EECE 695M CM#19d'

□ Vector-valued r.p. $\underline{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_N(t) \end{bmatrix} \in \mathbb{C}^N$

$E\{\underline{X}(t)\} = \mu_{\underline{X}}(t)$

$R_{\underline{X}\underline{X}}(t_1,t_2) \triangleq E\{\underline{X}(t_1)\underline{X}(t_2)^H\}$

$\tilde{R}_{\underline{X}\underline{X}}(t_1,t_2) \triangleq E\{\underline{X}(t_1)\underline{X}(t_2)^T\}$

$C_{\underline{X}\underline{X}}(t_1,t_2) \triangleq \cdots$

$\tilde{C}_{\underline{X}\underline{X}}(t_1,t_2) \triangleq \cdots$

(i) If $\tilde{C}_{\underline{X}\underline{X}}(t_1,t_2) = 0, \ \forall (t_1,t_2) \in \mathbb{R}^2$

then $\underline{X}(t)$ is proper.

improper.

(ii) WSS, SOS ... ← '' '' SOS

if - $R^{(\sim)}_{\underline{X}\underline{X}}(t+\tau,t) = R^{(\sim)}_{\underline{X}\underline{X}}(\tau,0) = R^{(\sim)}_{\underline{X}\underline{X}}(\tau)$ ← pairwise jointly WSS

- $\mu_{\underline{X}}(t) = \mu_{\underline{X}}$.

(iii) WSCS, SOCS ← '' '' WSCS ←

'' SOCS

if $\begin{cases} C_{\underline{X}\underline{X}}(t+\tau,t) = C_{\underline{X}\underline{X}}(t+T+\tau,t+T), \ \forall t. \\ \tilde{C}_{\underline{X}\underline{X}}(\quad) = \tilde{C}_{\underline{X}\underline{X}} \end{cases}$

- $\mu_{\underline{X}}(t) = \mu_{\underline{X}}(t+T)$. $\forall t$.

Theorem

□ $\mathcal{F}_{\tau}\left[\mathcal{A}_{t}\{R_{\underline{X}\underline{X}}(t+\tau,t)\}\right]$

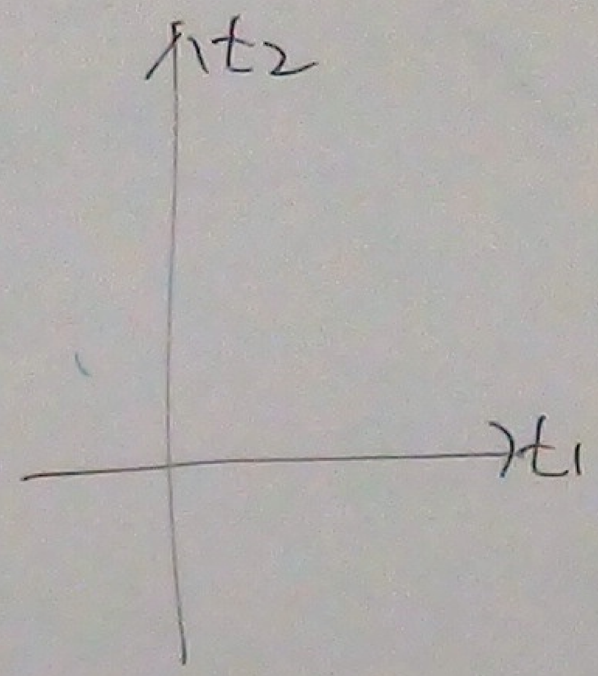
$= S_{\underline{X}\underline{X}}(f)$ PSD matrix, matrix-valued PSD.

$\mathcal{F}\left[\mathcal{A}\{\tilde{R}_{\underline{X}\underline{X}}(t+\tau,t)\}\right]$

$= \tilde{S}_{\underline{X}\underline{X}}(f)$ c. PSD matrix

□ $S_{\underline{X}\underline{X}}(f_1,f_2) \triangleq D\mathcal{F}^{*}\{R_{\underline{X}\underline{X}}(t_1,t_2)\}$ Bi-F. spec. matrix

$\tilde{S}_{\underline{X}\underline{X}}(f_1,f_2) \triangleq D\mathcal{F}\{\tilde{R}_{\underline{X}\underline{X}}(t_1,t_2)\}$ C.B.F.S. matrix



$[S_{\underline{X}\underline{X}}(f)]_{(i,j)} = S_{X_iX_j}(f)$ ← × PSD.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{R_{\underline{X}\underline{X}}(t_1,t_2)}_{E\{\underline{X}(t_1)\underline{X}(t_2)^H\}} e^{-j2\pi f_1 t_1} e^{+j2\pi f_2 t_2} dt_1 dt_2$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\tilde{R}_{\underline{X}\underline{X}}(t_1,t_2)}_{E\{\underline{X}(t_1)\underline{X}(t_2)^T\}} e^{-j2\pi f_1 t_1} e^{-j2\pi f_2 t_2} dt_1 dt_2$

