EECE695M CM#13a

□ $X \in \mathbb{C}^1$.

$\mu_X \triangleq E\{X\}$

$C_{XX} = \sigma_X^2 \triangleq E\{|X-\mu_X|^2\} = \sigma_{X_c}^2 + \sigma_{X_s}^2$

$\tilde{C}_{XX} = \tilde{\sigma}_X^2 \triangleq E\{(X-\mu_X)^2\} = \sigma_{X_c}^2 - \sigma_{X_s}^2 + j2\rho_{X_cX_s}\sigma_{X_c}\sigma_{X_s}$

□ Def. The complex correlation coefficient of X between X & X^*

$\rho_X \triangleq \frac{\tilde{C}_{XX}}{C_{XX}} = \frac{\tilde{\sigma}_X^2}{\sigma_X^2} \in \mathbb{C}$

$\triangleq \underset{\uparrow}{k} e^{j\theta}$ $\quad 0 \le k \in \mathbb{R}$

circularity coefficient $\quad \theta \in [0, 2\pi)$

cf) real X & Y

$-1 \le \rho_{XY} \le 1$ $\quad |\rho_{XY}| \le$

$\rho_{XY} \triangleq \frac{C_{XY}}{\sqrt{C_{XX}C_{YY}}}$

$E\{(a\frac{X-\mu_X}{\sigma_X} - \frac{Y-\mu_Y}{\sigma_Y})^2\} \ge 0$

$\forall a \in \mathbb{R}$

$D/4 = \cdots \le 0$

□ Properties of ρ & k.

(i) $0 \le |\rho| \le 1$

$\Leftrightarrow 0 \le k \le 1$

(ii) k is invariant under strongly Linear/Affine transformation.

proof 1) $X = X_c + jX_s$; $\rho_{X_cX_s}$, $\sigma_{X_c}^2, \sigma_{X_s}^2, \mu_{X_c}, \mu_{X_s}$.

$\rho = \frac{\sigma_{X_c}^2 - \sigma_{X_s}^2 + j2\rho_{X_cX_s}\sigma_{X_c}\sigma_{X_s}}{\sigma_{X_c}^2 + \sigma_{X_s}^2}$

$k^2 = |\rho|^2 = \cdots = \le 1.$

$\therefore 0 \le |\rho| \le 1$

$0 \le k \le 1$

proof 2) $Y = aX + b$, $a, b \in \mathbb{C}^1$

$a = re^{j\phi}$

$\frac{\tilde{\sigma}_Y^2}{\sigma_Y^2} = \frac{a^2\tilde{\sigma}_X^2}{|a|^2\sigma_X^2} = \rho_X e^{j2\phi} = k e^{j(\theta+2\phi)}$

$\Rightarrow \left|\frac{\tilde{\sigma}_Y^2}{\sigma_Y^2}\right| = \left|\frac{\tilde{\sigma}_X^2}{\sigma_X^2}\right| = k.$

