRONG

A. $\tilde{Var}(X) = (\sigma_c^2 - \sigma_s^2) + j\,2\rho\sigma_c\sigma_s \Big|_{\rho=0}$

$= \sigma_c^2 - \sigma_s^2 \ne 0$ in general.

"$\rho = 0,\ \sigma_c = \sigma_s \Leftrightarrow X$ is proper" ⇐ Correct.

□ $0 < |\rho| < 1$.



$\rho = 0$ & $\sigma_c = \sigma_s \Leftrightarrow$ proper

□ Circularity Coefficient (Δ) ρ $\quad -1 \le \rho \le 1$

Impropriety 〃 (0) k $\quad 0 \le k \le 1$

$\Updownarrow$ proper, $\Updownarrow$ maximally improper

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☐ White? $X \in \mathbb{C}^1$. Gaussian

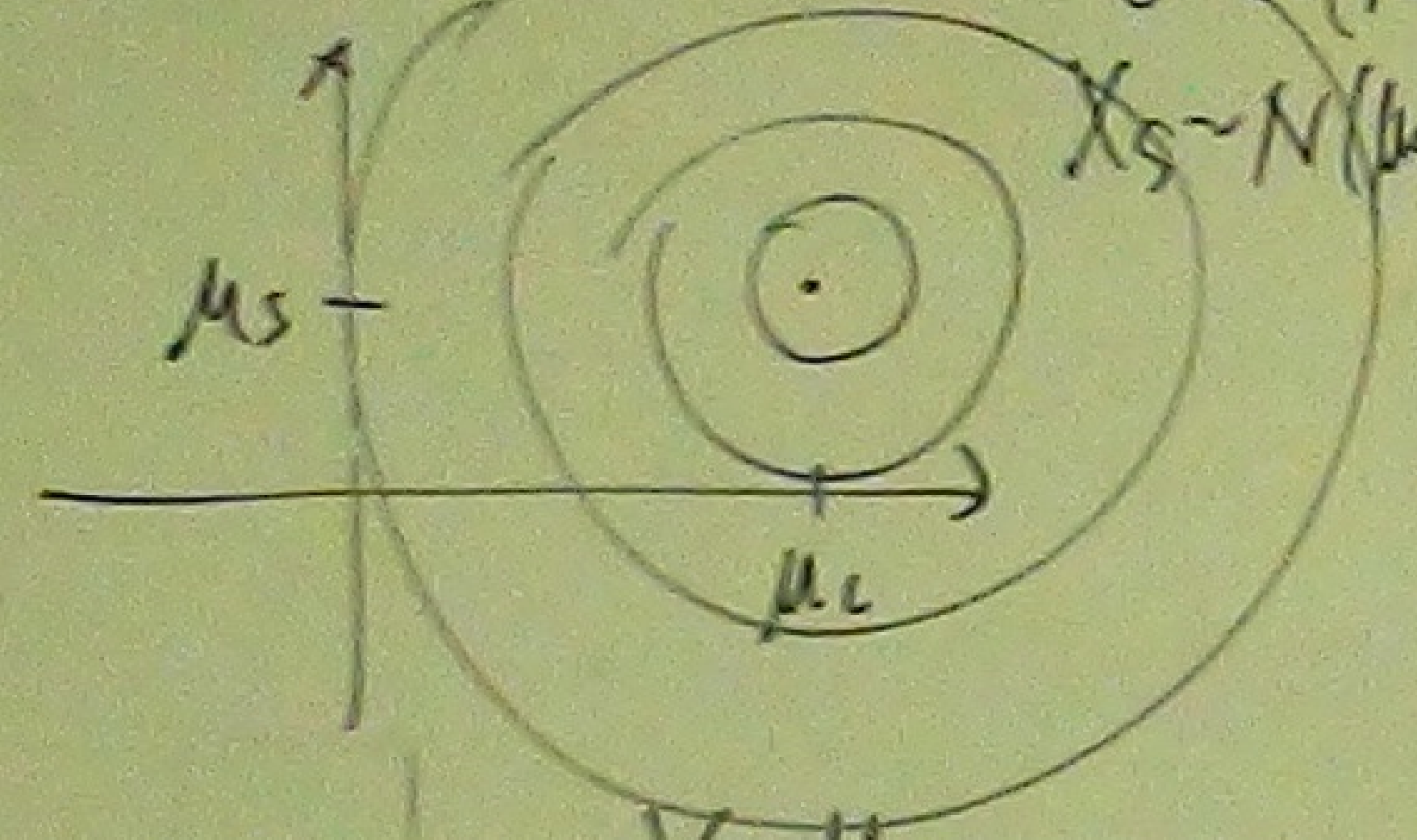
$E\{X\} = \mu_X$

$Var\{X\} = \sigma_X^2 = 2\sigma_c^2 = 2\sigma_s^2,\ \rho = 0$ ← Proper

$\widetilde{Var}\{X\} = 0$

$X_c \sim N(\mu_c, \sigma_c^2)$

$X_s \sim N(\mu_s, \sigma_c^2)$



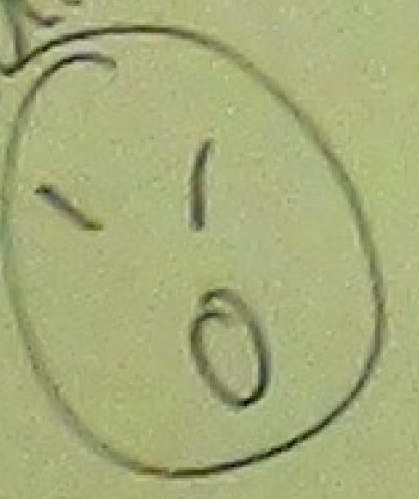
☐ $Z \triangleq \frac{X - \mu_X}{\sigma_X}$ X improper