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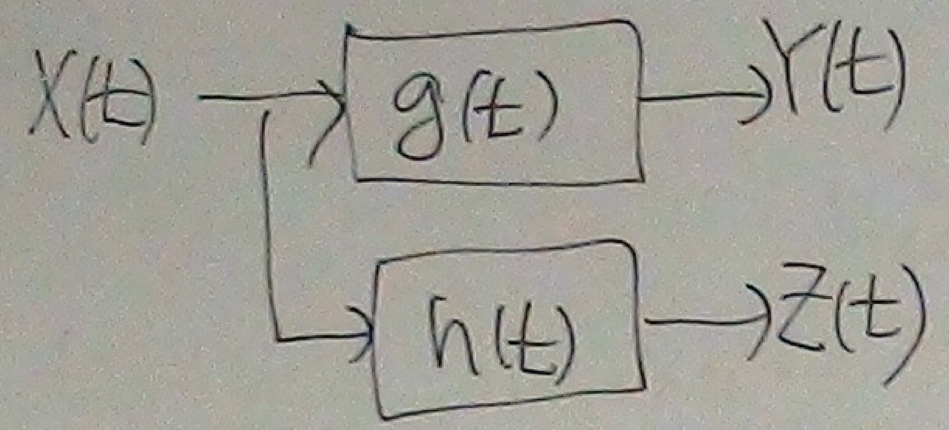
□



$$R_{YX}(t_1,t_2) = E\left[\left(g(t_1) * X(t_1)\right) X^*(t_2)\right] = E\left[\int_{-\infty}^{\infty} g(t_1-\tau_1) X(\tau_1)\, d\tau_1\, X^*(t_2)\right]$$

$$= \int_{-\infty}^{\infty} g(t_1-\tau_1) R_{XX}(\tau_1,t_2)\, d\tau_1$$

$$= \boxed{g(t_1) * R_{XX}(t_1,t_2)}$$

$$\boxed{R_{XZ}(t_1,t_2) = E\left[X(t_1)\left(h(t_2) * X(t_2)\right)^*\right] = \cdots = R_{XX}(t_1,t_2) * h^*(t_2)}$$

$\mathcal{F}$ $h^*(t_2) \leftrightarrow H^*(-f_2)$

$$\tilde{R}_{XY}(t_1,t_2) = \tilde{R}_{XX}(t_1,t_2) * g(t_2) \rightarrow \boxed{\tilde{S}_{XY}(f_1,f_2) = \tilde{S}_{YX}(f_1,f_2) G(f_2)}$$

$$\tilde{R}_{ZX}(t_1,t_2) = h(t_1) * \tilde{R}_{XX}(t_1,t_2) \rightarrow \boxed{\tilde{S}_{ZX}(f_1,f_2) = H(f_1) \tilde{S}_{YX}(f_1,f_2)}$$

$$R_{YZ}(t_1,t_2) = g(t_1) * R_{XX}(t_1,t_2) * h^*(t_2)$$

$$\tilde{R}_{YZ}(t_1,t_2) = g(t_1) * \tilde{R}_{XX}(t_1,t_2) * h(t_2)$$

□ DF

$$S_{XX}(f_1,f_2) \triangleq DF^*\{R_{XX}(t_1,t_2)\} \simeq E\{X(f_1) X^*(f_2)\}$$

bi-frequency spectrum

spectral correlation ft

( ) " " "

co-intensity spectrum

generalized "

Loève bi-freq spectrum

$$\tilde{S}_{XX}(f_1,f_2) \triangleq D\tilde{F}\{\tilde{R}_{XX}(t_1,t_2)\} \simeq E\{X(f_1) X(f_2)\}$$

$$\boxed{S_{YX}(f_1,f_2) = G(f_1) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t_1,t_2) e^{-j2\pi f_1 t_1} dt_1\, e^{j2\pi f_2 t_2} dt_2 = G(f_1) S_{XX}(f_1,f_2)}$$

$$\int_{-\infty}^{\infty} x(t_2) * h^*(t_2)\; e^{j2\pi f t_2} dt_2 = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau) h^*(t_2-\tau)\, d\tau\right) e^{j2\pi f t_2} dt$$

$e^{j2\pi f \tau}\, e^{-j2\pi f \tau}$

$$= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(t_2-\tau)\, e^{-j2\pi f(t_2-\tau)} dt\right)^* e^{j2\pi f \tau} d\tau$$

$$= \int_{-\infty}^{\infty} x(t_2) e^{j2\pi f t_2} dt_2\, H^*(f)$$

$$\therefore \boxed{S_{XZ}(f_1,f_2) = S_{XX}(f_1,f_2) H^*(f_2)}$$

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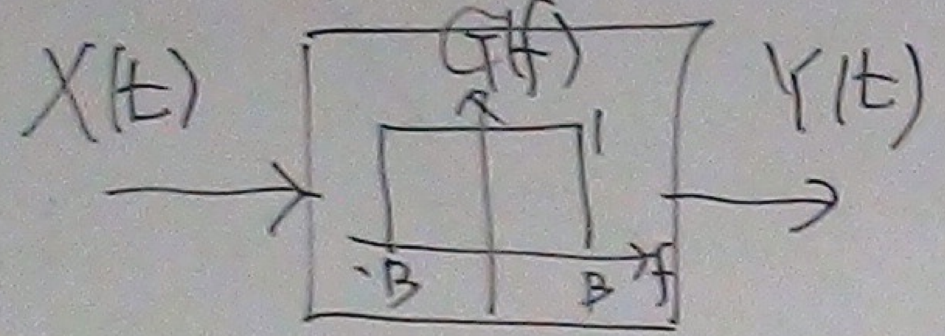
□ Band-limited r.p. $X(t)$