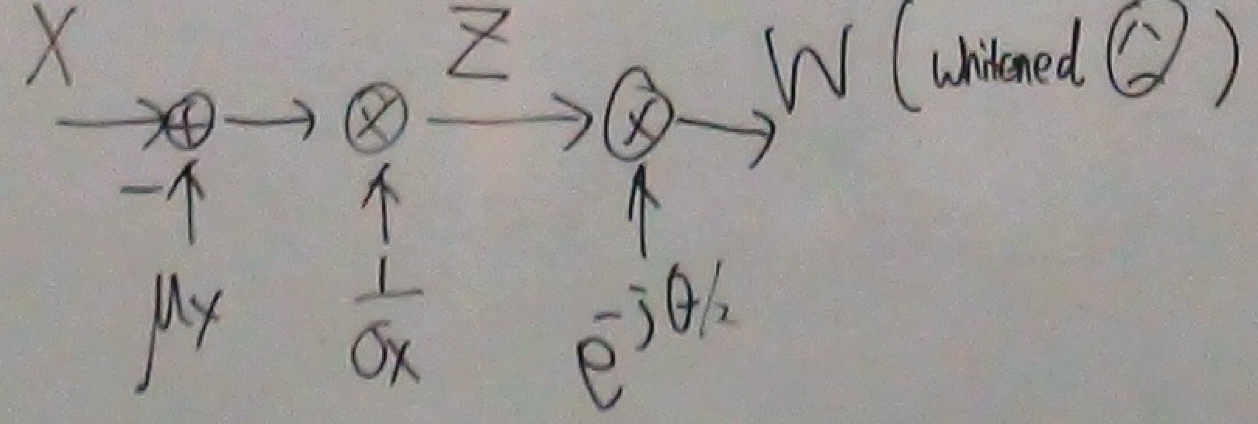
□ Example

$X \sim \tilde{\mathcal{C}}N(\mu_X, \sigma_X^2, \tilde{\sigma}_X^2)$

$X \sim \tilde{\mathcal{C}}N(\mu_X, \sigma_X^2, \rho_X)$

~~$X \sim \tilde{\mathcal{C}}N(\mu_X, \sigma_X^2, k)$~~

□ UT (uncorrelating transformation)



$E\{Z\} = 0$

$\sigma_Z^2 = 1$

$\tilde{\sigma}_Z^2 = \tilde{\sigma}_X^2/\sigma_X^2 = \rho_X = ke^{j\theta}$

$E\{W\} = 0$

$\sigma_W^2 = 1 = \sigma_{W_c}^2 + \sigma_{W_s}^2$

$\tilde{\sigma}_W^2 = k = \frac{\sigma_{W_c}^2 - \sigma_{W_s}^2}{\sigma_{W_c}^2 + \sigma_{W_s}^2} = \frac{\text{Difference}}{\text{SUM}}$, $\sigma_{W_c} \ge \sigma_{W_s}$

$\rho_{W_rW_i} = 0$ ∴ "uncorrelating" T.

