EECE 695M CM#19a

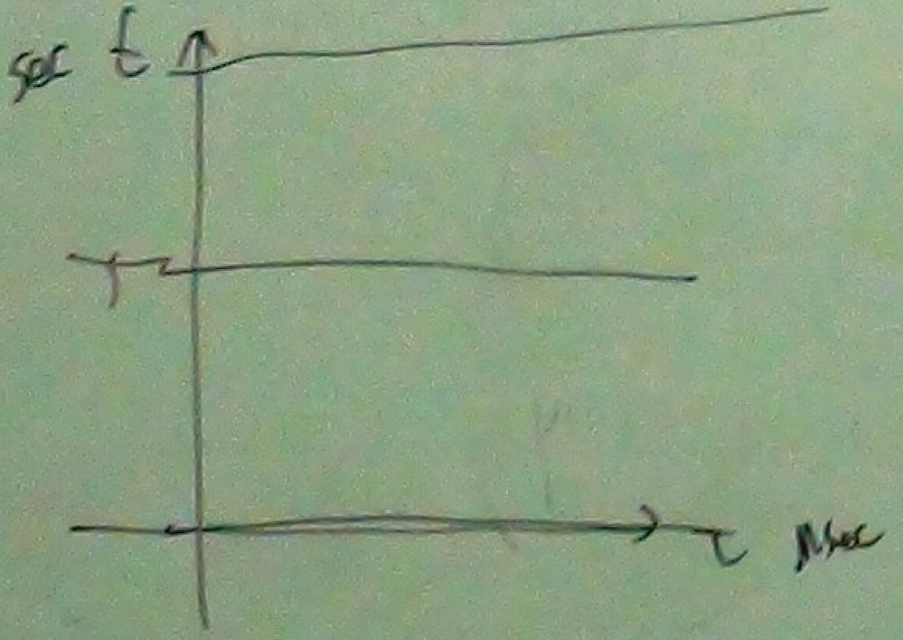
□ Two-variable functions

(i) LTV

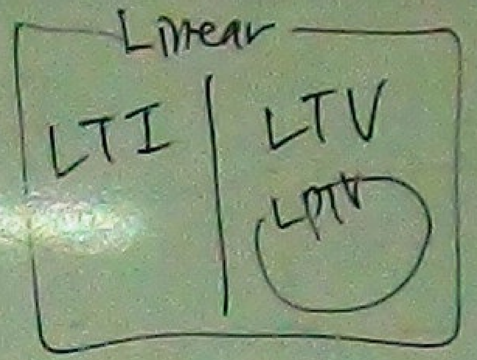
$x(t) \rightarrow \boxed{\mathcal{L}} \rightarrow y(t)$

$$y(t) = \mathcal{L}\{x(t)\}$$
$$= \mathcal{L}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}$$
$$= \int_{-\infty}^{\infty} \mathcal{L}(\delta(t-\tau))x(\tau)d\tau \qquad h_\tau(t)$$
$$= \int_{-\infty}^{\infty} h_\tau(t)x(\tau)d\tau$$
$$= \int_{-\infty}^{\infty} h(t,\tau)x(\tau)d\tau \qquad \tau = t-\tau'$$
$$= \int_{-\infty}^{\infty} h(t,t-\tau')x(t-\tau')d\tau'$$
$$= \int_{-\infty}^{\infty} \underbrace{\boxed{h(t,t-\tau)}}_{C_t(\tau)}x(t-\tau)d\tau \quad \leftarrow$$
$$= \int_{-\infty}^{\infty} C_t(\tau)x(t-\tau)d\tau$$