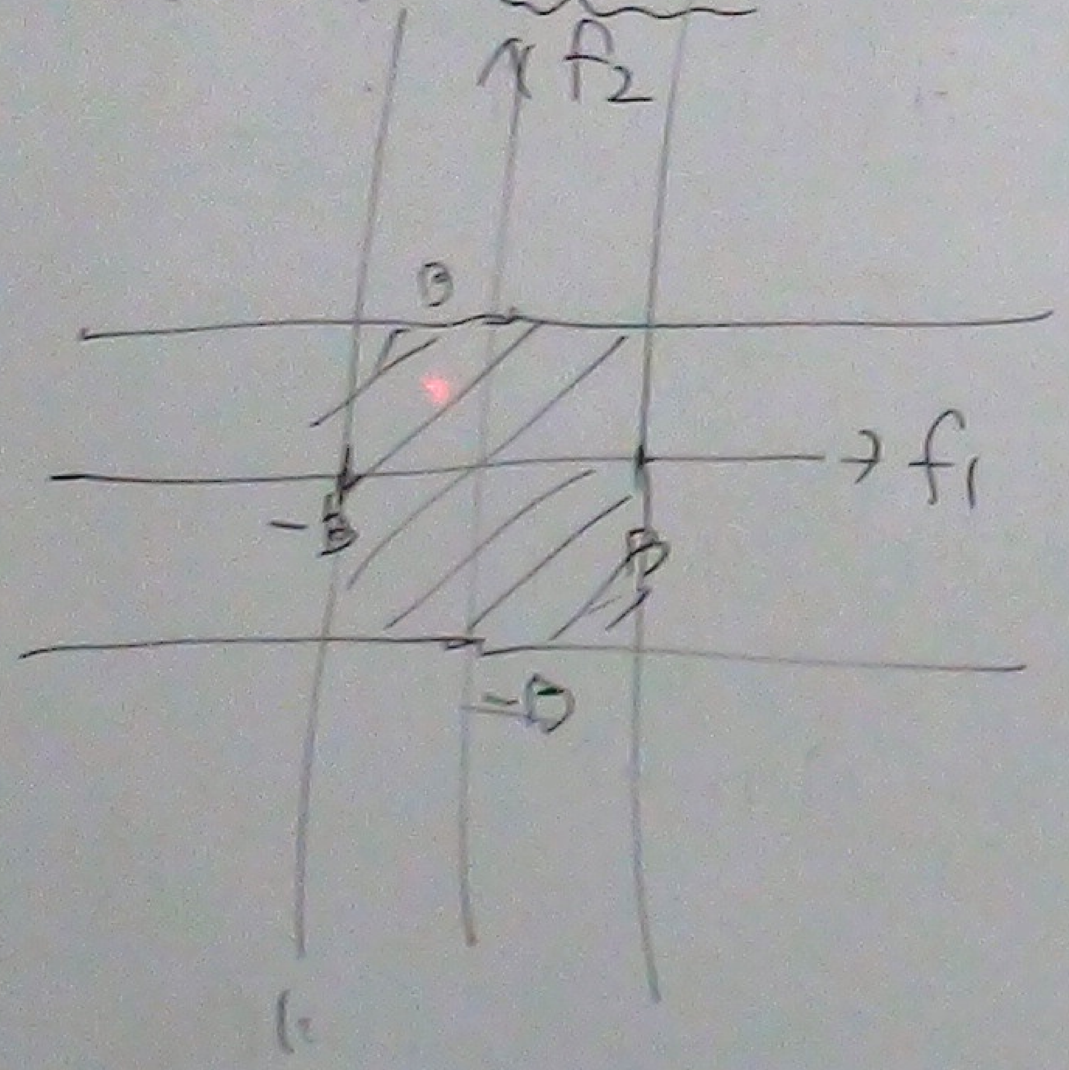
$f \in [-B, B]$.



$Y(t) = X(t)$ in some sense.

$$\Rightarrow R_{YY}(t_1,t_2) = g(t_1) * R_{XX}(t_1,t_2) * g^*(t_2) = R_{XX}(t_1,t_2)$$

$$\Rightarrow S_{YY}(f_1,f_2) = G(f_1)\, S_{XX}(f_1,f_2)\, G^*(f_2) = S_{XX}(f_1,f_2)$$



$$\Rightarrow \tilde{R}_{YY}(t_1,t_2) = g(t_1) * \tilde{R}_{XX}(t_1,t_2) * g(t_2)$$

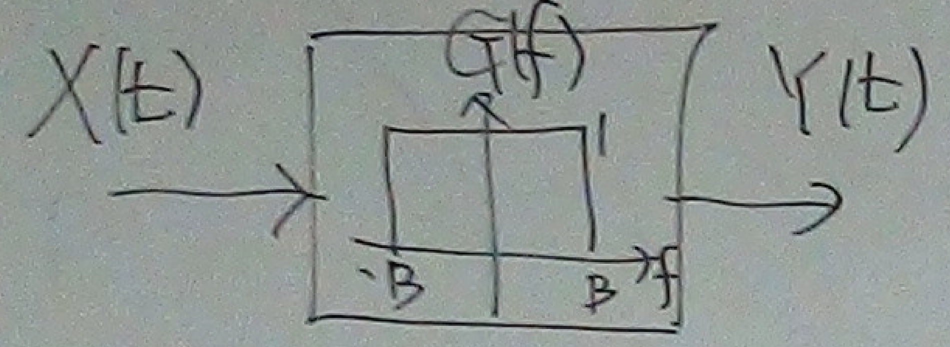
$$\Rightarrow \tilde{S}_{YY}(f_1,f_2) = G(f_1)\, \tilde{S}_{XX}(f_1,f_2)\, G(f_2)$$
$$= \tilde{S}_{XX}(f_1,f_2)$$