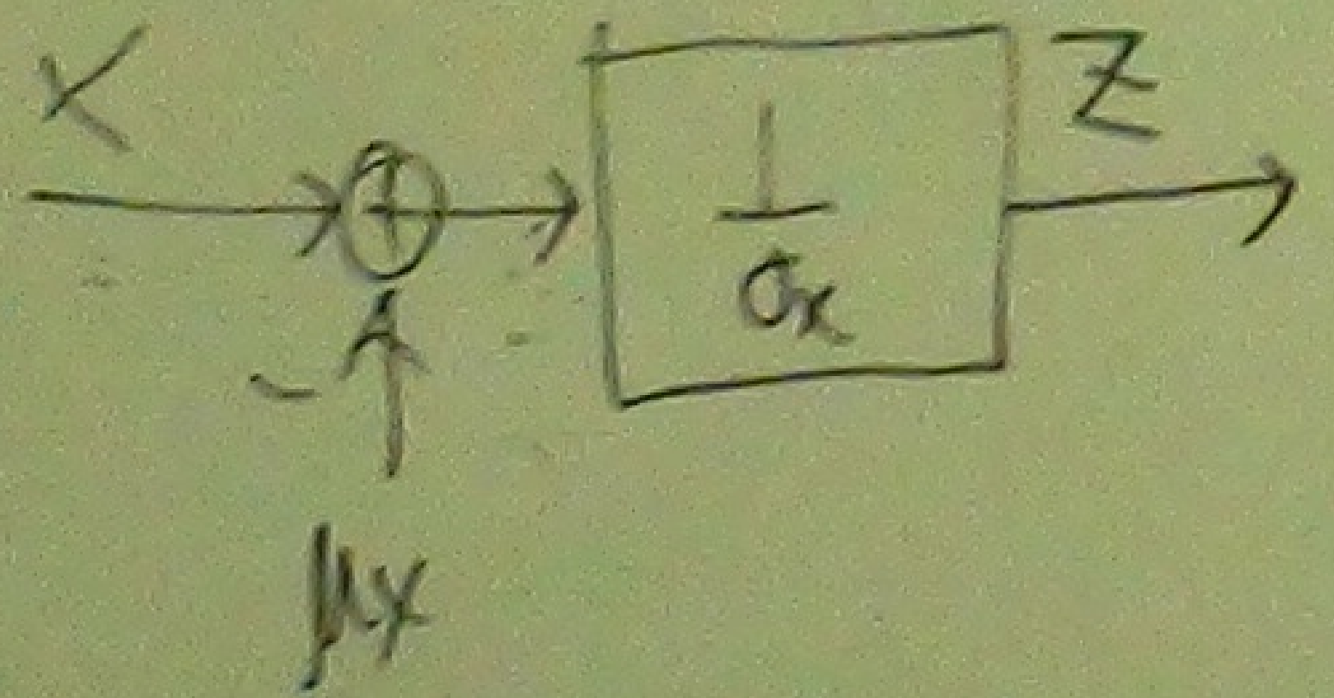
$E\{Z\} = 0$

$Var\{Z\} = 1$

$\widetilde{Var}\{Z\} = ke^{j\theta},\ 0 \le k \le 1$?

Claim.





$$\widetilde{Var}(Z) = \widetilde{Var}\left(\frac{X-\mu_X}{\sigma_X}\right)$$
$$= E\left\{\left(\frac{X-\mu_X}{\sigma_X}\right)^2\right\}$$
$$= \frac{(\sigma_c^2 - \sigma_s^2) + j(2\rho\sigma_c\sigma_s)}{\sigma_c^2 + \sigma_s^2}$$

1) $$k^2 = \frac{\sigma_c^4 - 2\sigma_c^2\sigma_s^2 + \sigma_s^4 + 4\rho^2\sigma_c^2\sigma_s^2}{(\sigma_c^2 + \sigma_s^2)^2}$$
$$= \frac{\sigma_c^4 + 2(2\rho^2 - 1)\sigma_c^2\sigma_s^2 + \sigma_s^4}{(\sigma_c^2 + \sigma_s^2)^2}$$
$$\le \frac{(\sigma_c^2 + \sigma_s^2)^2}{(\sigma_c^2 + \sigma_s^2)^2} = 1$$