cf) LTI
$$y(t) = \int_{-\infty}^{\infty} \underline{h(\tau)}x(t-\tau)d\tau$$



(ii) LPTV



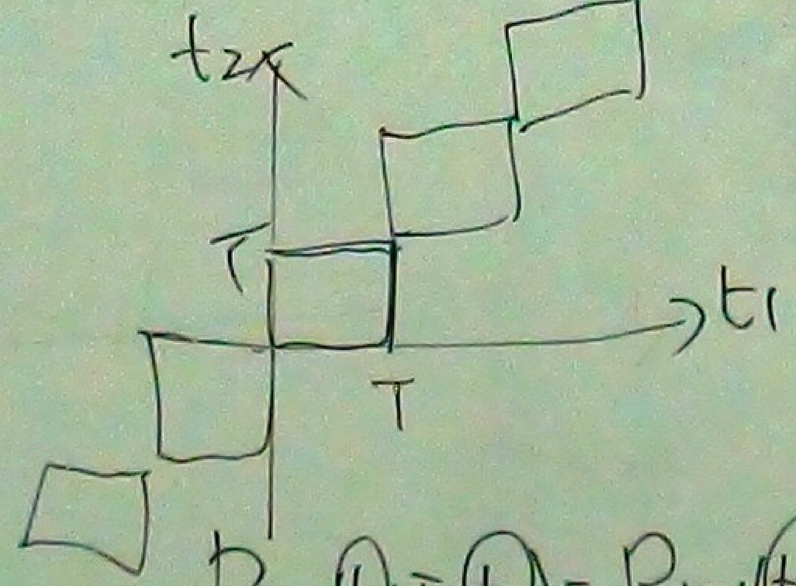
$$y(t) = \int_{-\infty}^{\infty} \underbrace{h(t,t-\tau)}_{C_t(\tau) = C_{t+T}(\tau)}x(t-\tau)d\tau$$
$$= \int_{-\infty}^{\infty} h(t+T,t+T-\tau)x(t-\tau)d\tau$$