□

EECE 695M CM#20b

□ Band-limited r.p. $X(t)$

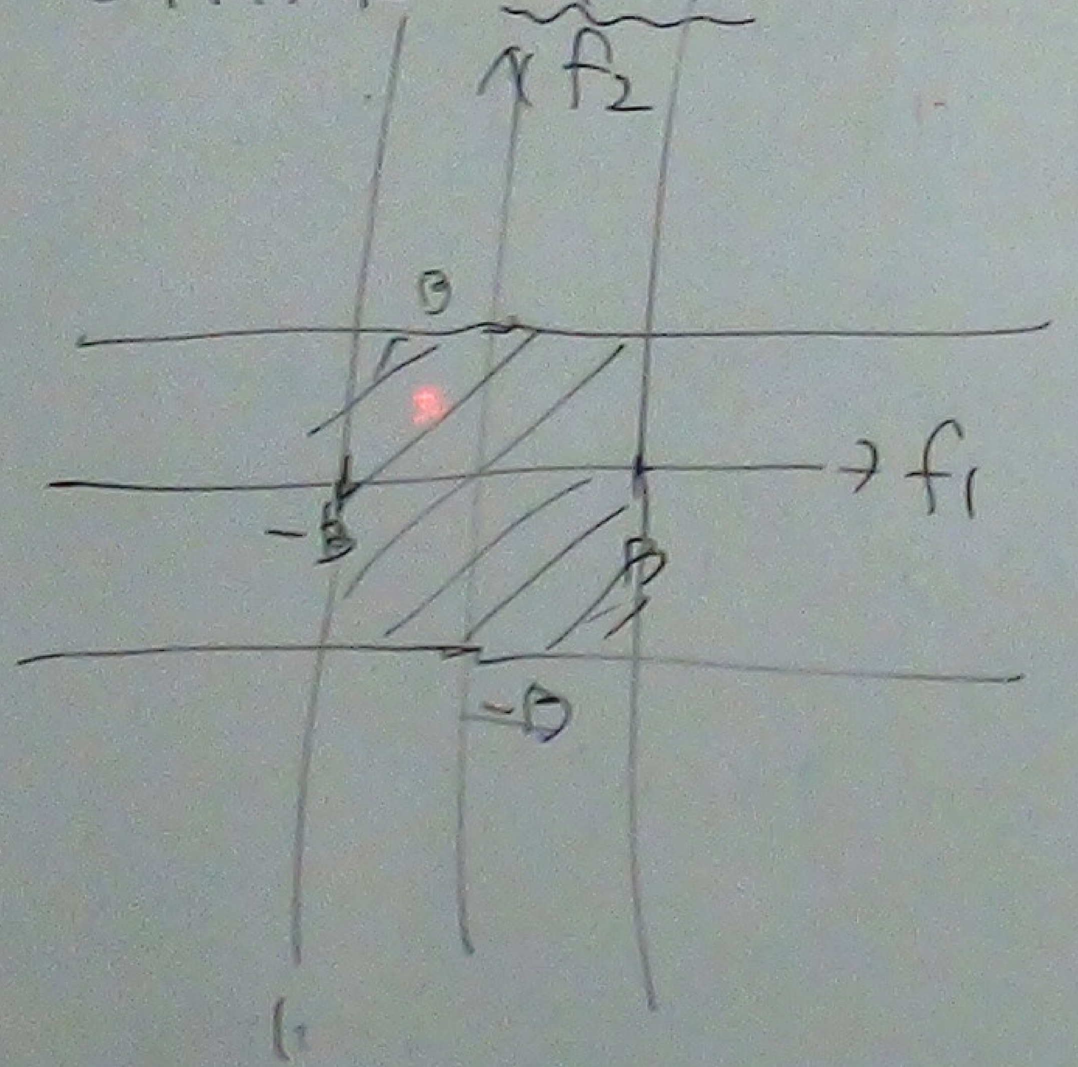
$f \in [-B, B)$.



$Y(t) = X(t)$ in some sense.

$\Rightarrow R_{YY}(t_1,t_2) = g(t_1) * R_{XX}(t_1,t_2) * g^*(t_2) = R_{XX}(t_1,t_2)$

$\Leftrightarrow S_{YY}(f_1,f_2) = G(f_1)\, S_{XX}(f_1,f_2)\, G^*(f_2) = S_{XX}(f_1,f_2)$