$0 \le \rho^2 \le 1$

$\rho^2 = 1$

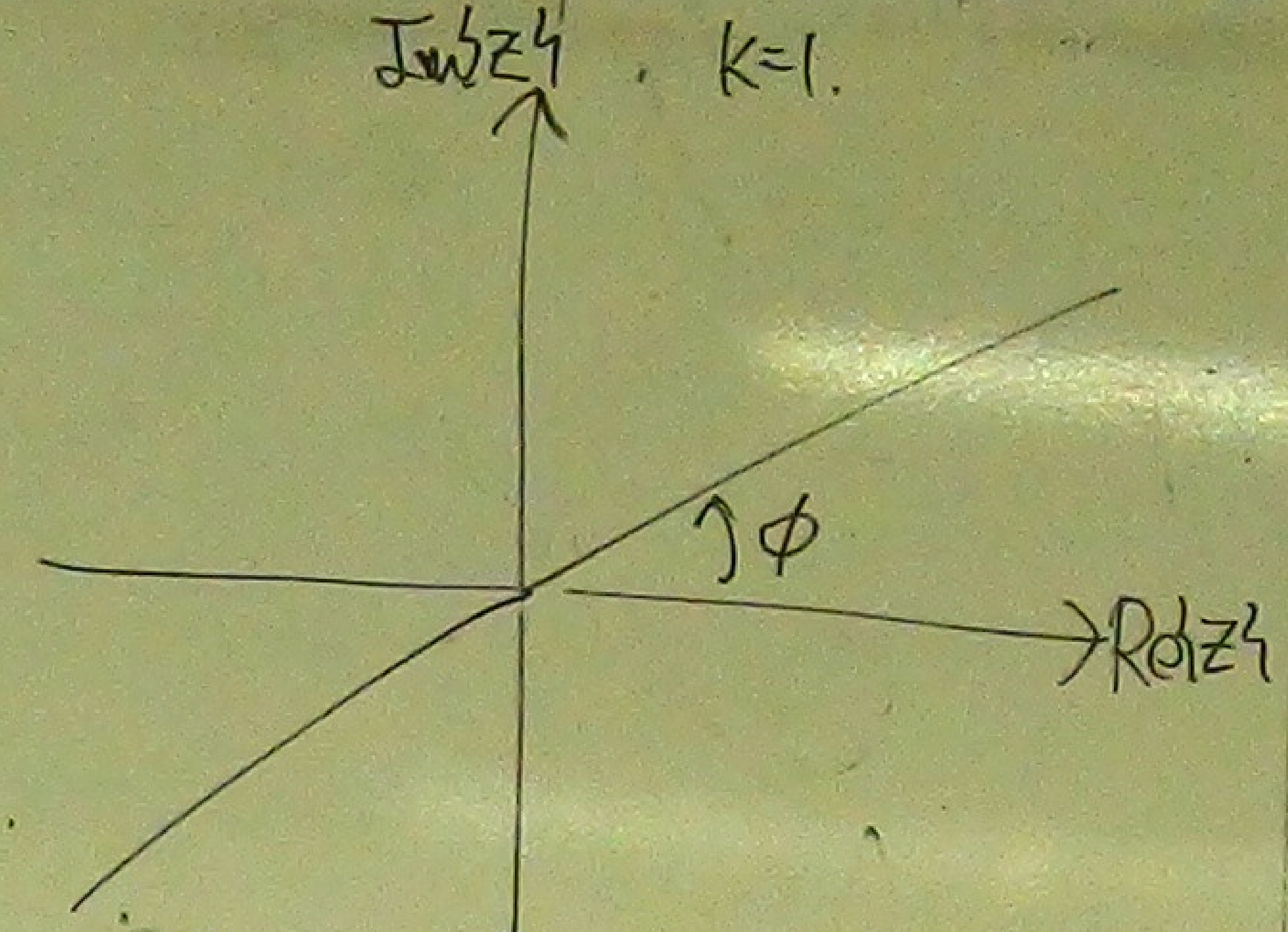
$\rho^2 = 0$

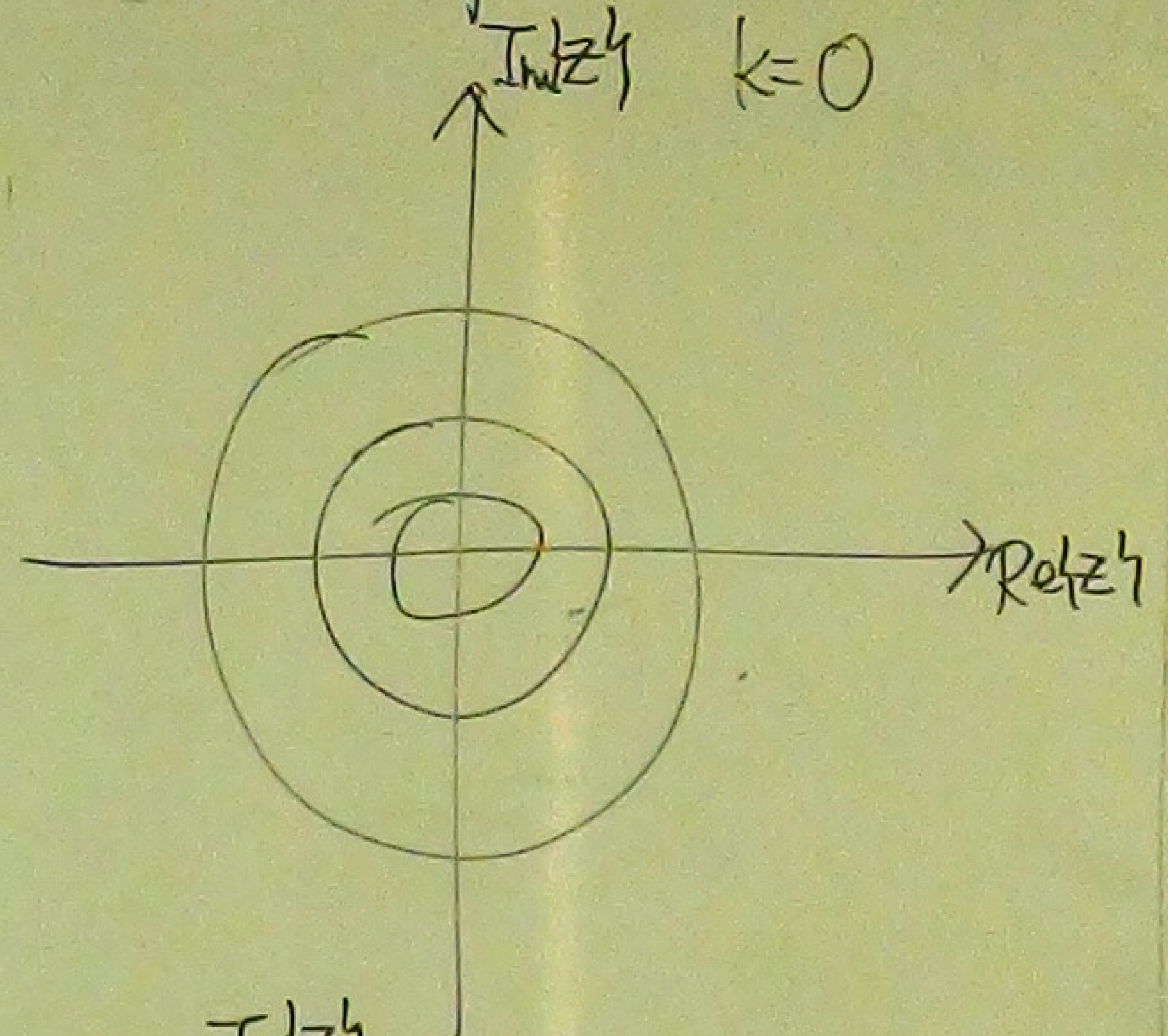