$\therefore\ h(t,t-\tau) = h(t+T,t+T-\tau),\ \forall t,\tau$

$\Leftrightarrow\ h(t+\tau,t) = h(t+T+\tau,t+T),\ \forall t,\tau$

$\Leftrightarrow\ h(t,\tau) = h(t+T,\tau+T),\ \forall t,\tau$

(iii) $R_{xx}(t_1,t_2) = R_{xx}(t_1+T,t_2+T)\quad \forall t_1,t_2$

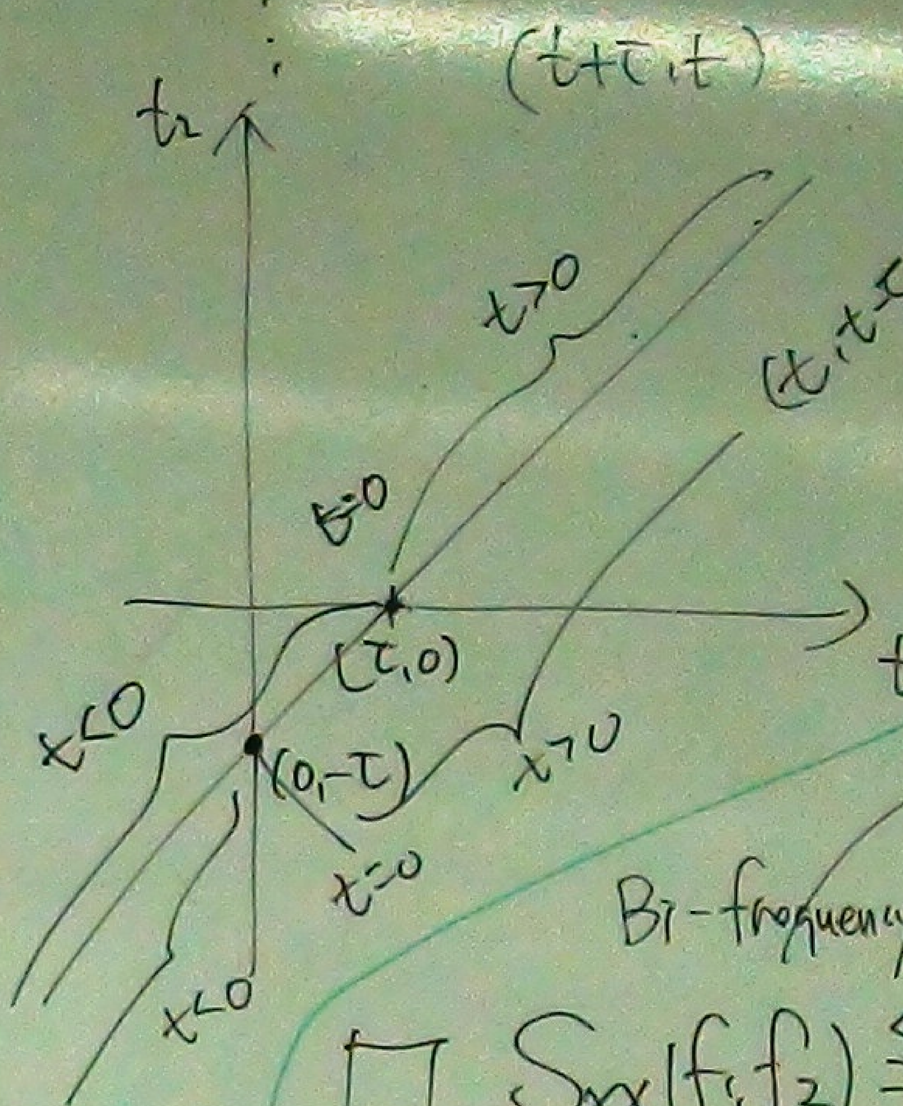


$R_{xx}(t+\tau,t) = R_{xx}(t+T+\tau,t+T),\ \forall t,\tau$

$R_{xx}(t,t-\tau) = R_{xx}(t+T,t+T-\tau),\ \forall t,\tau$

periodic in t

$\tilde{R}_{xx}(t_1,t_2) = R_{xx}(t_1+T,t_2+T),\ \forall t_1,t_2$



$\simeq E\{X(f_1)X^*(f_2)\}$

Bi-frequency Spectrum

□ $S_{xx}(f_1,f_2) \triangleq DFT\{R_{xx}(t_1,t_2)\}$

$\tilde{S}_{xx}(f_1,f_2) \triangleq DFT^*\{\tilde{R}_{xx}(t_1,t_2)\}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi f_1 t_1} R_{xx}(t_1,t_2)\, e^{+j2\pi f_2 t_2}\, dt_1 dt_2$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi f_1 t_1} \tilde{R}_{xx}(t_1,t_2)\, e^{-j2\pi f_2 t_2}\, dt_1 dt_2$$

$\simeq E\{X(f_1)X(f_2)\}$

C.B.F.S.

EECE 695M CM#19b

□ When $X(t)$ is WSCS w/ c.p. T

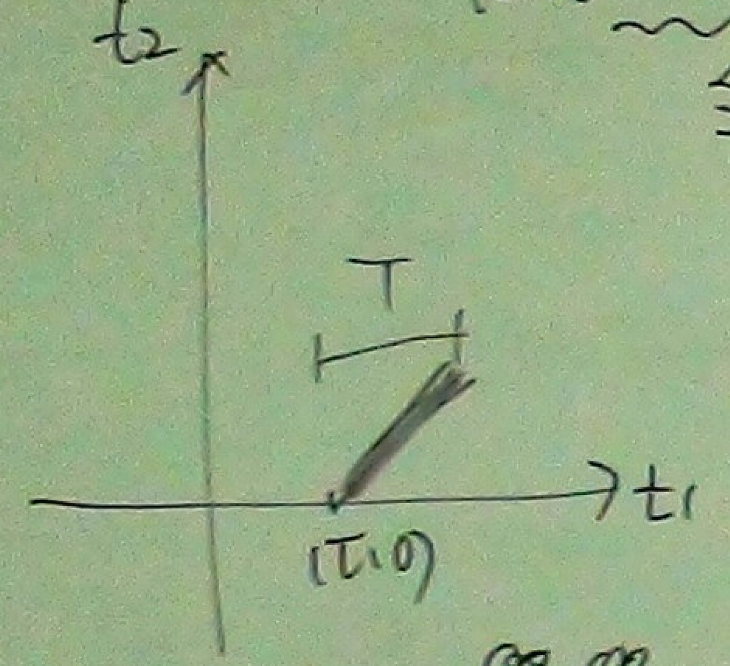
$R_{xx}(t_1,t_2)=R_{xx}(t_1+T,t_2+T),\ \forall (t_1,t_2)\in\mathbb{R}^2$

$R_{xx}(t+\tau,t)=R_{xx}(t+T+\tau,t+T)$, periodic in t w/ period T.

$$=\sum_{k=-\infty}^{\infty} \underbrace{R_{xx}^{(k)}(\tau)}\, e^{j\frac{2\pi k t}{T}}.$$

$$\triangleq \frac{1}{T}\int_0^T R_{xx}(t+\tau,t)\, e^{-j\frac{2\pi k t}{T}}\, dt$$

$\tau = t_1 - t_2$.
$t = t_2$



$$S_{xx}(f_1,f_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1,t_2)\, e^{-j2\pi f_1 t_1}\, e^{j2\pi f_2 t_2}\, dt_1\, dt_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{R_{xx}(t+\tau,t)}\, e^{-j2\pi f_1(\tau+t)}\, e^{j2\pi f_2 t}\, d\tau\, dt$$