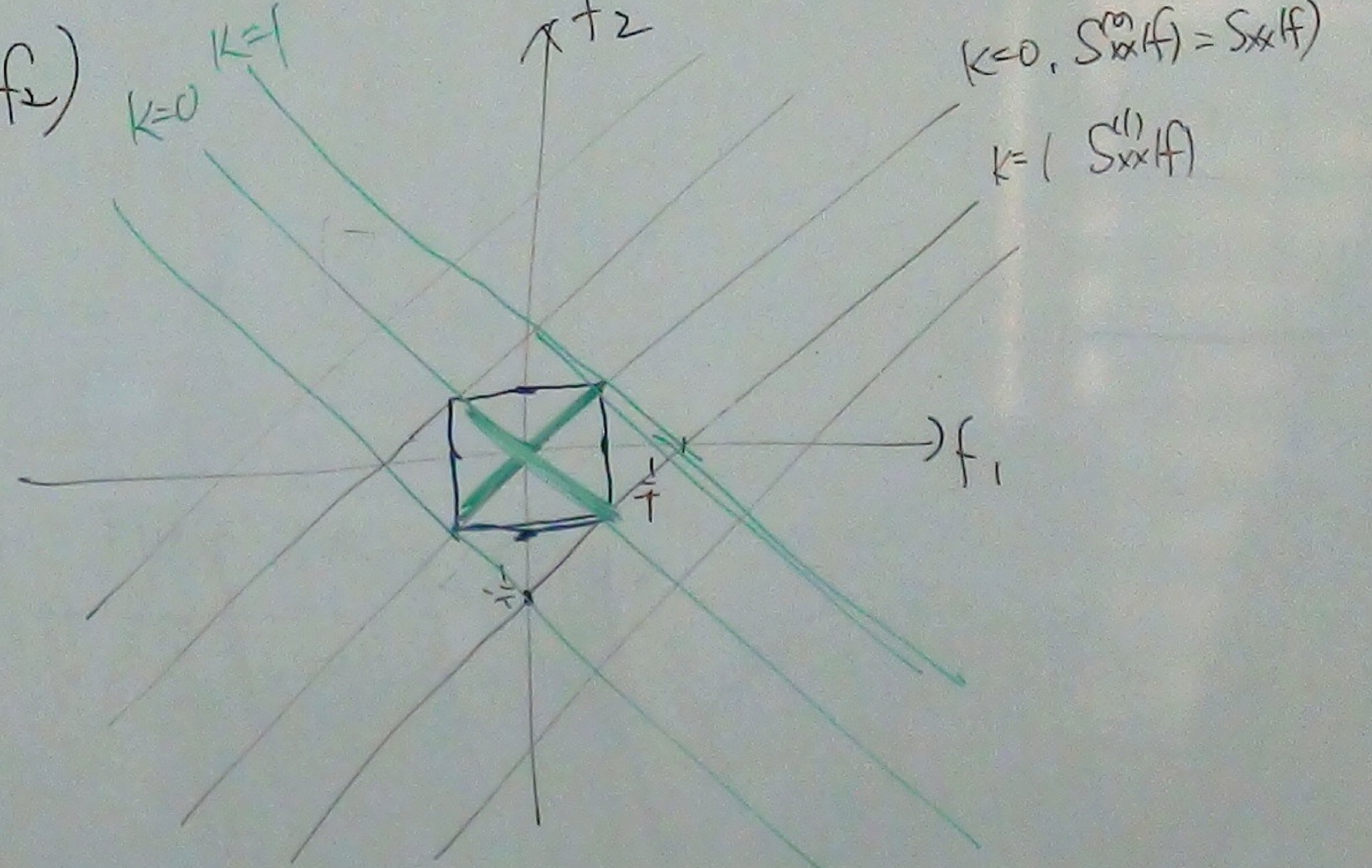
$\Rightarrow \tilde{R}_{YY}(t_1,t_2) = g(t_1) * \tilde{R}_{XX}(t_1,t_2) * g(t_2)$

$\Leftrightarrow \tilde{S}_{YY}(f_1,f_2) = G(f_1)\, \tilde{S}_{XX}(f_1,f_2)\, G(f_2)$
$= \tilde{S}_{XX}(f_1,f_2)$

□ $X(t)$ complex SOCS w/ T $\quad f_2 = f_1 - \frac{k}{T} \quad\quad f_2 = -f_1 + \frac{k}{T}$

$S_{XX}(f_1,f_2) = \sum_{k=-\infty}^{\infty} S_{XX}^{(k)}(f_1)\, \delta\!\left(f_2 - f_1 + \frac{k}{T}\right)$, $\quad \tilde{S}_{XX}(f_1,f_2) = \sum_{k=-\infty}^{\infty} \tilde{S}_{XX}^{(k)}(f_1)\, \delta\!\left(f_2 + f_1 - \frac{k}{T}\right)$

If $X(t)$ is BL to $\left[-\frac{1}{2T}, \frac{1}{2T}\right)$, then $X(t)$ is SOS.