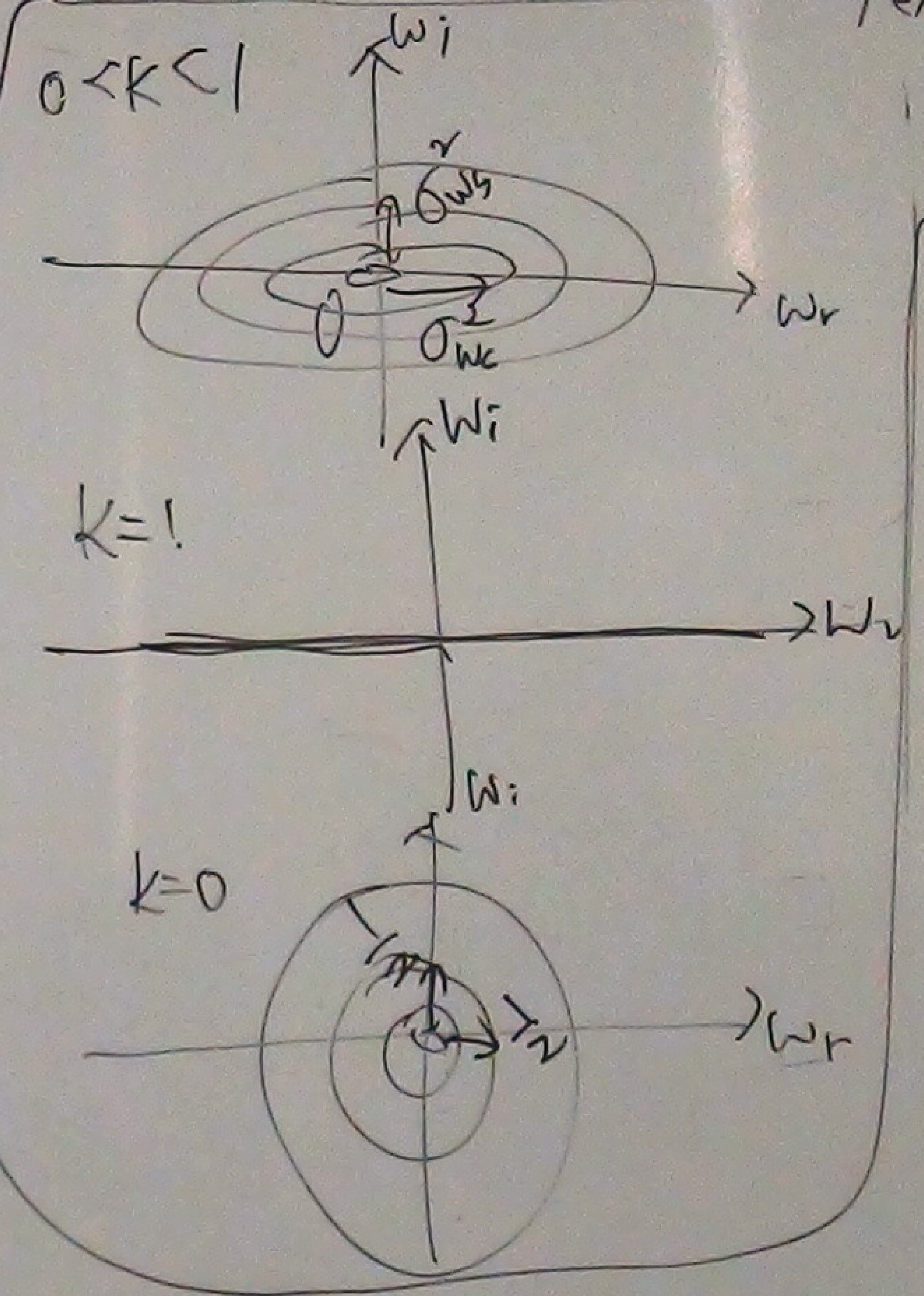
$\sigma_{W_c}^2 = \frac{1+k}{2}$

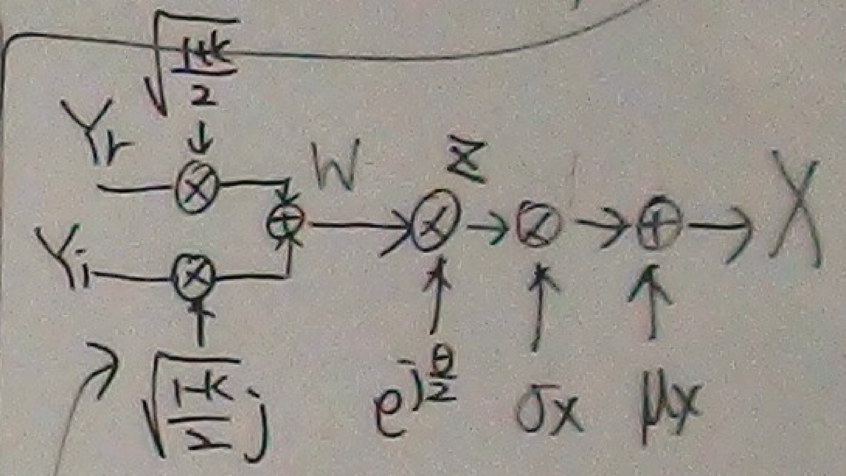
$\sigma_{W_s}^2 = \frac{1-k}{2}$

/ex/

$W \sim \tilde{\mathcal{C}}N(0, 1, k)$

correlation coefficient





$ke^{j\theta} = \frac{\tilde{\sigma}_X^2}{\sigma_X^2}$

□ Generation of an improper complex G. r.v.

$X \sim \mathcal{C}N(\mu_X, \sigma_X^2, \tilde{\sigma}_X^2)$! from $Y_r, Y_i \sim N(0,1)$ indep.