$$=\sum_{k=-\infty}^{\infty} R^{(k)}_{xx}(\tau)\, e^{j\frac{2\pi k t}{T}}$$

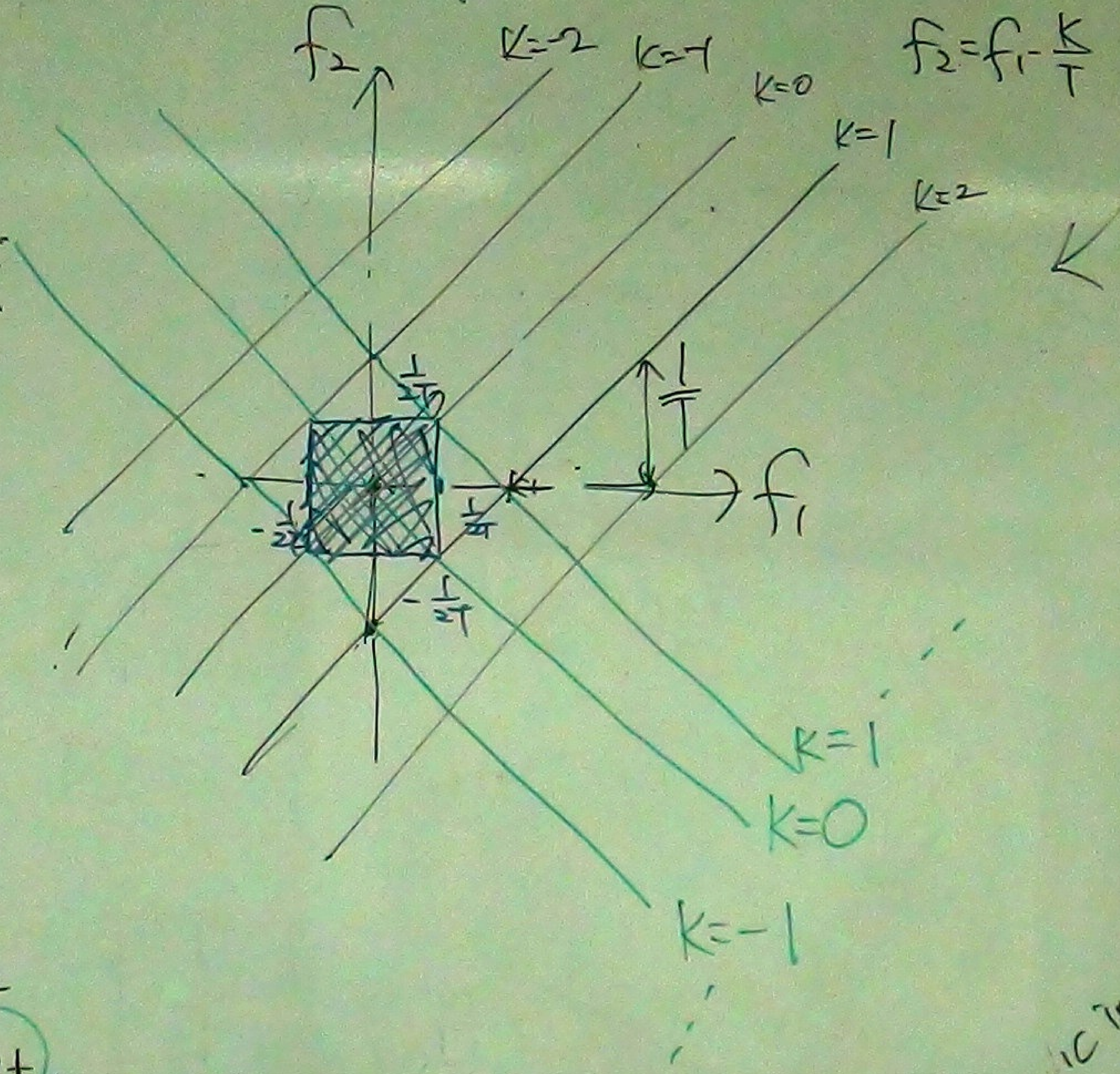
$$\int_{-\infty}^{\infty} e^{j\frac{2\pi k t}{T}}\, e^{-j2\pi f_1 t}\, e^{j2\pi f_2 t}\, dt$$

$$=\delta\left(f_2-f_1+\frac{k}{T}\right)$$

$$=\sum_{k=-\infty}^{\infty} \boxed{\int_{-\infty}^{\infty} R^{(k)}_{xx}(\tau)\, e^{-j2\pi f_1 \tau}\, d\tau} \triangleq S^{(k)}_{xx}(f_1)$$

$$\delta\left(f_2-f_1+\frac{k}{T}\right)$$

$$S_{xx}(f_1,f_2)=\sum_{k=-\infty}^{\infty} S^{(k)}_{xx}(f_1)\,\delta\left(f_2-f_1+\frac{k}{T}\right).$$



□ When $X(t)$ is SOCS.

$$\tilde S_{xx}(f_1,f_2)=\sum_{k=-\infty}^{\infty} \tilde S^{(k)}_{xx}(f_1)\,\delta\left(f_2+f_1-\frac{k}{T}\right)$$