□ $X(t)$ SOS

$R_{XX}(t_1,t_2) = R_{XX}^{(0)}(t_1 - t_2)$

$\tilde{R}_{XX}(t_1,t_2) = \tilde{R}_{XX}^{(0)}(t_1 - t_2)$

$S_{XX}(f_1,f_2) = S_{XX}^{(0)}(f_1)\, \delta(f_2 - f_1)$

$\tilde{S}_{XX}(f_1,f_2) = \tilde{S}_{XX}^{(0)}(f_1)\, \delta(f_2 + f_1)$

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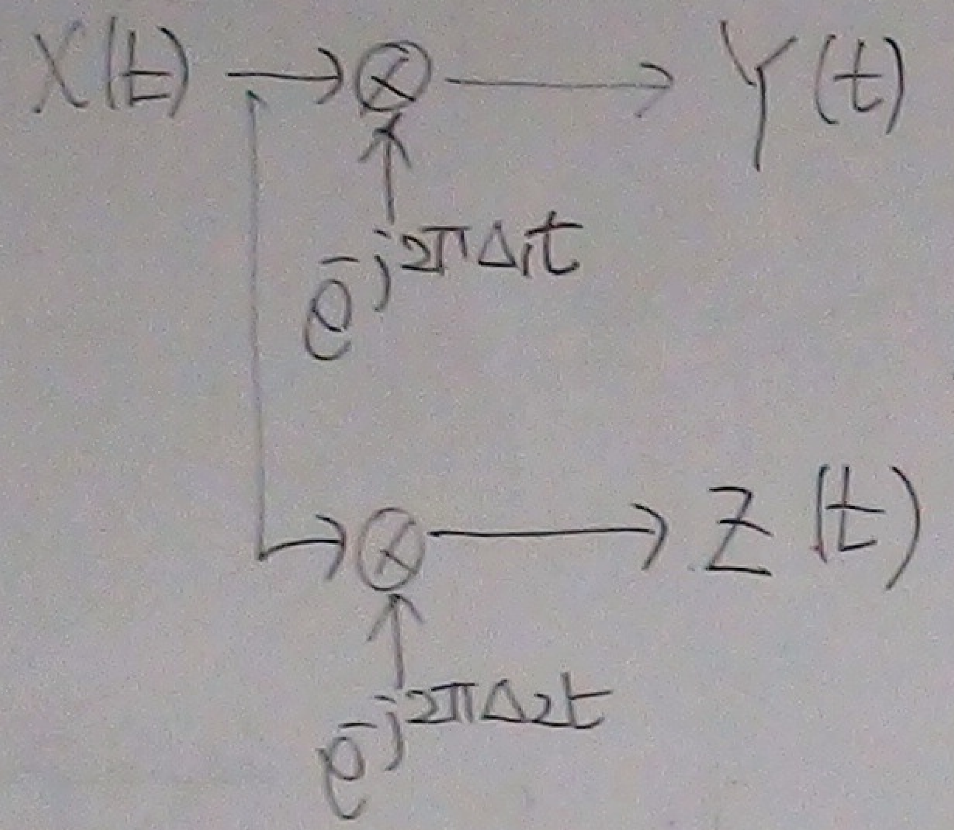
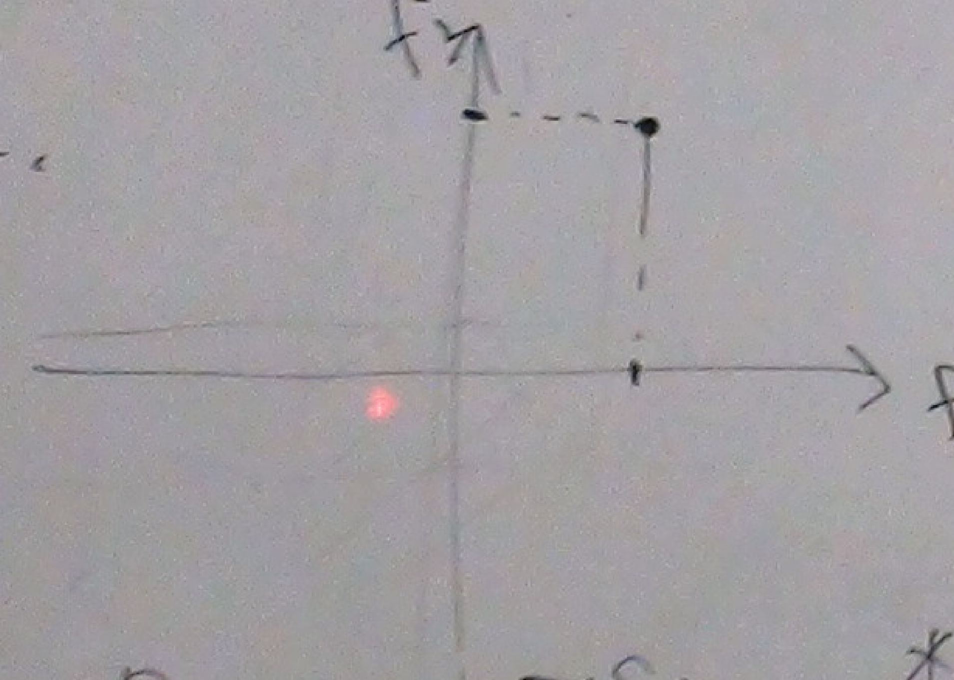
☐ Modulation & $\tilde{S}_{XX}(f_1,f_2)$.



Q. $\tilde{S}_{YZ}(f_1,f_2)$ in terms of $\tilde{S}_{XX}(f_1,f_2)$?

$\tilde{S}_{XX}(f_1+\Delta_1, f_2+\Delta_2)$.

A.



$R_{YZ}(t_1,t_2)=E\{Y(t_1)Z^*(t_2)\}=E\{(X(t_1)e^{-j2\pi\Delta_1 t_1})(X(t_2)e^{-j2\pi\Delta_2 t_2})^*\}$