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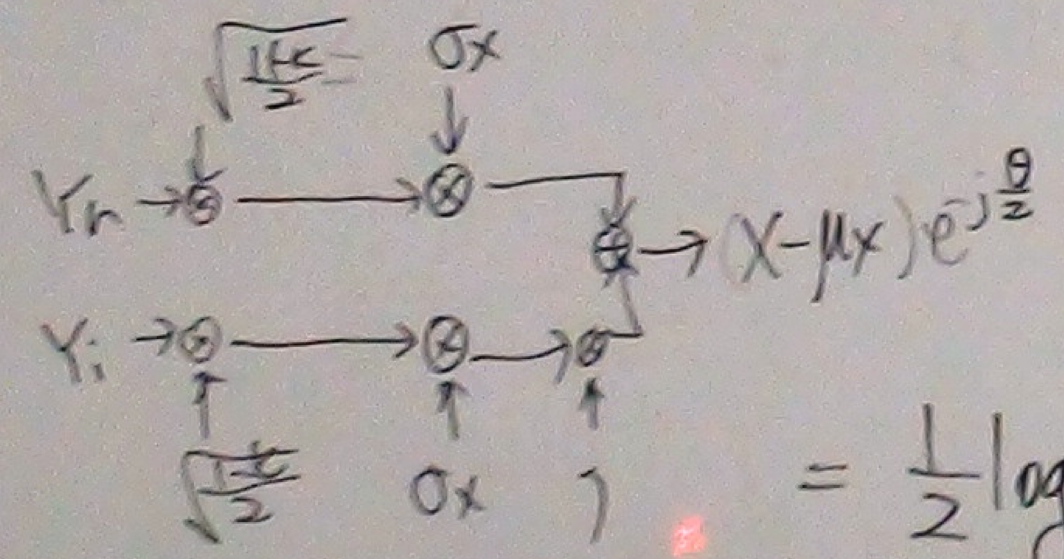
□ $X \sim \tilde{\mathcal{CN}}(\mu_X, \sigma_X^2, \rho_X = \kappa e^{j\theta})$

pdf

$$f_X(x) = \frac{1}{\pi\sigma_X^2\sqrt{1-\kappa^2}} \exp\left(-\frac{|x-\mu_X|^2 - \mathrm{Re}\{\rho_X((x-\mu_X)^*)^2\}}{\sigma_X^2(1-\kappa^2)}\right)$$

□ Differential Entropy of $X \sim \tilde{\mathcal{CN}}(\mu_X, \sigma_X^2, \tilde{\sigma}_X^2 = \rho_X\sigma_X^2)$

$$h(X) = h\left((X-\mu_X)e^{-j\frac{\theta}{2}}\right) = h\left(\sigma_X\sqrt{\tfrac{1+\kappa}{2}}\,Y_r\right) + h\left(\sigma_X\sqrt{\tfrac{1-\kappa}{2}}\,Y_i\right)$$



$$= \frac{1}{2}\log\left(2\pi e\,\frac{\sigma_X^2(1+\kappa)}{2}\right) + \frac{1}{2}\log\left(2\pi e\,\frac{\sigma_X^2(1-\kappa)}{2}\right)$$

$$= \frac{1}{2}\log \pi e\sigma_X^2 + \frac{1}{2}\log(1+\kappa) + \frac{1}{2}\log \pi e\sigma_X^2 + \frac{1}{2}\log(1-\kappa)$$

$$= \underbrace{\log(\pi e\sigma_X^2)}_{h(X)\text{ when }X\text{ is proper}} + \underbrace{\frac{1}{2}\log(1-\kappa^2)}_{(-)\text{ due to impropriety}} \le \log(\pi e\sigma_X^2)$$

where equality holds iff X is proper Gaussian.

□ $\underline{X} \sim \mathbb{C}^N$ improper

circularity coefficients ?

correlation 〃 ? $\rho \triangleq \frac{\tilde{\sigma}_X^2}{\sigma_X^2} = \frac{\tilde{C}_{XX}}{C_{XX}}$ coherence coefficient.