$f_2=-f_1+\frac{k}{T}$

$\tilde R_{xx}(t_1,t_2)$

$$\tilde R_{xx}(t+\tau,t)=\sum_{k=-\infty}^{\infty} \underbrace{\tilde R^{(k)}_{xx}(\tau)}\, e^{j\frac{2\pi k t}{T}}$$

$$\triangleq \frac{1}{T}\int_0^T \tilde R_{xx}(t+\tau,t)\, e^{-j\frac{2\pi k t}{T}}\, dt.$$

$$\simeq E\{X(f_1)X^*(f_2)\}$$

Bi-frequency Spectrum

□ $S_{xx}(f_1,f_2)\triangleq \mathrm{DFT}\{R_{xx}(t_1,t_2)\}$

$\tilde S_{xx}(f_1,f_2)\triangleq \mathrm{DFT}^*\{\tilde R_{xx}(t_1,t_2)\}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi f_1 t_1}\, R_{xx}(t_1,t_2)\, e^{+j2\pi f_2 t_2}\, dt_1\, dt_2$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi f_1 t_1}\, \tilde R_{xx}(t_1,t_2)\, e^{-j2\pi f_2 t_2}\, dt_1\, dt_2$$

ic in t

$$\simeq E\{X(f_1)X(f_2)\}$$

C.B-F.S.

EECE 695M CM#19C

□ When $X(t)$ is WSS

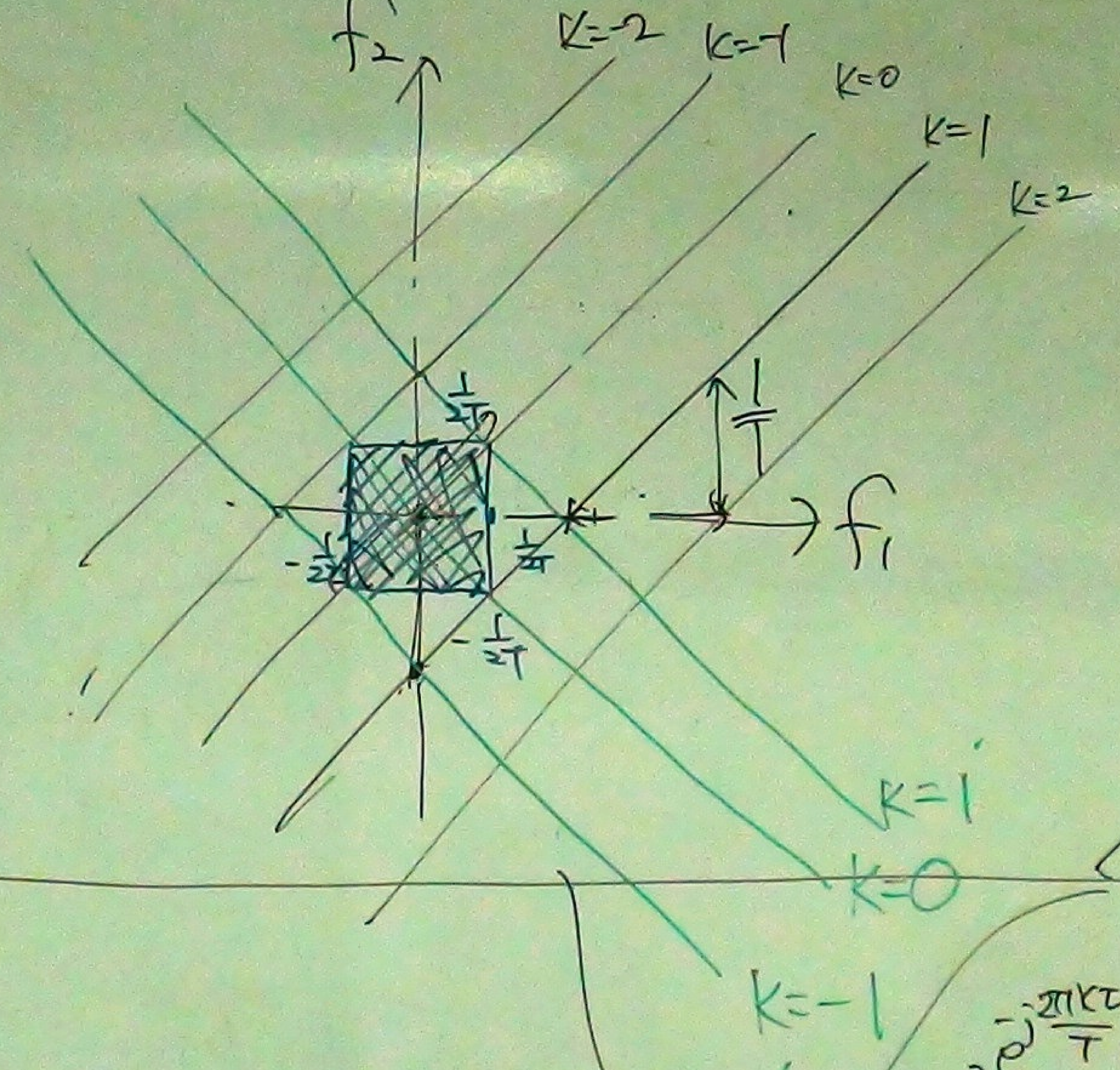
$$R_{XX}(t+\tau,t) = R_{XX}(\tau) = R^{(0)}_{XX}(\tau)$$