$\therefore 0 \le k \le 1$

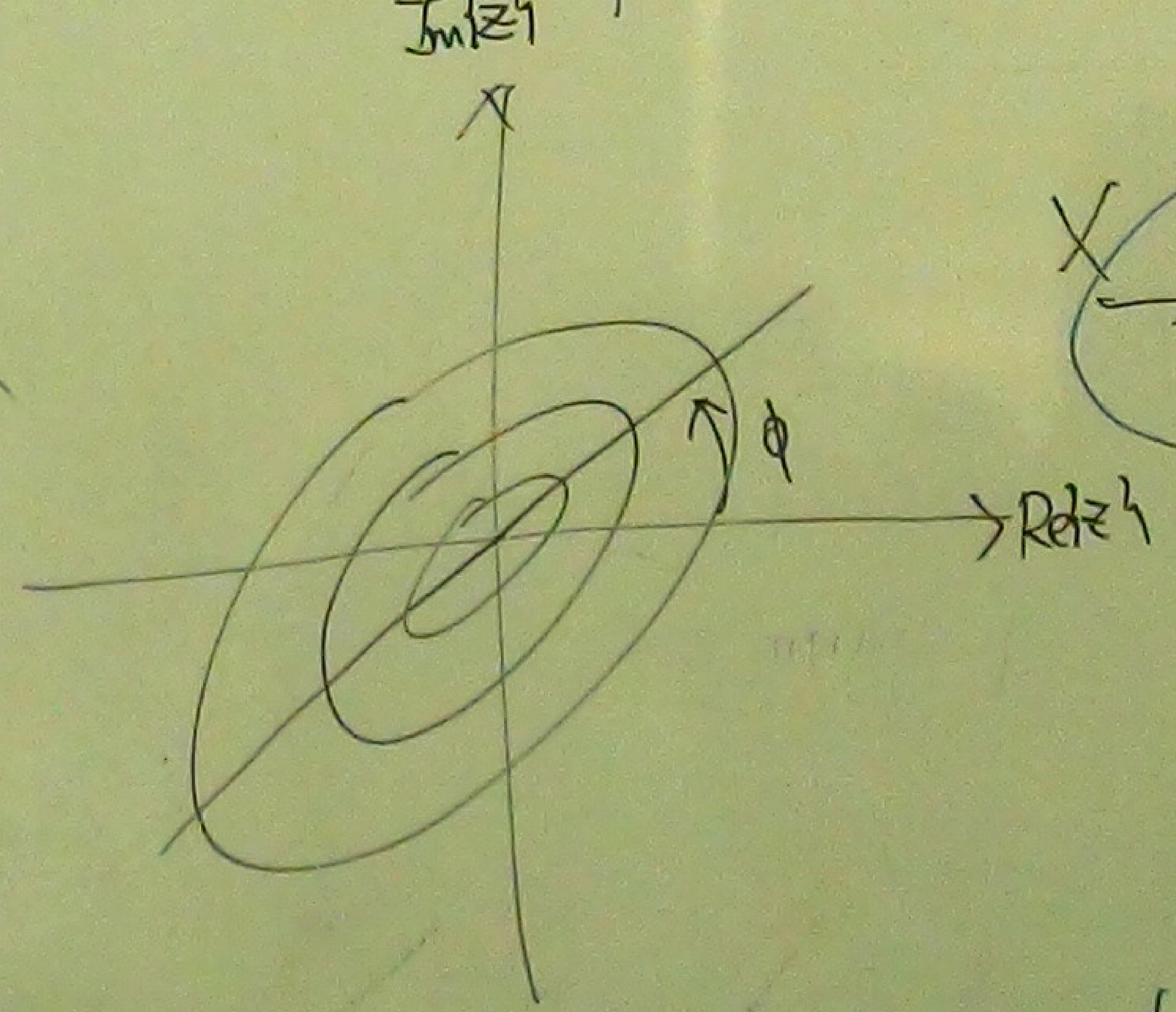
$\Updownarrow$ $\rho = 0$, $\sigma_c = \sigma_s$ $\Updownarrow$ proper

