EECE695M CM#13C

☐ Takagi Factorization of a complex symmetric matrix $\tilde{C} = \tilde{C}^T \in \mathbb{C}^{N \times N}$

$$\tilde{C} = FKF^T$$

where F unitary, $K = \text{diag}\{k_1, k_2, \cdots, k_N\}$ w/ $0 \le k_n$

☐ $\tilde{C} = UKV^H$ (SVD)

$$= C^T$$

$$= V^* K U^T$$

$$\tilde{C}\tilde{C}^H = UK\cancel{V^H V}KU^H = UK^2U^H$$

$$= V^*KU^T(\quad)^H = V^*K^2V^T))$$

$$UK^2U^H = V^*K^2V^T$$

$$\boxed{K^2}\boxed{U^HV^*} = \boxed{U^HV^*}\boxed{K^2}$$

diagonal $\quad\quad$ diagonal

$\|$

A

[illegible]

$D = U^HV^*$ is diagonal.

$D = \text{diag}(d_1, d_2, \cdots, d_N)$

$|d_n| = 1, \forall n$

$= \text{diag}(e^{j\theta_1}, e^{j\theta_2}, \cdots, e^{j\theta_N})$

$$KD = DK = KU^HV^*$$

$\Rightarrow$

$$\tilde{C} = UKDU^T$$

$$= \boxed{UD^{\frac{1}{2}}}K\boxed{D^{\frac{1}{2}}U^T}$$

where $D^{\frac{1}{2}} = \text{diag}(e^{j\frac{\theta_1}{2}}, e^{j\frac{\theta_2}{2}}, \cdots)$

$$= FKF^T$$

☐ Invariance of K matrix

under invertible strongly linear/Affine TF.

.....

☐ $h(\underline{X})$ $\quad \underline{X} \sim \tilde{C}N(\mu_{\underline{X}}, C_{\underline{XX}}, \tilde{C}_{\underline{XX}})$

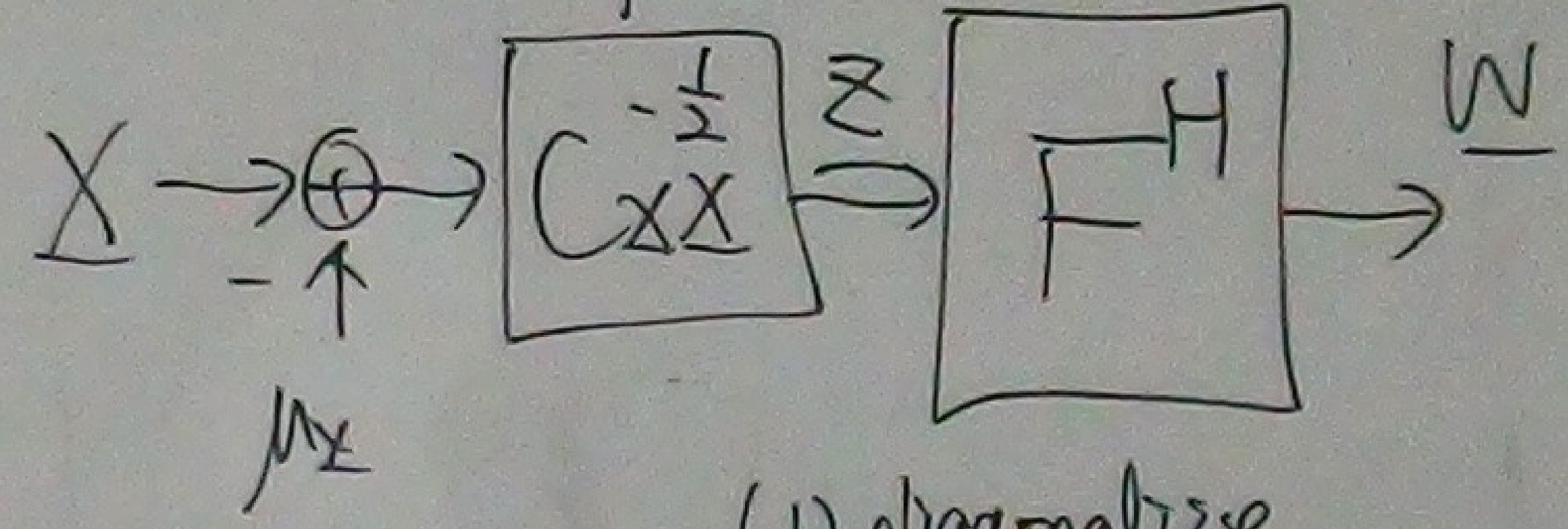
$$= \text{(proper)} + \frac{1}{2}\log\prod_{n=1}^{N}(1-k_n^2). \quad = K..$$

$$\boxed{= C_{\underline{XX}}^{-\frac{1}{2}}\tilde{C}_{\underline{XX}}(C_{\underline{XX}}^{-\frac{1}{2}})^T}$$