$= R_{XX}(t_1,t_2)e^{-j2\pi\Delta_1 t_1}e^{+j2\pi\Delta_2 t_2}$

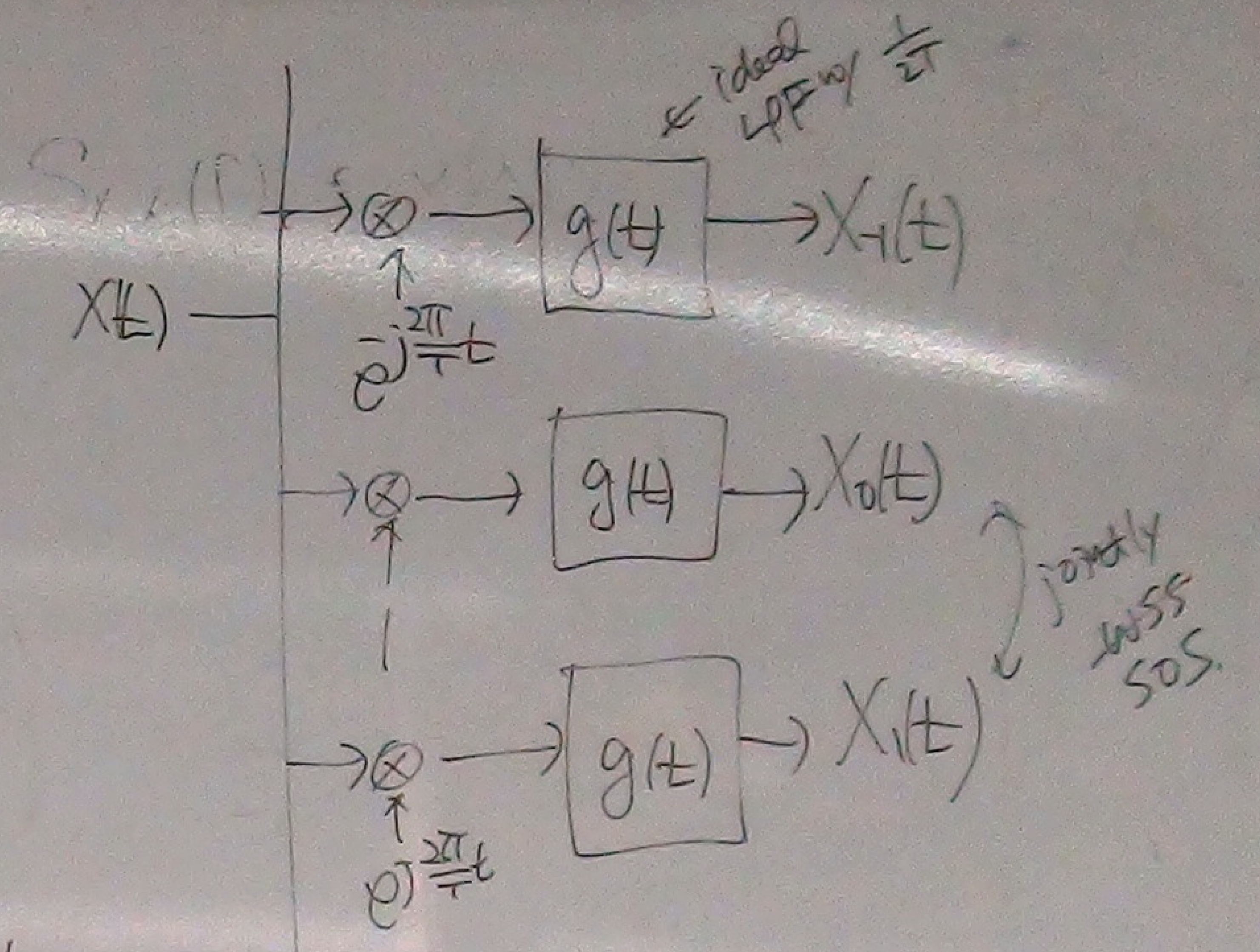
$S_{YZ}(f_1,f_2)=\int_{-\infty}^{\infty} R_{XX}(t_1,t_2)e^{-j2\pi\Delta_1 t_1}e^{+j2\pi\Delta_2 t_2}e^{-j2\pi f_1 t}e^{+j2\pi f_2 t}\,dt_1dt_2$

$=\iint R_{XX}(t_1,t_2)e^{-j2\pi(f_1+\Delta_1)t_1}e^{j2\pi(f_2+\Delta_2)t_2}\,dt_1dt_2$