$$\boxed{R_{XX}(t+\tau,t) = \sum_{k=-\infty}^{\infty} R^{(k)}_{XX}(\tau)\, e^{j\frac{2\pi k t}{T}}}$$

$\mathcal{F}_\tau$

$$S_{XX}(f) = S^{(0)}_{XX}(f)$$

$$S_{XX}(f_1,f_2) = \sum_{k=-\infty}^{\infty} S^{(k)}_{XX}(f_1)\,\delta\!\left(f_2 - f_1 + \frac{k}{T}\right)$$



□ When $X(t)$ is SOS

$$\tilde{S}_{XX}(f_1,f_2) = \sum_{k=-\infty}^{\infty} \tilde{S}^{(k)}_{XX}(f_1)\,\delta\!\left(f_2 + f_1 - \frac{k}{T}\right)$$

$$\tilde{R}_{XX}(t+\tau,t) = \sum_{k=-\infty}^{\infty} \underbrace{\tilde{R}^{(k)}_{XX}(\tau)}\, e^{j\frac{2\pi k t}{T}}$$

$$\triangleq \frac{1}{T}\int_0^T \tilde{R}_{XX}(t+\tau,t)\, e^{-j\frac{2\pi k t}{T}}\, dt$$

$$= \tilde{R}_{XX}(\tau) \overset{\mathcal{F}_\tau}{\longleftrightarrow} \tilde{S}_{XX}(f)$$

$$= \tilde{R}^{(0)}_{XX}(\tau) \overset{\mathcal{F}_\tau}{\longleftrightarrow} \tilde{S}^{(0)}_{XX}(f)$$

□ LPTV revisited.

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)\,x(\tau)\,d\tau \quad \Big\} \text{ LTV}$$

$$= \int_{-\infty}^{\infty} h(t,t-\tau)\,x(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(t+T,t+T-\tau)\,x(t-\tau)\,d\tau \quad \Big\} \text{ LPTV w/ period } T$$

$$h(t+\tau,t) = \sum_{k=-\infty}^{\infty} h^{(k)}(\tau)\, e^{j\frac{2\pi k t}{T}}$$

$t = t' - \tau$ where $h^{(k)}(\tau) \triangleq \frac{1}{T}\int_0^T h(t+\tau,t)\, e^{-j\frac{2\pi k t}{T}}\, dt$.

$$h(t,t-\tau) = \sum_{k=-\infty}^{\infty} h^{(k)}(\tau)\, e^{j\frac{2\pi k(t-\tau)}{T}}$$

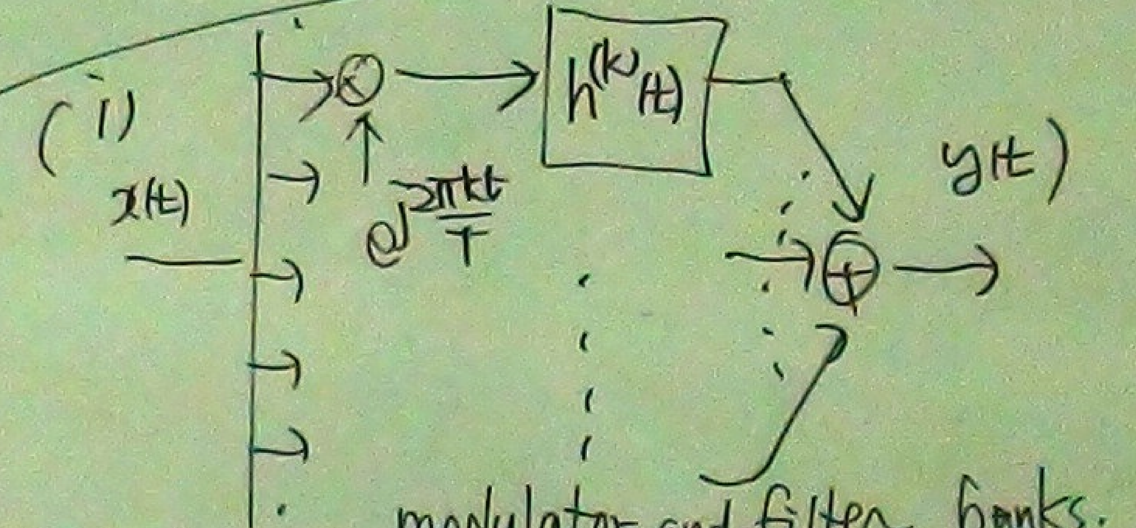
$$e^{-j\frac{2\pi k\tau}{T}}\, e^{j\frac{2\pi k t}{T}}$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} h^{(k)}(\tau)\, e^{j\frac{2\pi k(t-\tau)}{T}}\, x(t-\tau)\, d\tau$$