$\rightarrow$ coherence matrix

☐ What corresponds to k? SUT

of $\underline{X}$



(1) diagonalize

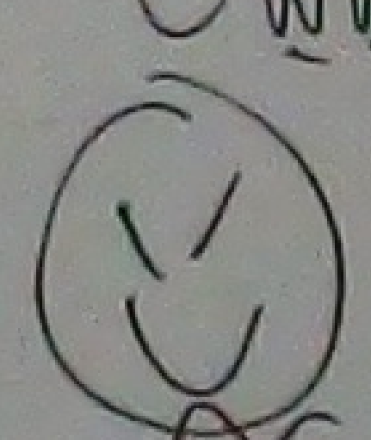
(2) phase rotation

$C_{\underline{WW}} = I$

$\tilde{C}_{\underline{WW}} = \text{diag}\{k_1, k_2, \cdots, k_N\}$

$0 \le k_n \le 1.$

$C_{\underline{WW}} = F^HF = I$



F must be unitary.

$$\tilde{C}_{\underline{WW}} = F^H\left(C_{\underline{XX}}^{-\frac{1}{2}}\tilde{C}_{\underline{XX}}(C_{\underline{XX}}^{-\frac{1}{2}})^T\right)F^* = K$$

$$FKF^T$$

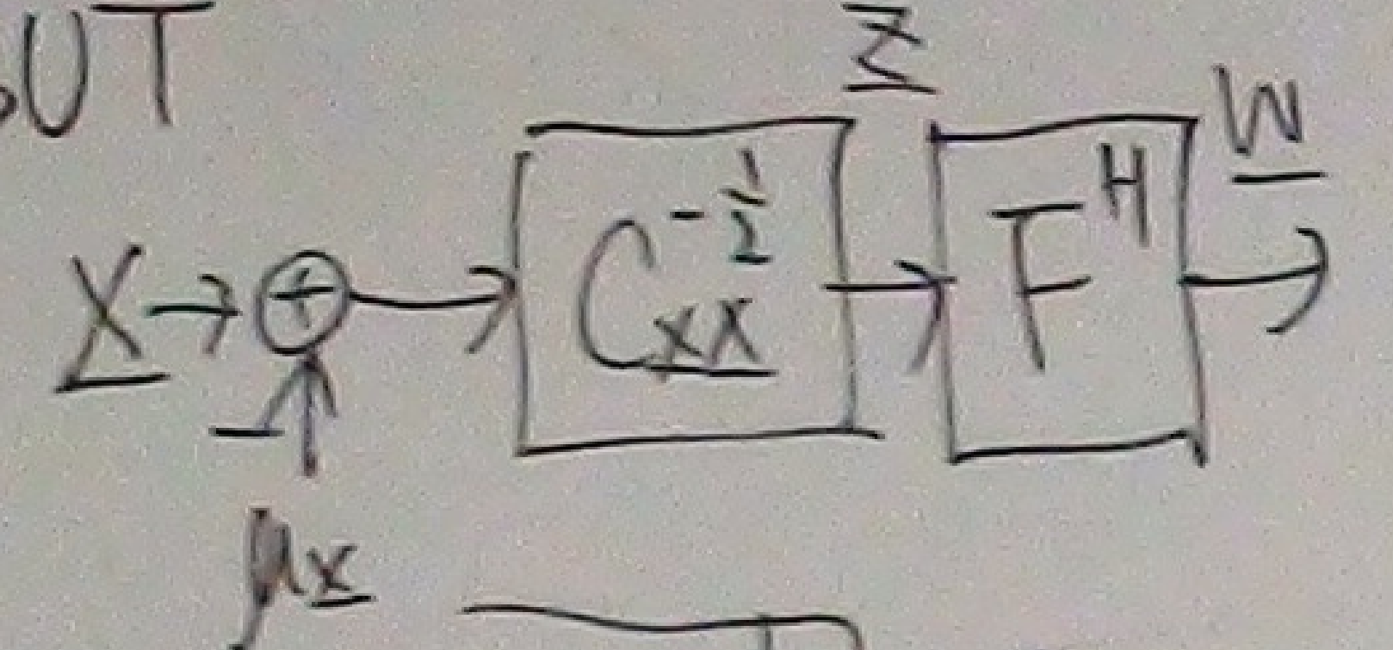
EECE695M CM#14a

□ Generation of Improper-Complex Gaussian

r. vector $\underline{X} \sim \tilde{C}N(\mu_{\underline{x}}, C_{\underline{xx}}, \tilde{C}_{\underline{xx}}) \in \mathbb{C}^N$

$$C_{\underline{xx}}^{-\frac{1}{2}} \tilde{C}_{\underline{xx}} (C_{\underline{xx}}^{-\frac{1}{2}})^T = FKF^T$$