$\leftrightarrow \rho = \pm 1$ $\Updownarrow$ maximally improper

2) $\theta = ?$







$W \hat{=} Ze^{j\psi}$

$E\{W\} = 0$

$Var\{W\} = Var\{Ze^{j\psi}\} = Var\{Z\} = 1$

$\widetilde{Var}\{W\} = \widetilde{Var}\{Ze^{j\psi}\} = e^{j2\psi}\widetilde{Var}\{Z\}$

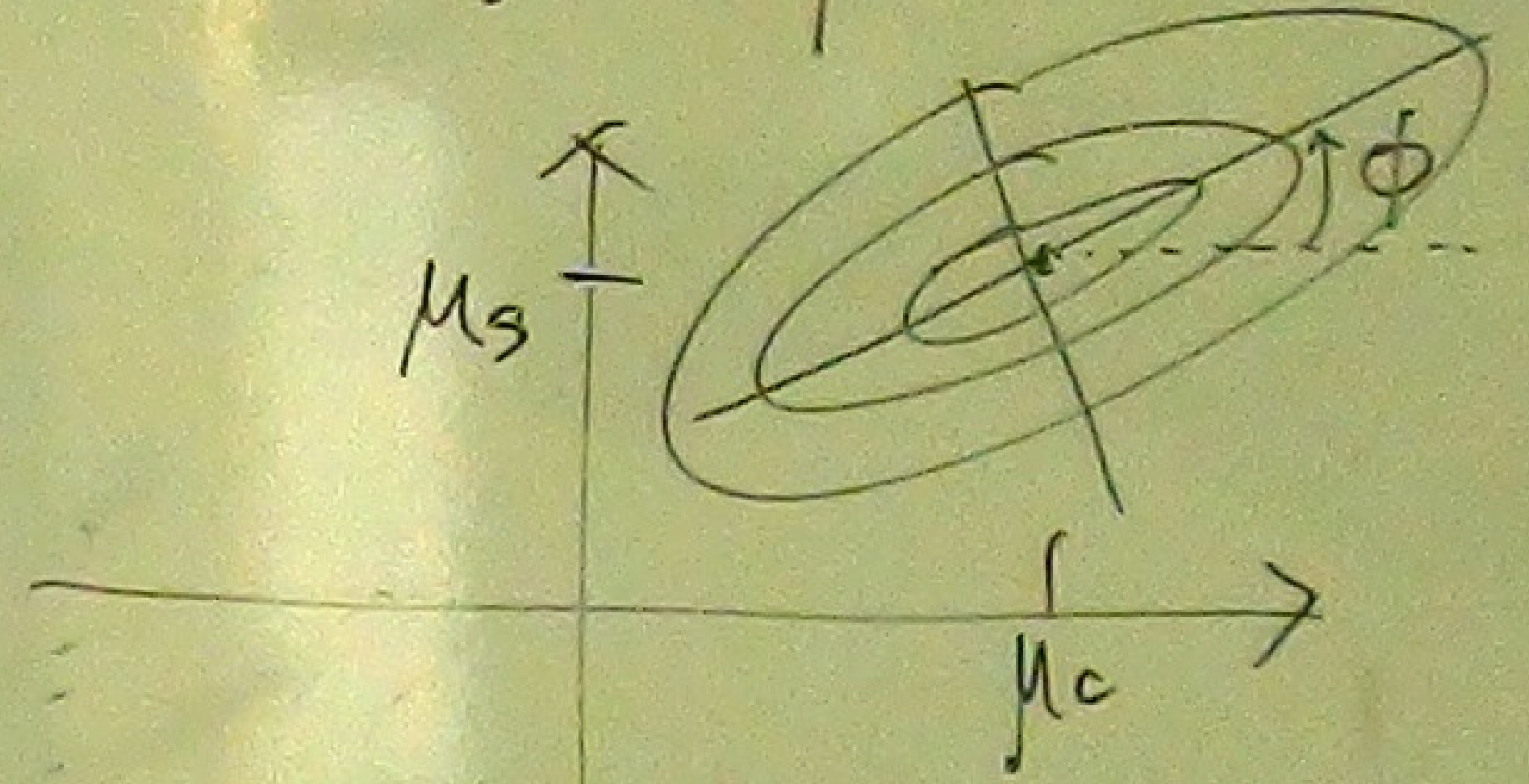
$= ke^{j(\theta + 2\psi)}$

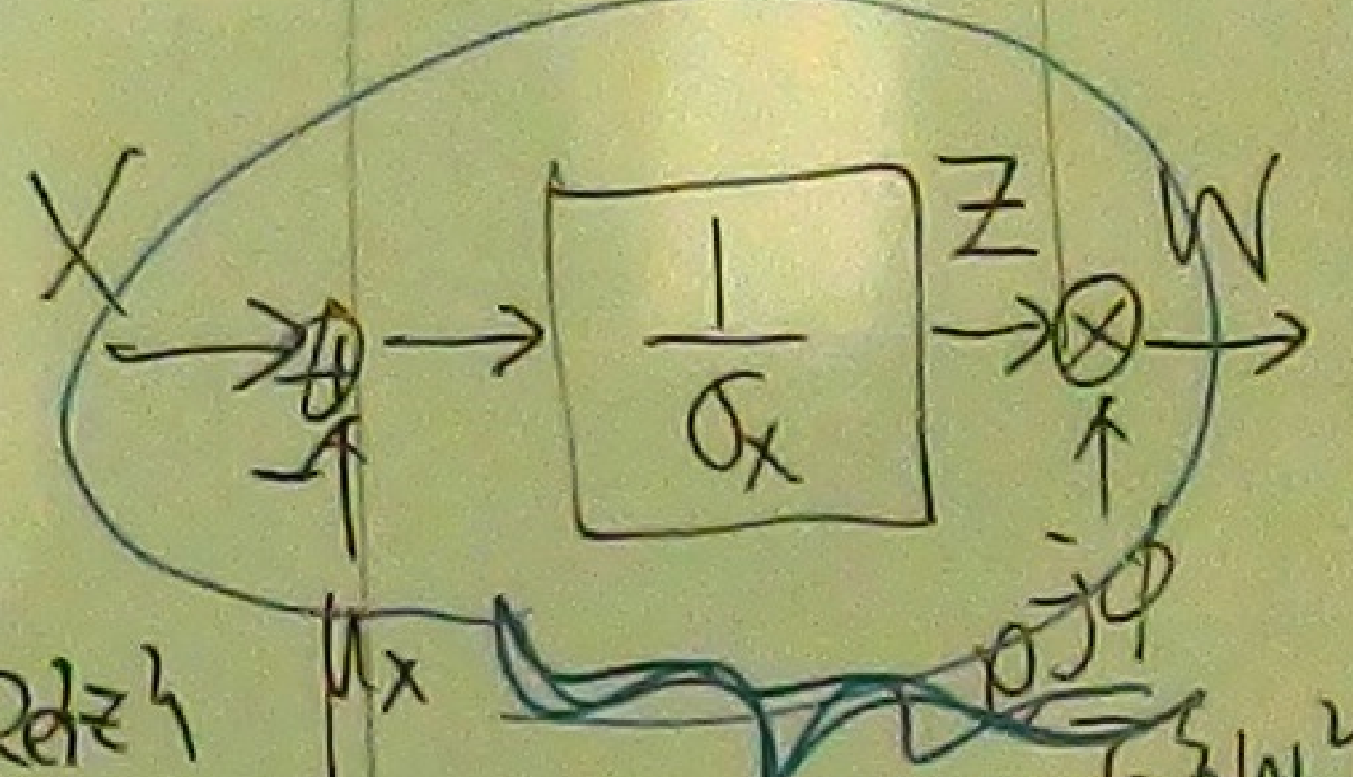
If we set

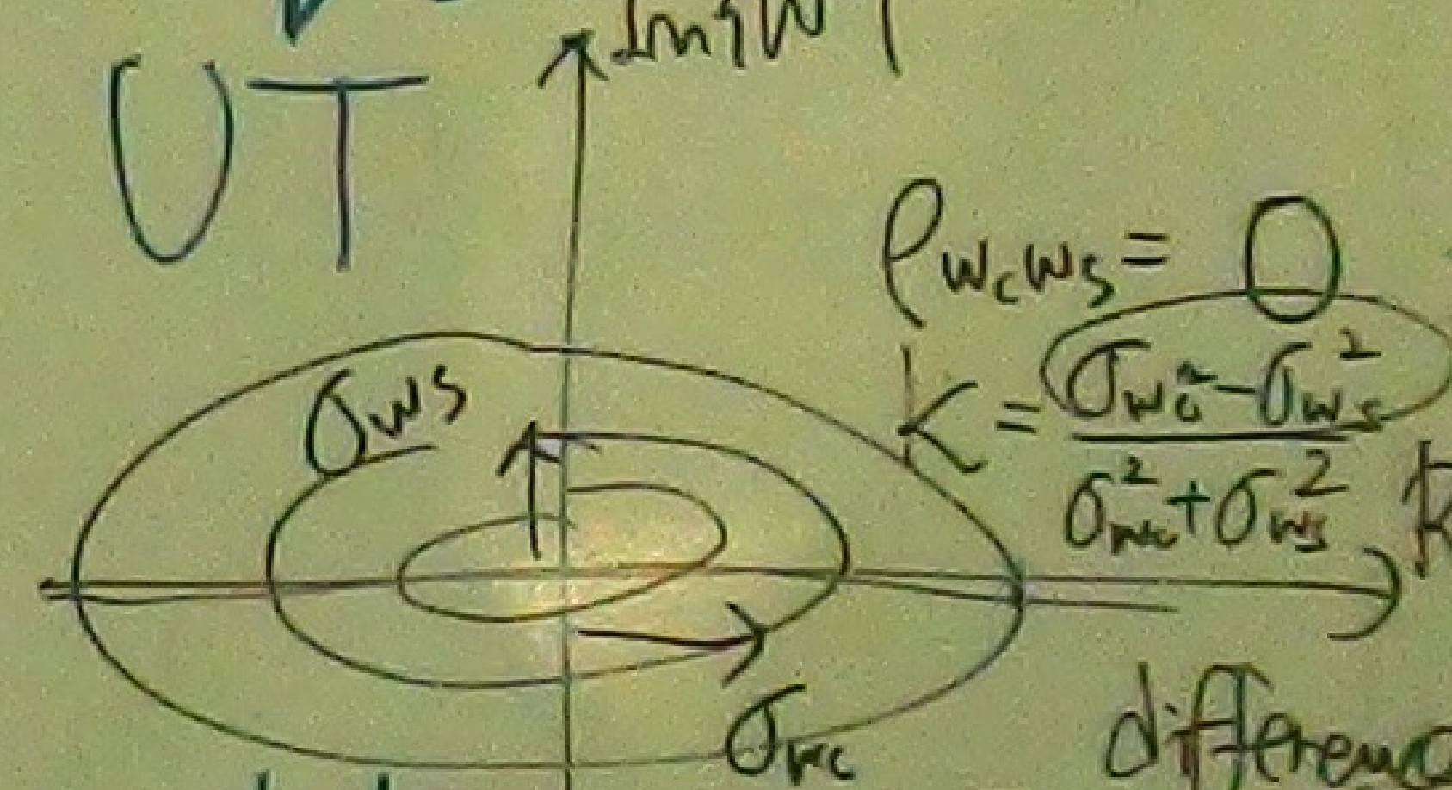
$\psi = -\frac{\phi}{2}$,

then $\widetilde{Var}\{W\} = k$.

$\therefore \theta = 2\phi$.