C_{XX}

$\tilde{C}_{XX}$

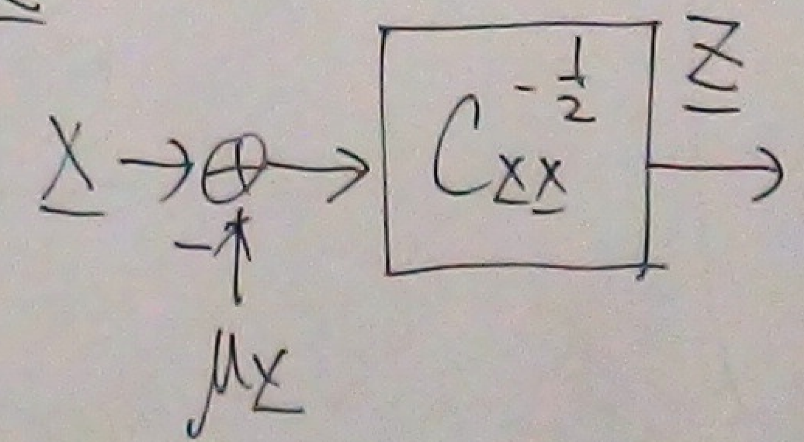
cf) real Gaussian X & Y

$$\mu_{X|Y}(y) = \mu_X + \frac{\sigma_{XY}}{\sigma_{YY}}(y-\mu_Y)$$

real Gaussian $\underline{X}$ & $\underline{Y}$

$$\mu_{\underline{X}|\underline{Y}}(\underline{y}) = \mu_{\underline{X}} + C_{XY}C_{YY}^{-1}(\underline{y}-\mu_{\underline{Y}})$$

□ $\underline{Z}$ vector & coherence Mx of $\underline{X}$



$$C_{\underline{X}\underline{X}} = C_{\underline{X}\underline{X}}^H = V\Lambda V^H$$

where $VV^H = V^HV = I$

$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$

$\lambda_i \ge 0, \forall i$

$$C_{\underline{X}\underline{X}}^{\frac{1}{2}} \triangleq V\Lambda^{\frac{1}{2}}V^H, \quad C_{\underline{X}\underline{X}}^{-\frac{1}{2}} = V\Lambda^{-\frac{1}{2}}V^H = \left(C_{\underline{X}\underline{X}}^{-\frac{1}{2}}\right)^H$$

$E\{\underline{Z}\} = \underline{0}$

$$C_{\underline{Z}\underline{Z}} = E\{\underline{Z}\underline{Z}^H\} = E\{C_{\underline{X}\underline{X}}^{-\frac{1}{2}}(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})^H C_{\underline{X}\underline{X}}^{-\frac{1}{2}}\}$$

$$= C_{\underline{X}\underline{X}}^{-\frac{1}{2}}C_{XX}C_{\underline{X}\underline{X}}^{-\frac{1}{2}} = V\Lambda^{-\frac{1}{2}}V^H V\Lambda V^H V\Lambda^{-\frac{1}{2}}V^H = I$$

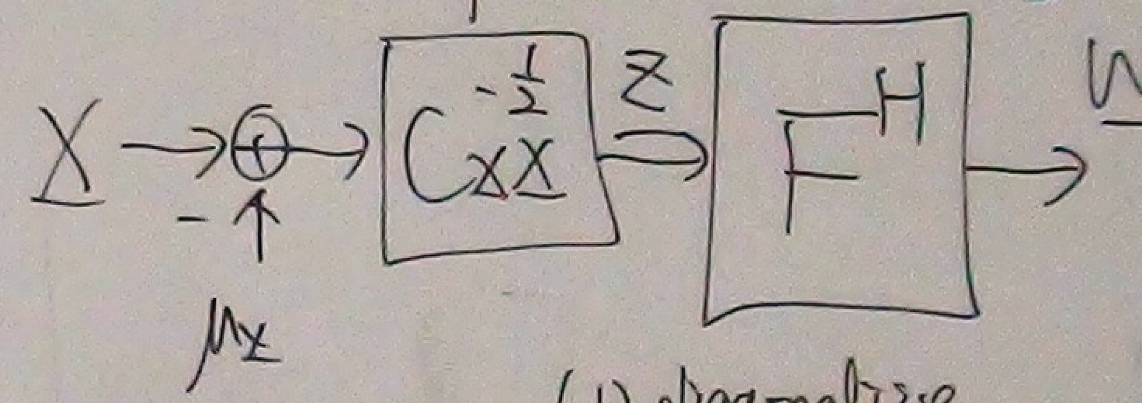
$$\tilde{C}_{\underline{Z}\underline{Z}} = E\{\underline{Z}\underline{Z}^T\} = \cdots = \boxed{C_{\underline{X}\underline{X}}^{-\frac{1}{2}}\tilde{C}_{\underline{X}\underline{X}}\left(C_{\underline{X}\underline{X}}^{-\frac{1}{2}}\right)^T}$$