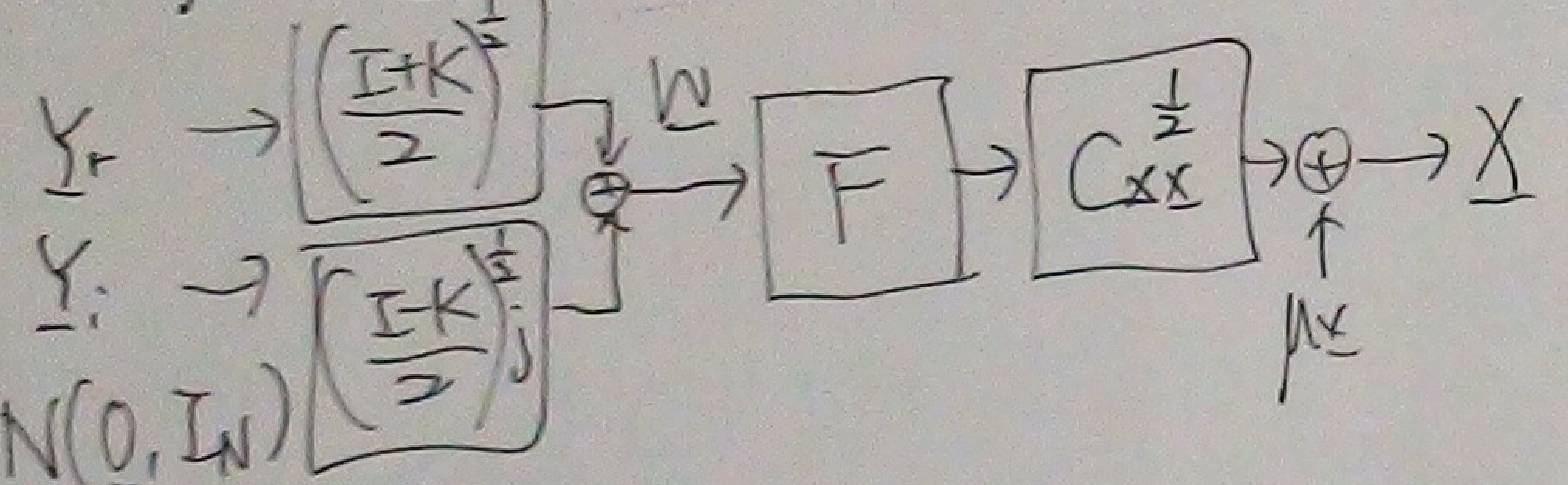
SUT



$E\{\underline{W}\} = \underline{0}$

$C_{\underline{ww}} = I$

$\tilde{C}_{\underline{ww}} = K.$



$\sim N(\underline{0}, I_N)$

i.i.d.

□ h(X), $\underline{X} \sim$ "

$$h(\underline{X}) = \underbrace{\log\{(\pi e)^N \det C_{\underline{xx}}\}}_{\text{proper 일때의 differential entropy}} + \underbrace{\frac{1}{2}\log\left\{\prod_{n=1}^{N}(1-k_n^2)\right\}}_{(-)} \le \log\{(\pi e)^N \det C_{\underline{xx}}\}$$