$\rho_{w_c w_s} = 0$ ≥ 0

$$k = \frac{\sigma_{wc}^2 - \sigma_{ws}^2}{\sigma_{wc}^2 + \sigma_{ws}^2}$$

$= \frac{\text{difference}}{\text{sum}}$

Uncorrelating Transform

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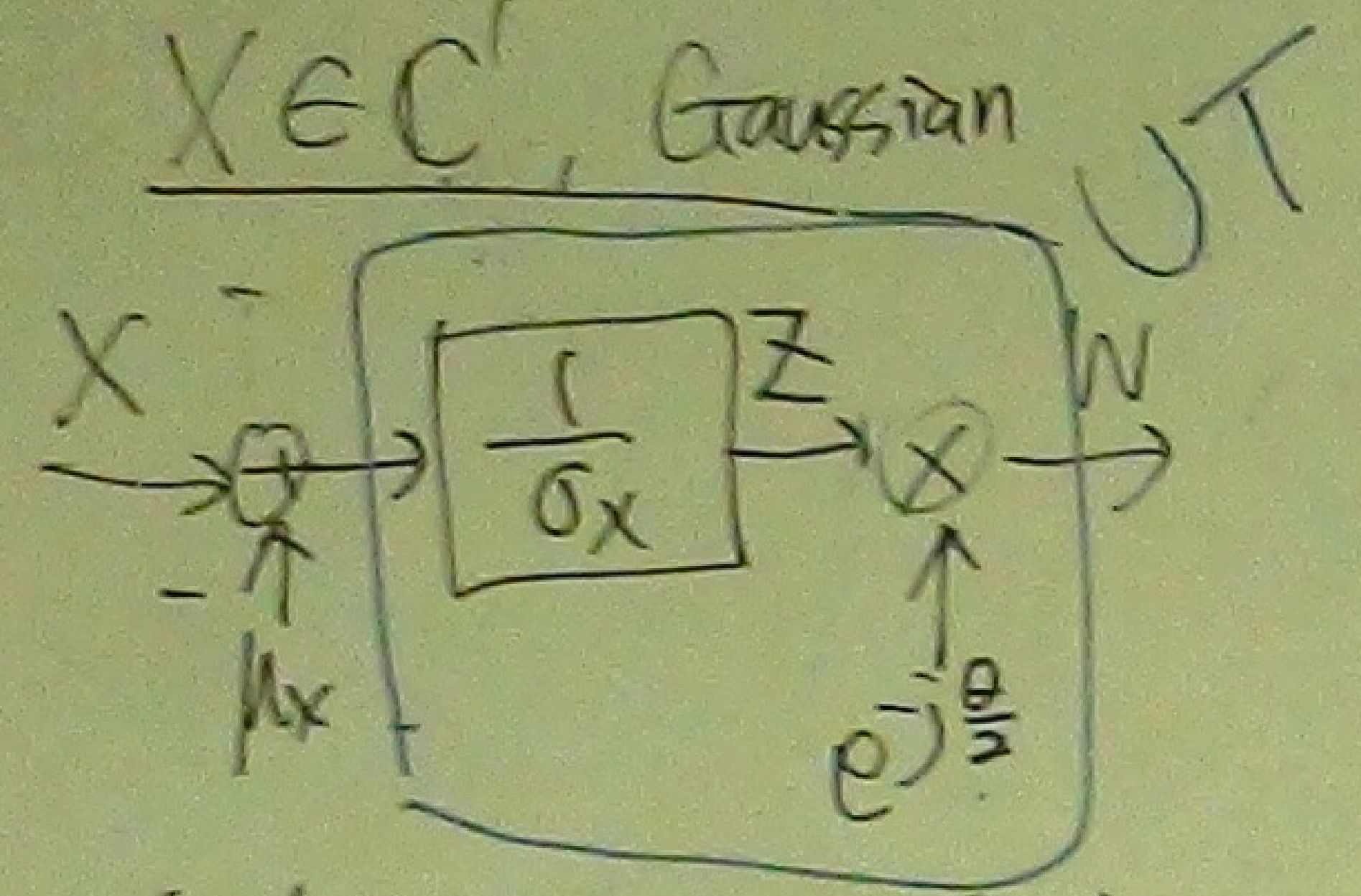
☐ $X \in \mathbb{C}^1$, Gaussian



$$Z = (\sigma_X^2)^{-\frac{1}{2}} (X - \mu_X)$$

$E\{Z\} = 0$

$\mathrm{Var}\{Z\} = 1$

$\widetilde{\mathrm{Var}}\{Z\} = k e^{j\theta}$

$E\{W\} = 0$

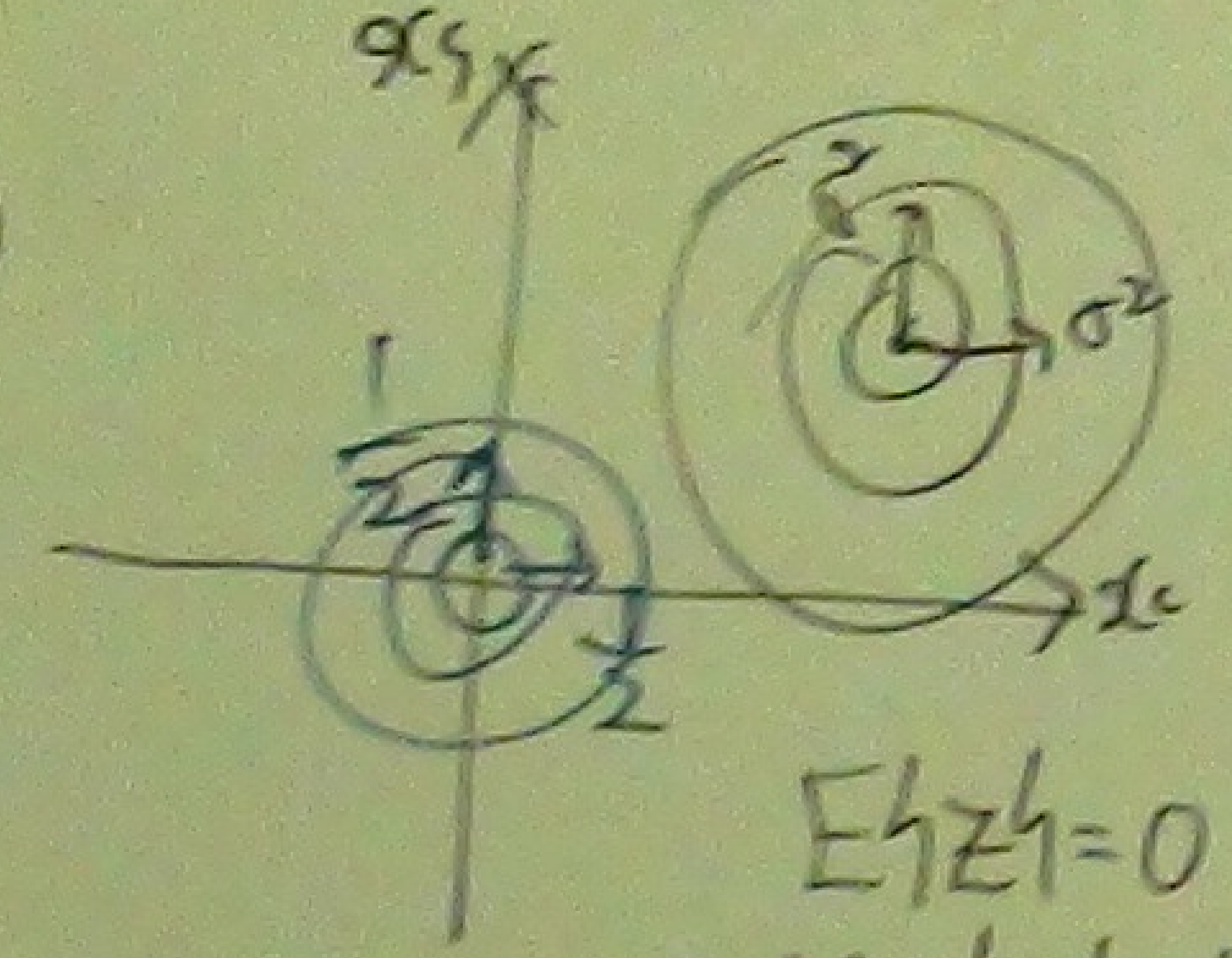
$\mathrm{Var}\{W\} = \mathrm{Var}\{Z e^{j\phi}\} = \mathrm{Var}\{Z\} = 1$