$$= \sum_{k=-\infty}^{\infty} h^{(k)}(t) * \left(x(t)\, e^{j\frac{2\pi k t}{T}}\right) \quad \cdots (i)$$

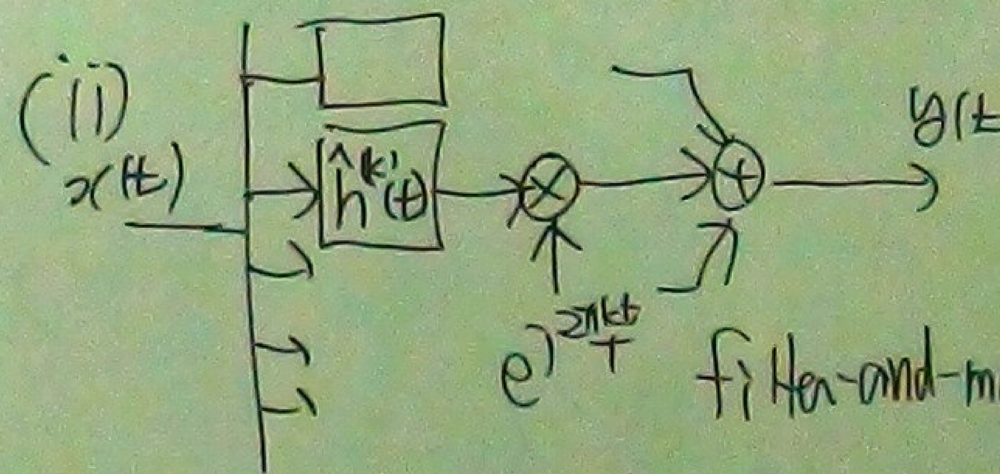
$$= \sum_{k=-\infty}^{\infty} \Big[\underbrace{\left(h^{(k)}(t)\, e^{-j\frac{2\pi k t}{T}}\right)}_{\hat{h}^{(k)}(t)} * x(t) \Big]\, e^{j\frac{2\pi k t}{T}} \quad \cdots (ii) = \sum_{k=-\infty}^{\infty} \left(\hat{h}^{(k)}(t) * x(t)\right) e^{j\frac{2\pi k t}{T}}$$

(i)



modulator-and-filter banks.

(ii)



filter-and-modulator banks.

EECE 6965M CM#19d

□ Vector-valued r.p. $\underline{X}(t) \in \mathbb{C}^N$

$E\{\underline{X}(t)\}$

$R_{\underline{XX}}(t_1,t_2) \triangleq E\{\underline{X}(t_1)\underline{X}(t_2)^H\}$

$\tilde{R}_{\underline{XX}}(t_1,t_2) \triangleq E\{\underline{X}(t_1)\underline{X}(t_2)^T\}$

$C_{\underline{XX}}(t_1,t_2) \triangleq \cdots$

$\tilde{C}_{\underline{XX}}(t_1,t_2) \triangleq \cdots$

(i) If $\tilde{C}_{\underline{XX}}(t_1,t_2)=0,\ \forall (t_1,t_2)\in\mathbb{R}^2$

then $\underline{X}(t)$ is proper.

improper.