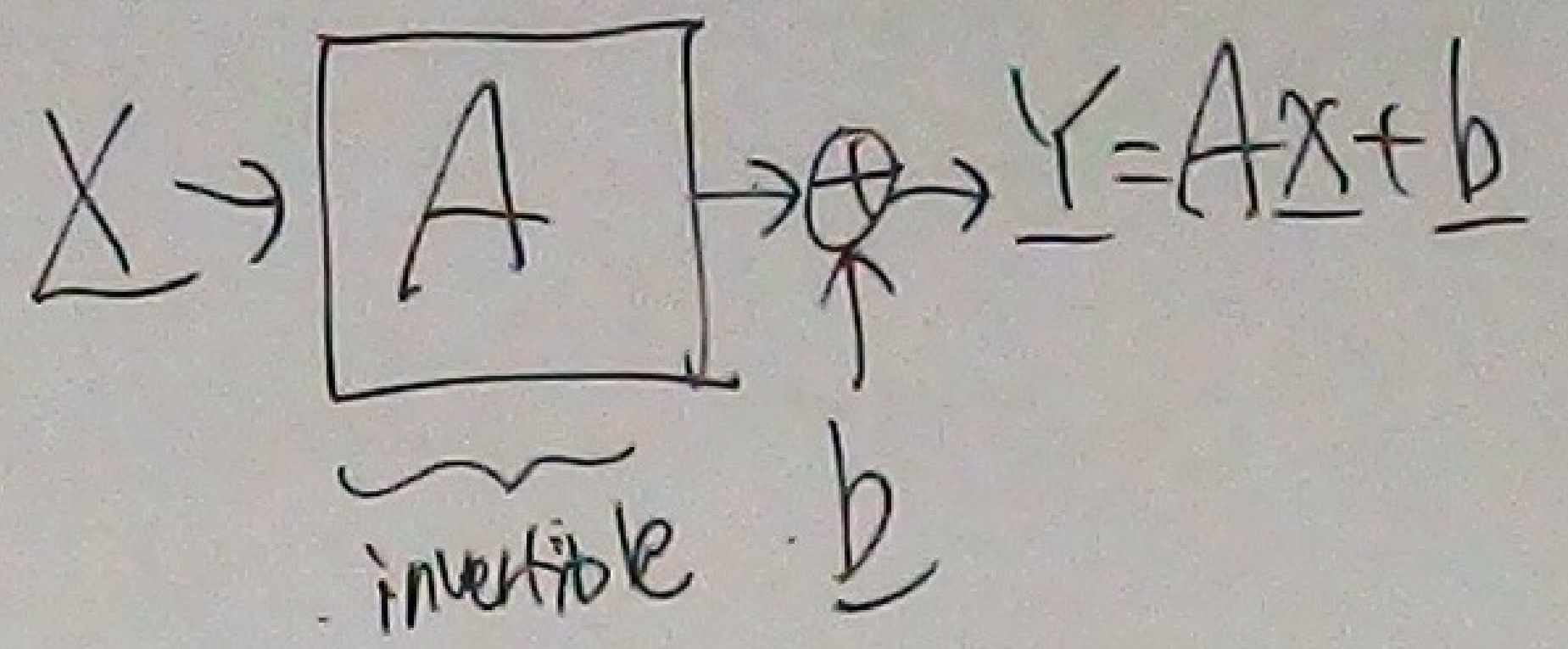
where equality holds iff

$K = 0$ .. proper

" $K = I$ maximally improper r. vector "

□ Invariance of K under invertible Strongly Linear/Affine TF

unitary



$E\{\underline{Y}\} = A\mu_{\underline{x}} + \underline{b}$

$C_{\underline{YY}} = AC_{\underline{xx}}A^H$

$\tilde{C}_{\underline{YY}} = A\tilde{C}_{\underline{xx}}A^T$

$$C_{\underline{YY}}^{-\frac{1}{2}} \tilde{C}_{\underline{YY}} (C_{\underline{YY}}^{-\frac{1}{2}})^T = \tilde{F}\,\textcircled{K}\,\tilde{F}^T$$

$$C_{\underline{xx}}^{-\frac{1}{2}} \tilde{C}_{\underline{xx}} (C_{\underline{xx}}^{-\frac{1}{2}})^T = FKF^T$$

□ Degree of Impropriety.

$K \doteq \mathrm{diag}(k_1, k_2, \cdots, k_N)$

$1 - \prod_{i=1}^{N}(1-k_i^2)$ = 0 iff K = 0. =1 iff $\exists k_i = 0$

$\prod_{i=1}^{N} k_i^2$ = 0 iff at least one $k_i = 0$ = 1 iff $\forall k_i = 1$

$\frac{1}{N}\sum_{i=1}^{N} k_i^2$ $\boxed{\begin{array}{l} = 0 \text{ iff } K = 0. \\ = 1 \text{ iff } K = \underline{I}. \end{array}}$

$0 \le \quad \le 1.$

□ W/O structure to be exploited working w/ complex r.vector is messy.

$$\begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix} = T \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \qquad T = \begin{bmatrix} I & jI \\ I & -jI \end{bmatrix}$$

$$\frac{1}{2}T^H T = I$$

□ Unitary TF

$\underline{x} = \underline{x}_c + j\underline{x}_s$

$$\begin{bmatrix} \underline{y}_c \\ \underline{y}_s \end{bmatrix} = U \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix} \iff \begin{bmatrix} \underline{y} \\ \underline{y}^* \end{bmatrix} = \boxed{\begin{matrix} A & B \\ B^* & A^* \end{matrix}} \begin{bmatrix} \underline{x} \\ \underline{x}^* \end{bmatrix}$$

$UU^H = U^H U = \underline{I}$

$\iff \underline{y} = A\underline{x} + B\underline{x}^*$

$$\boxed{\frac{1}{2}TUT^H}$$

unitary

EECE695M CM#14b §3.1-3.3

☐ augmented EVD.