$\widetilde{\mathrm{Var}}\{W\} = e^{-j2\phi} \widetilde{\mathrm{Var}}\{Z\} = k.$

$\underset{\sim\sim\sim}{\theta = 2\phi.}$

$$= \frac{\sigma_{w_c}^2 - \sigma_{w_s}^2}{\sigma_{w_c}^2 + \sigma_{w_s}^2}$$

1)



$E\{X\} = \mu_X$

$\mathrm{Var}\{X_c\} = \mathrm{Var}\{X_s\} = \sigma^2 = \frac{\sigma_X^2}{2}$

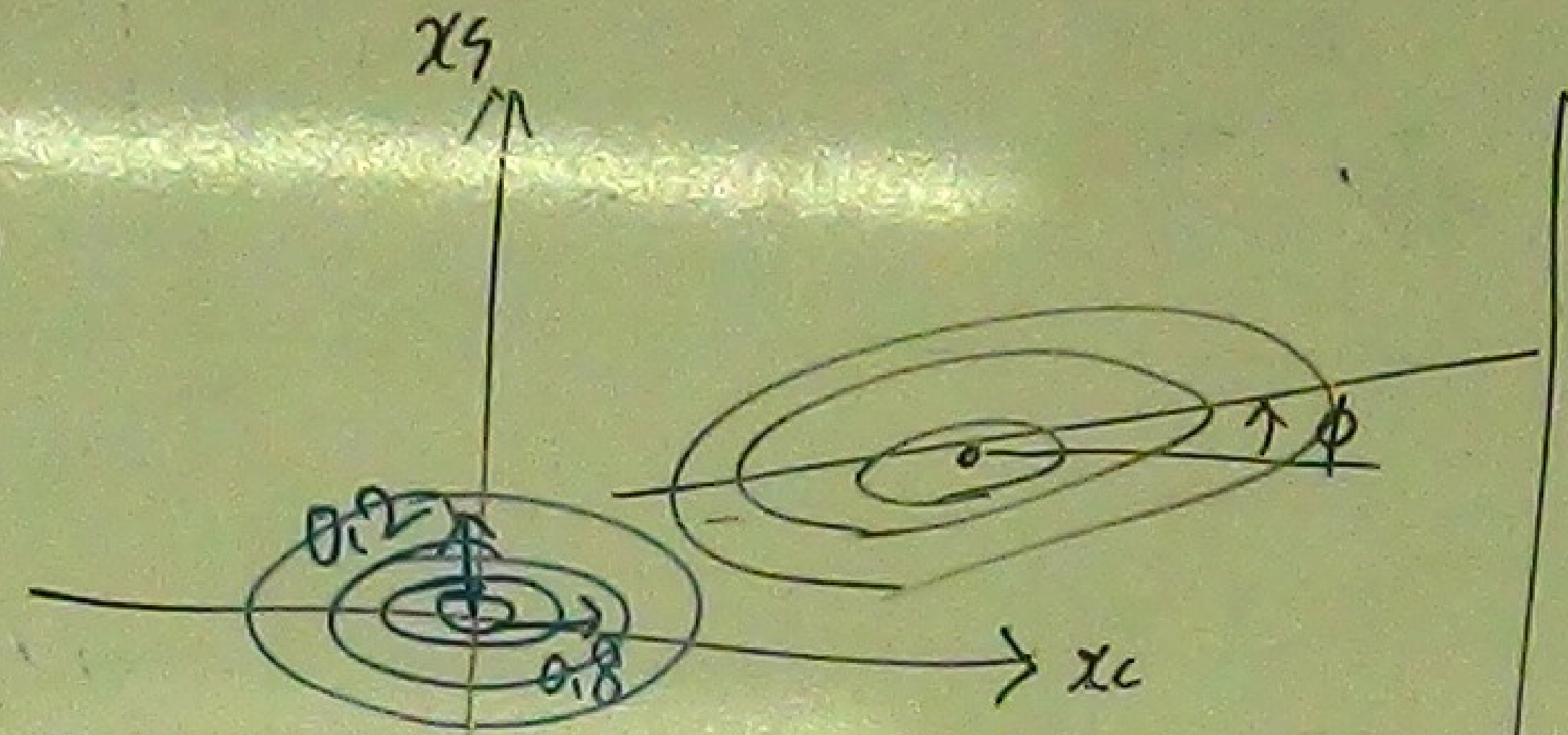
$\rho_{x_c x_s} = 0.$

$E\{Z\} = 0$

$\mathrm{Var}\{Z_c\} = \mathrm{Var}\{Z_s\} = \frac{1}{2} = \frac{1}{2} \underset{\overset{\sim}{1}}{\sigma_Z^2}$