(ii) WSS, SOS.... ← " " SOS

if $\overset{(\sim)}{R}_{\underline{XX}}(t+\tau,t) = \overset{(\sim)}{R}_{\underline{XX}}(\tau,0) = \overset{(\sim)}{R}_{\underline{XX}}(\tau)$ ← pairwise jointly WSS

(iii) WSCS, SOCS ← " " WSCS

if $\begin{cases}\cdots \\ \cdots\end{cases}$ ← " " SOCS

□ $\mathcal{F}_{\tau}\left[A_t\{R_{\underline{XX}}(t+\tau,t)\}\right]$

$= S_{\underline{XX}}(f)$ PSD matrix

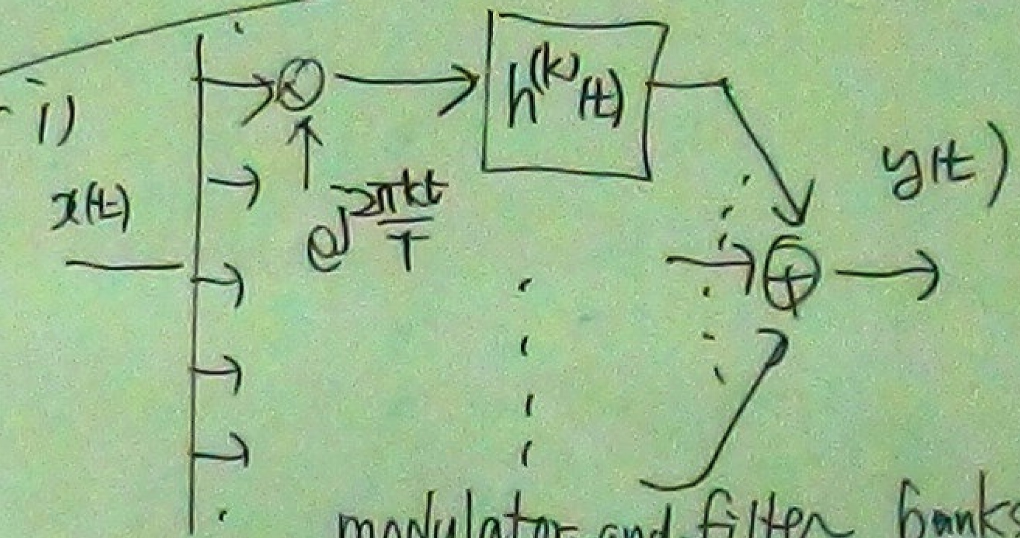
$\mathcal{F}\left[A\{\tilde{R}_{\underline{XX}}(t+\tau,t)\}\right]$

$= \tilde{S}_{\underline{XX}}(f)$ c. PSD matrix

□ $S_{\underline{XX}}(f_1,f_2) \triangleq D\mathcal{F}\{R_{\underline{XX}}(t_1,t_2)\}$ Bi-F. spec. matrix

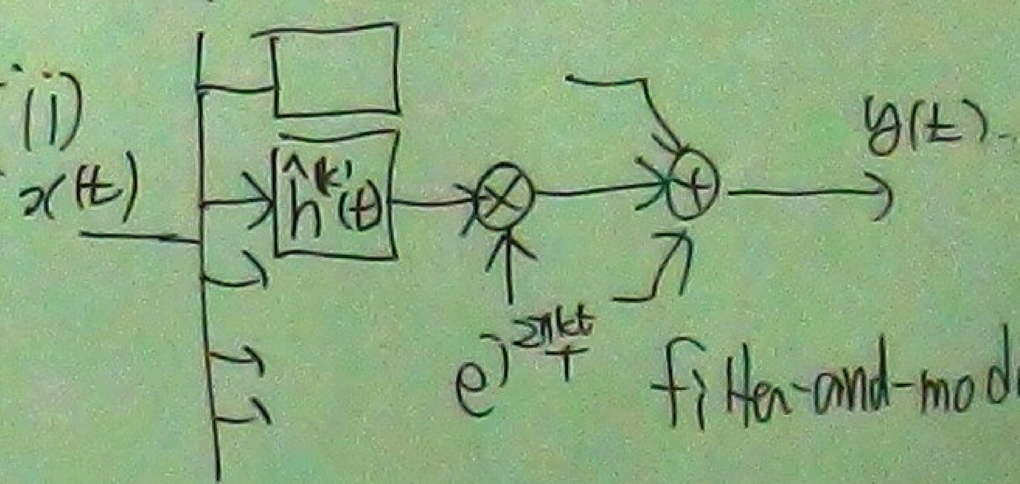
$\tilde{S}_{\underline{XX}}(f_1,f_2) \triangleq D\mathcal{F}^{*}\{R_{\underline{XX}}(t_1,t_2)\}$ C. B.F.S. matrix

(i)



modulator-and-filter banks.

(ii)



filter-and-modulator banks

$\therefore (ii) = \sum_{k=-\infty}^{\infty}\left(\hat{h}^{(k)}(t) * x(t)\right) e^{\frac{j2\pi kt}{T}}$