$\underline{X} = \underline{X}_c + j\underline{X}_s$

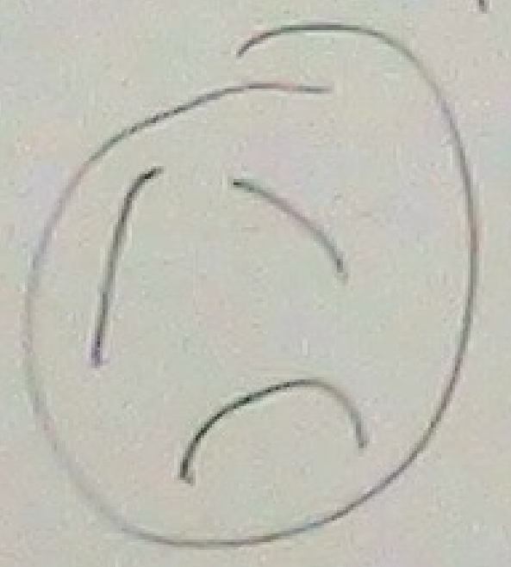
$$\mathrm{Cov}\left(\begin{bmatrix}\underline{X}_c\\ \underline{X}_s\end{bmatrix}\right) = V\Lambda V^T = V\begin{bmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{bmatrix}V^T$$

$VV^T = V^TV = I$

$C_{\underline{XX}}, \tilde{C}_{\underline{XX}}$

$$\mathrm{Cov}\left(\begin{bmatrix}\underline{X}\\ \underline{X}^*\end{bmatrix}\right) = \begin{bmatrix}C_{\underline{XX}} & \tilde{C}_{\underline{XX}}\\ \tilde{C}^*_{\underline{XX}} & C^T_{\underline{XX}}\end{bmatrix} = \underbrace{\left(\frac{1}{2}TVT^H\right)}_{\text{W unitary TF matrix}} \frac{1}{2}\begin{pmatrix}\Lambda_1+\Lambda_2 & \Lambda_1-\Lambda_2\\ \Lambda_1-\Lambda_2 & \Lambda_1+\Lambda_2\end{pmatrix}\left(\qquad\right)^H$$

NOT diagonal.
BUT diagonal blockmatrices.



☐ PCA

$\underline{X} \in \mathbb{R}^N$

$\mu_{\underline{X}}$

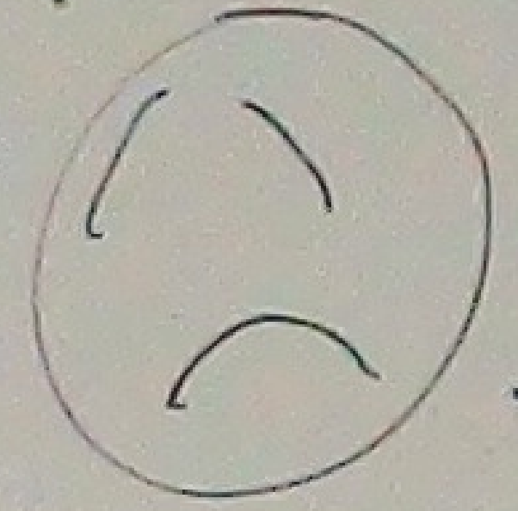
$C_{\underline{XX}} = V\Lambda V^T$

$$\underline{X} \longrightarrow \boxed{V^T} \longrightarrow \underline{Y} = \begin{bmatrix}Y_1\\ Y_2\\ \vdots\\ Y_N\end{bmatrix}$$

$$R_{\underline{XX}} = \mu_X\mu_X^T + V\Lambda V^T = \tilde{V}\tilde{\Lambda}\tilde{V}^T$$

$$\underline{X} \longrightarrow \boxed{\tilde{V}^T} \longrightarrow \tilde{\underline{Y}}$$

Improper-Complex



☐ ICA

$$\underline{X} \to \boxed{M} \to \underline{Y}\ \boxed{M^{+}} \to \underline{Z}$$

✡ KL Expansion $X(t), t\in[0,T)$

↓

Widely $W(t)$

☐ W/O structure to be exploited
working w/ complex r.vector is messy.

$$\begin{bmatrix}\underline{x}\\ \underline{x}^*\end{bmatrix} = T\begin{bmatrix}\underline{x}_c\\ \underline{x}_s\end{bmatrix} \qquad T = \begin{bmatrix}I & jI\\ I & -jI\end{bmatrix}$$

$\frac{1}{2}T^HT = I$

☐ Unitary TF

$\underline{x} = \underline{x}_c + j\underline{x}_s$

$$\begin{bmatrix}\underline{y}_c\\ \underline{y}_s\end{bmatrix} = U\begin{bmatrix}\underline{x}_c\\ \underline{x}_s\end{bmatrix} \Longleftrightarrow \begin{bmatrix}\underline{y}\\ \underline{y}^*\end{bmatrix} = \boxed{\begin{matrix}A & B\\ B^* & A^*\end{matrix}}\begin{bmatrix}\underline{x}\\ \underline{x}^*\end{bmatrix}$$