$k = 0.$

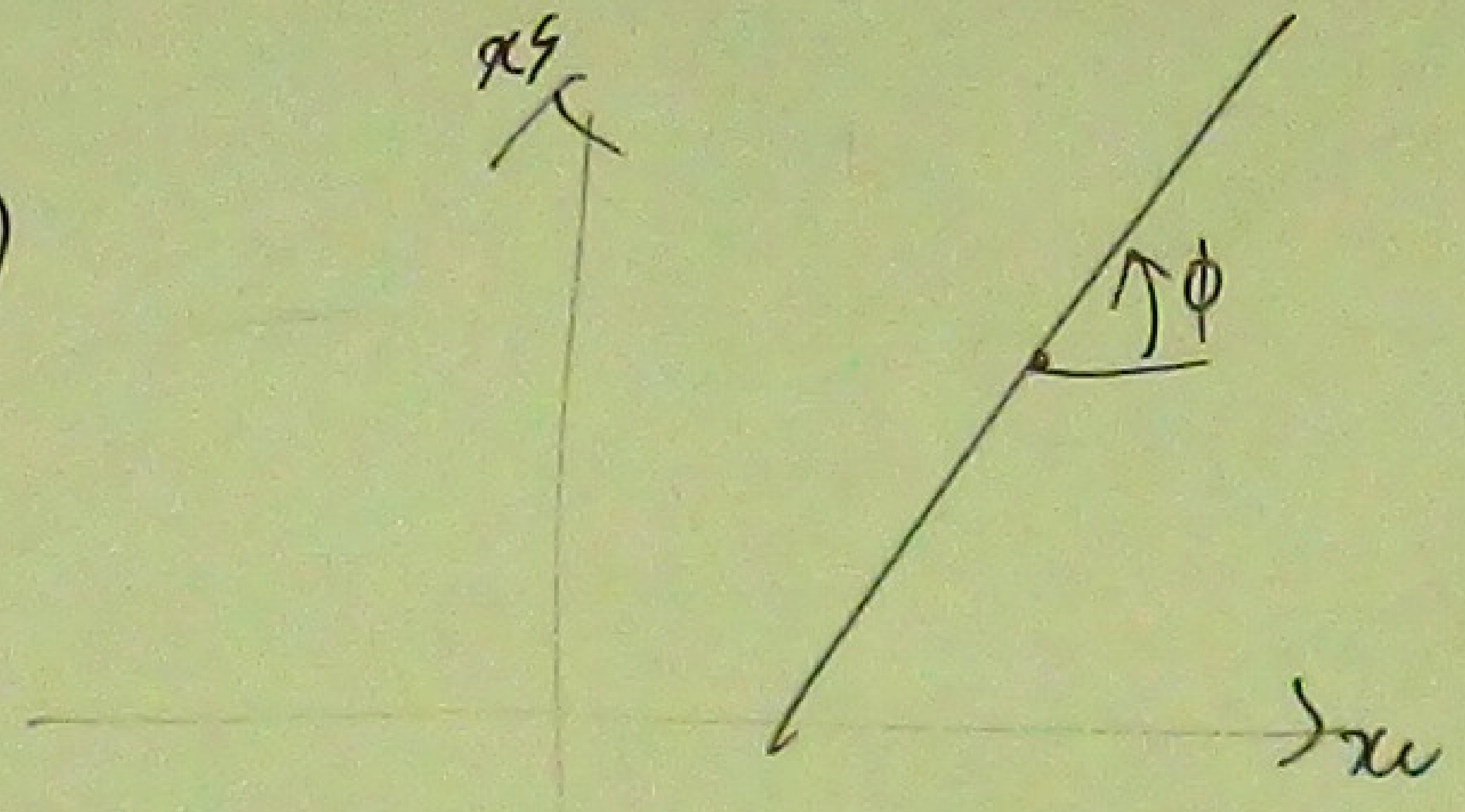
2)



$0 < |\rho_{x_c x_s}| < 1$

$0 < k = \frac{0.8 - 0.2}{1} = 0.6 < 1$

3)



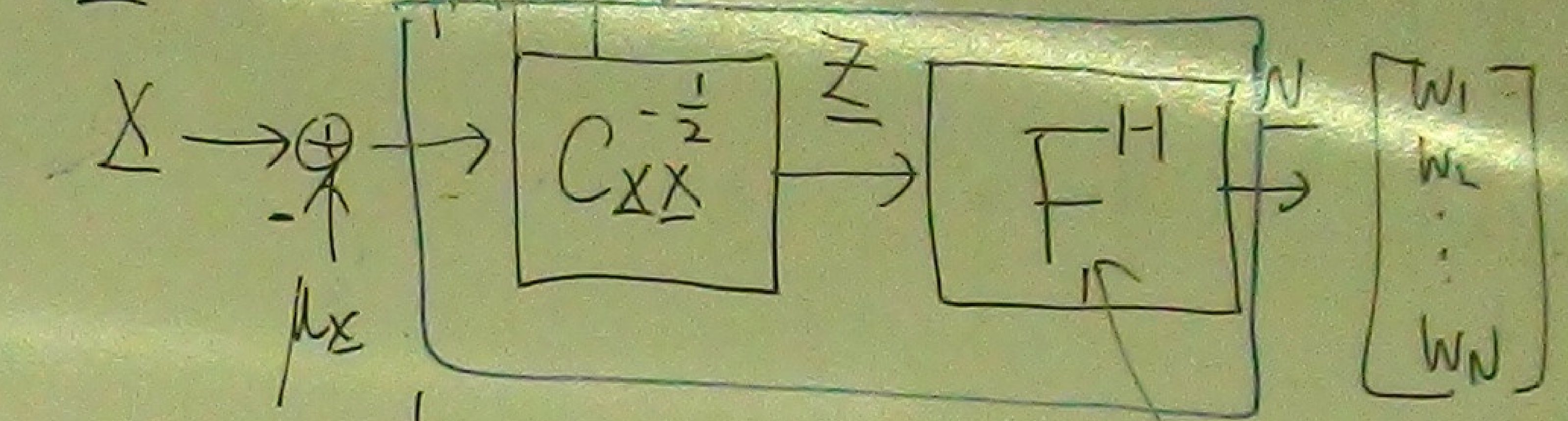
$k = 1$

$\widetilde{\mathrm{Var}}\{Z\} = e^{j2\phi}$

$\widetilde{\mathrm{Var}}\{W\} = 1$

☐ $\underline{X} \in \mathbb{C}^N$ improper

SUT



$\underline{Z} = C_{\underline{X}\underline{X}}^{-\frac{1}{2}} (\underline{X} - \mu_{\underline{X}})$

$E\{\underline{Z}\} = \underline{0}$

$\mathrm{Cov}\{\underline{Z}\} = E\{C_{\underline{X}\underline{X}}^{-\frac{1}{2}} (\underline{X} - \mu_{\underline{X}}) (\underline{X} - \mu_{\underline{X}})^H (C_{\underline{X}\underline{X}}^{-\frac{1}{2}})^H\}$

$= C_{\underline{X}\underline{X}}^{-\frac{1}{2}} C_{\underline{X}\underline{X}} (C_{\underline{X}\underline{X}}^{-\frac{1}{2}})^H = I.$

$\widetilde{\mathrm{Cov}}\{\underline{Z}\} = C_{\underline{X}\underline{X}}^{-\frac{1}{2}} \tilde{C}_{\underline{X}\underline{X}} (C_{\underline{X}\underline{X}}^{-\frac{1}{2}})^T \neq 0.$

Ⓠ. $E\{\underline{W}\} = \underline{0},$?

$\mathrm{Cov}\{\underline{W}\} = \boxed{I,}$

$\widetilde{\mathrm{Cov}}\{\underline{W}\} = \mathrm{diag}(k_1, k_2, \cdots, k_N)$

where $1 \geq k_1 \geq k_2 \geq \cdots \geq k_N \geq 0.$

Find F! if exits.

A. Yes!

$\mathrm{Cov}(w_i, w_j) = \delta_{i,j}$ ← Hermitian uncorrelated

$\widetilde{\mathrm{Cov}}(w_i. w_j) = k_i \delta_{i,j}$

strongly uncorrelated