coherence matrix

Recall if $X \in \mathbb{C}^1$

$$\tilde{C}_{ZZ} = \frac{\tilde{\sigma}_X^2}{\sigma_X^2} = \rho$$

□ What corresponds to κ? SUT



(1) diagonalize

(2) phase rotation

$C_{\underline{W}\underline{W}} = I$

$\tilde{C}_{\underline{W}\underline{W}} = \mathrm{diag}\{\kappa_1, \kappa_2, \cdots, \kappa_N\}$

$0 \le \kappa_n \le 1.$

$C_{\underline{W}\underline{W}} = F^HF = I$

☺ F must be unitary.

$$\tilde{C}_{\underline{W}\underline{W}} = F^H C_{\underline{X}\underline{X}}^{-\frac{1}{2}}\tilde{C}_{\underline{X}\underline{X}}\left(C_{\underline{X}\underline{X}}^{-\frac{1}{2}}\right)^T F^* = K$$

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□ Takagi Factorization of a complex symmetric matrix $\tilde{C} = \tilde{C}^T \in \mathbb{C}^{N \times N}$

$$\tilde{C} = FKF^T$$

where F unitary, $K = \mathrm{diag}\{k_1, k_2, \cdots, k_N\}$ w/ $0 \le k_n$

□ $\tilde{C} = UKV^H$ (SVD)

$= \tilde{C}^T$

$= V^* K U^T$

$\tilde{C}\tilde{C}^H = UKV^H VKU^H = UK^2U^H$

$= V^* K U^T (\quad)^H = V^* K^2 V^T$

$UK^2U^H = V^*K^2V^T$

$\boxed{K^2}\boxed{U^HV^*} = \boxed{U^HV^*}\boxed{K^2}$

diagonal $\quad \underset{A}{\|} \quad$ diagonal

$D = U^H V^*$ is diagonal.

$D = \mathrm{diag}(d_1, d_2, \cdots, d_N)$

$|d_n| = 1, \forall n$

$= \mathrm{diag}(e^{j\theta_1}, e^{j\theta_2}, \cdots, e^{j\theta_N})$

$KD = DK = KU^HV^*$

$\Rightarrow \tilde{C} = UKDU^T$

$= \boxed{UD^{\frac{1}{2}}}K\boxed{D^{\frac{1}{2}}U^T}$

where $D^{\frac{1}{2}} = \mathrm{diag}(e^{j\frac{\theta_1}{2}}, e^{j\frac{\theta_2}{2}}, \cdots)$

$= FKF^T$

□ Invariance of K matrix under strongly linear/Affine TF.

.....

□ pdf

□ $h(\underline{X}) \quad \underline{X} \sim \tilde{\mathcal{CN}}(\mu_x, C_{xx}, \tilde{C}_{xx})$

$=$ (proper) $+ \frac{1}{2}\log \prod_{n=1}^{N}(1-k_n^2). \quad = K..$

$\boxed{= C_{xx}^{-\frac{1}{2}} \tilde{C}_{xx} (C_{xx}^{-\frac{1}{2}})^T}$ → coherence matrix

□ What corresponds to K? of $\underline{X}$ SUT

$$\underline{X} \rightarrow \oplus \rightarrow \boxed{C_{xx}^{-\frac{1}{2}}} \xrightarrow{\underline{z}} \boxed{F^H} \rightarrow \underline{W}$$

$-\mu_x$

(1) diagonalize

(2) phase rotation

$C_{ww} = I$

$\tilde{C}_{ww} = \mathrm{diag}\{k_1, k_2, \cdots, k_N\}$

$0 \le k_n \le 1.$

$C_{ww} = F^H F = I$

☺ F must be unitary.

$\tilde{C}_{ww} = F^H C_{xx}^{-\frac{1}{2}} \tilde{C}_{xx} (C_{xx}^{-\frac{1}{2}})^T F^* = K$

FKF^T