$UU^H = U^HU = I$

$$\Longleftrightarrow \underline{y} = A\underline{x} + B\underline{x}^*$$

$$\boxed{\frac{1}{2}TUT^H}$$

unitary

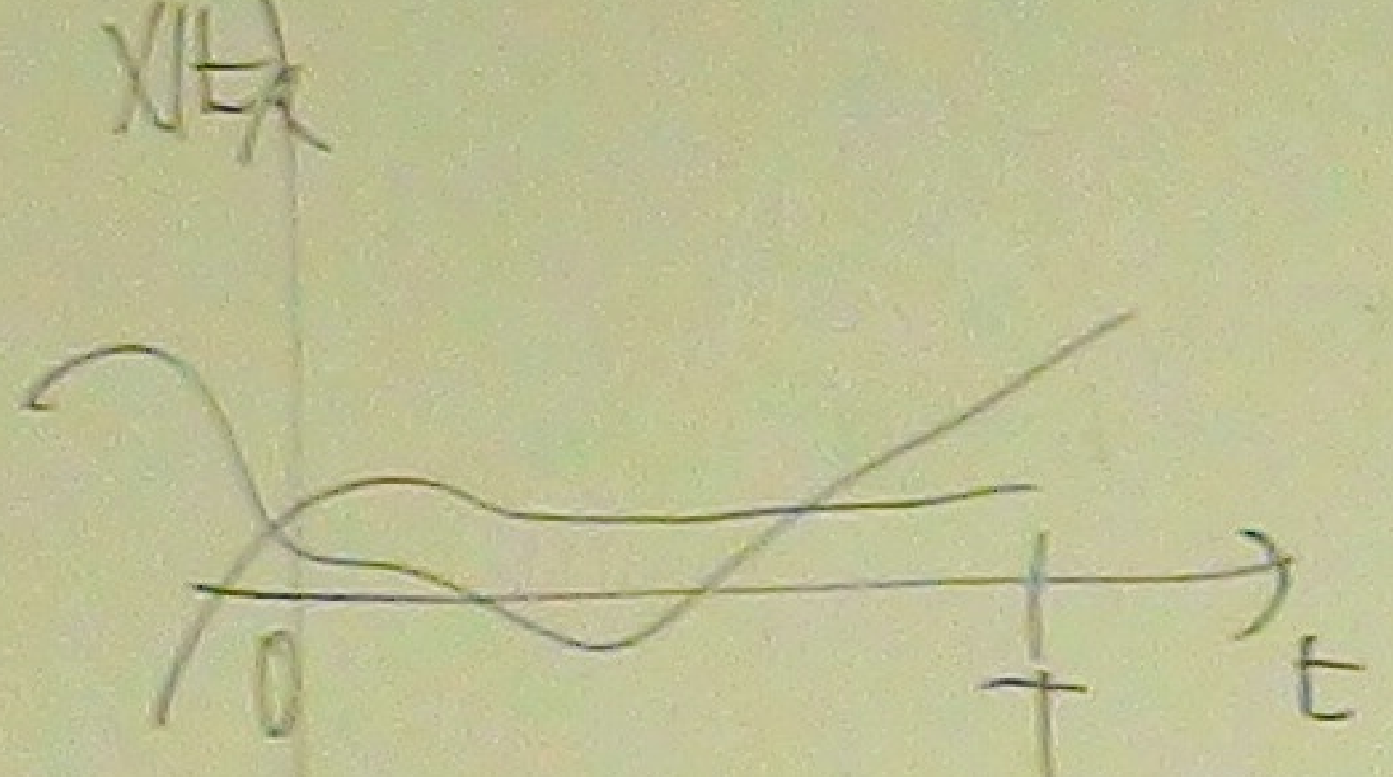
widely unitary TF.

EECE695M CM#14C

proper-complex
~~real~~
zero-mean

☐ KL Expansion of a CT r.p. $X(t)$ in $[0,T)$.

$(a+b)(c+d)=ac+\ldots$



$$\boxed{X(t)=\sum_{i=1}^{\infty} X_i\,\phi_i(t),}$$

$\{X_i\}_{i=1}^{\infty}$ X_i's are uncorrelated $E\{X_i X_j^*\}=\lambda_i\,\delta_{i,j}$

$\{\phi_i(t)\}_{i=1}^{\infty}$ ϕ_i's are orthonormal $\int_0^T \phi_i(t)\phi_j^*(t)\,dt=\delta_{i,j}$

$$\int_0^T \boxed{C_{XX}(t_1,t_2)}\,\phi_i(t_2)\,dt_2=\lambda_i\phi_i(t_1)$$

$\triangleq E\{(X(t_1)\quad)(X(t_2)\quad)^*\}$

$\{\phi_i(t)\}_{i=1}^{\infty}$

$$C_{XX}(t_1,t_2)=\sum_{i=1}^{\infty}\lambda_i\phi_i(t_1)\phi_i^*(t_2)$$

$$X(t)\longrightarrow\otimes\longrightarrow\boxed{\int_0^T\cdot\,dt}\longrightarrow X_1\triangleq\int_0^T X(t)\phi_1^*(t)\,dt$$

$\phi_1^*(t)$

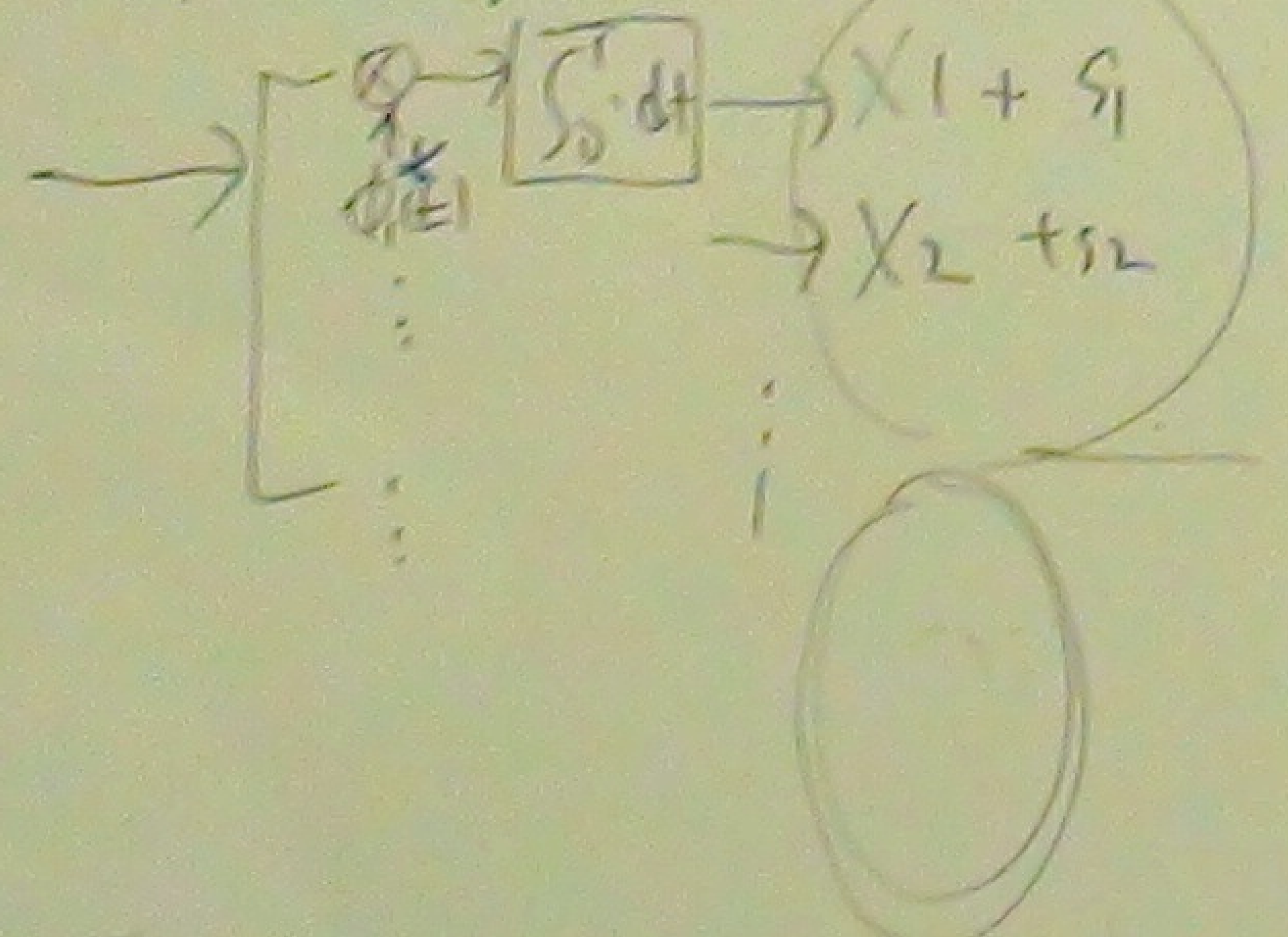
$\vdots$

$$E\{X_iX_j^*\}=\int_0^T\int_0^T \underbrace{E[X(t_1)X^*(t_2)]}_{C_{XX}(t_1,t_2)}\phi_i(t_1)\phi_j^*(t_2)\,dt_1\,dt_2$$

$$=\lambda_i\,\delta_{i,j}$$

$$E\{X_iX_j\}=0$$

$Y(t)=s(t)+X(t)$



cf)

$$\underline{X}\rightarrow\boxed{U^H}\rightarrow\underline{Y}\qquad E\{\underline{Y}\}=\underline{0}\qquad C_{\underline{Y}\underline{Y}}=\Lambda.$$

$E\{\underline{X}\}=0.$

$=\sum(\underline{u}_i^H\underline{x})\,\underline{e}_i$

$$C_{\underline{X}\underline{X}}=U\Lambda U^H=\sum_i\lambda_i\underline{u}_i\underline{u}_i^H$$

$$C_{\underline{X}\underline{X}}\underline{u}_i=\lambda_i\underline{u}_i$$

$$C_{\underline{X}\underline{X}}^{-\frac{1}{2}}=\sum\frac{1}{\sqrt{\lambda_i}}\underline{u}_i\underline{u}_i^H$$

$$X(t)\rightarrow\boxed{C_{XX}^{-\frac{1}{2}}(t,\tau)}\rightarrow Z(t)$$

$$C_{XX}^{-\frac{1}{2}}(t,\tau)\triangleq\sum_{i=1}^{\infty}\frac{1}{\sqrt{\lambda_i}}\phi_i(t)\phi_i^*(\tau)$$

LTV

$$Z(t)=\int_0^T C_{XX}^{-\frac{1}{2}}(t,\tau)X(\tau)\,d\tau$$

$$C_{ZZ}(t,\tau)=\sum_{i=1}^{\infty}\phi_i(t)\phi_i^*(\tau)$$

☐ Improper zero-mean r.p. $X(t)$

$$\xrightarrow{X(t)}\boxed{C_{XX}^{-\frac{1}{2}}(t,\tau)}\xrightarrow{Z(t)}\boxed{F^*(t,\tau)}\xrightarrow{W(t)}$$

$$\tilde{C}_{ZZ}(t,\tau)=\int_0^T\int_0^T C_{XX}^{-\frac{1}{2}}(t,\tau_1)\,\tilde{C}_{XX}(\tau_1,\tau_2)\,C_{XX}^{-\frac{1}{2}*}(\tau_2,\tau)\,d\tau_1\,d\tau_2$$

↑

coherence function

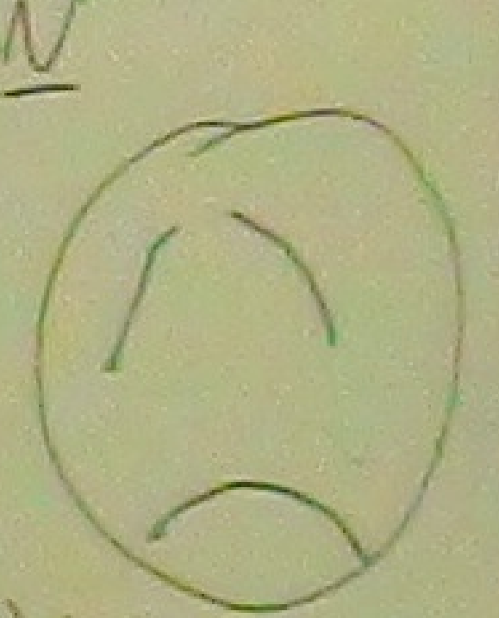
$$=\int_0^T\int_0^T F(t,\tau_1)\,K(\tau_1)\,\delta(\tau_1-\tau_2)\,F(\tau_2,\tau)\,d\tau_1\,d\tau_2$$

↑

Takagi Factorization

$$\therefore\ \tilde{C}_{WW}(t,\tau)=K(t)\,\delta(t-\tau)$$

$$\underline{X}\rightarrow\boxed{C_{\underline{X}\underline{X}}^{-\frac{1}{2}}}\rightarrow\boxed{F^H}\rightarrow\underline{W}$$



$$\underline{W}=F^HC_{\underline{X}\underline{X}}^{-\frac{1}{2}}\underline{X}$$

$$=\left(C_{\underline{X}\underline{X}}^{-\frac{1}{2}}\cdot[\underline{f}_1\,\underline{f}_2\cdots\underline{f}_N]\right)^H\underline{X}$$

$$=\left[C_{\underline{X}\underline{X}}^{-\frac{1}{2}}\underline{f}_1\ C_{\underline{X}\underline{X}}^{-\frac{1}{2}}\underline{f}_2\cdots\right]^H\underline{X}$$