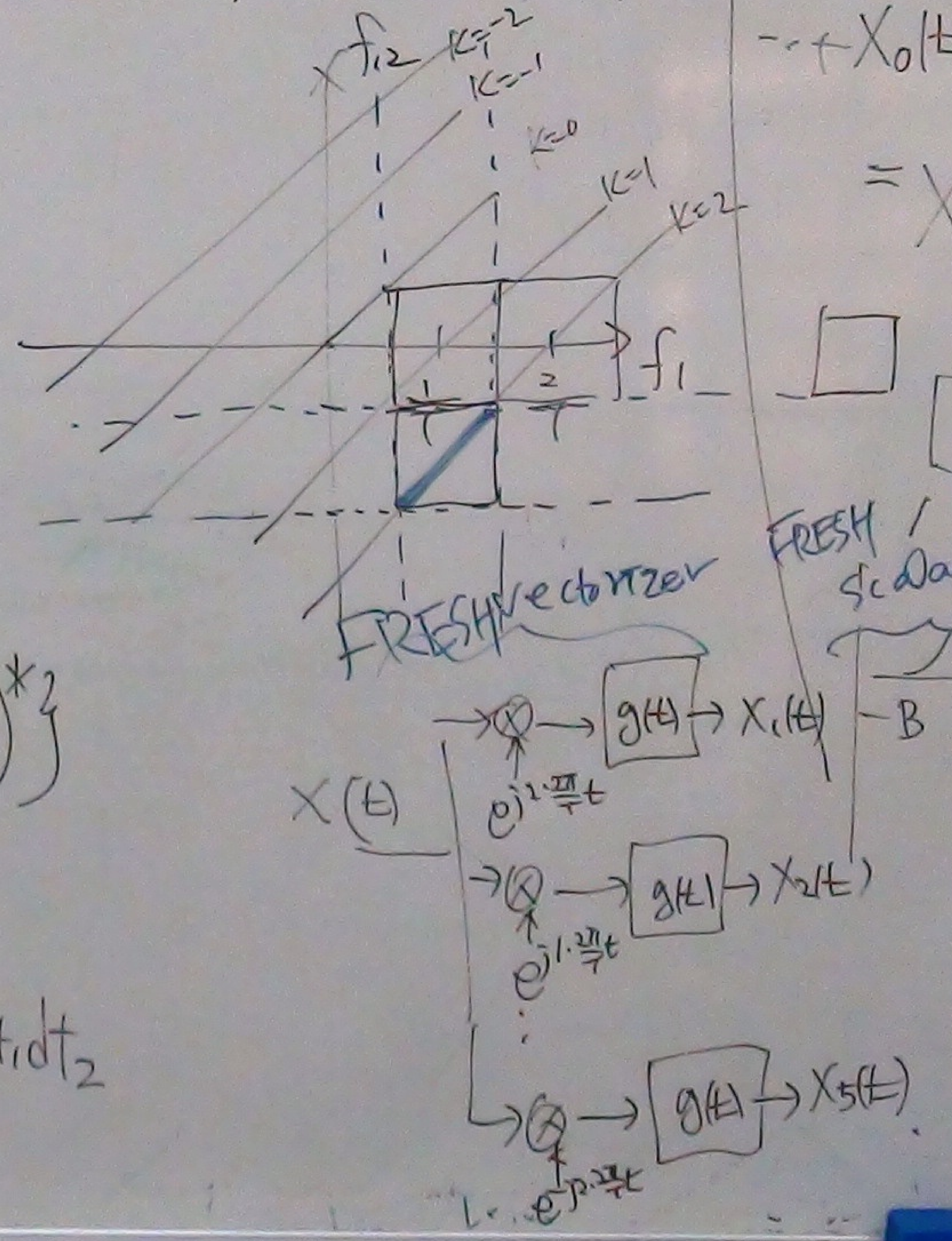
$= S_{XX}(f_1+\Delta_1, f_2+\Delta_2)$.

Similarly,

$S_{YZ}(f_1,f_2)=\tilde{S}_{XX}(f_1+\Delta_1,f_2+\Delta_2)$.



☐ X(t) SOCS w/ T



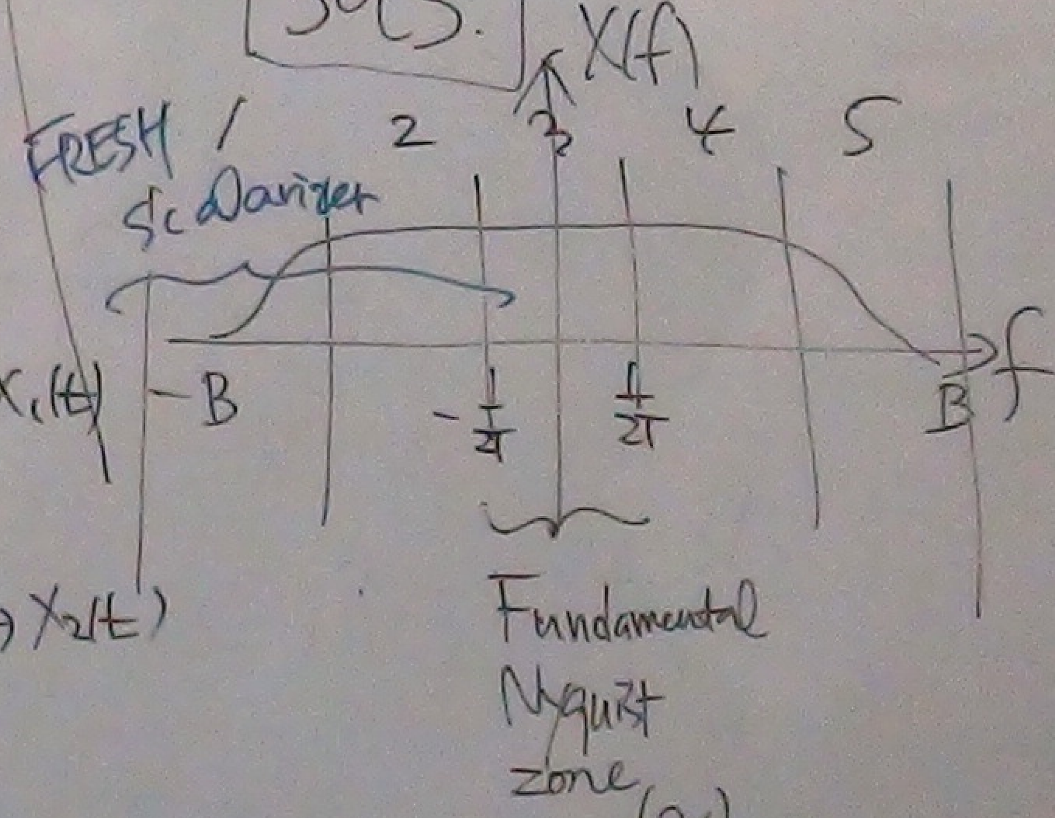
$\cdots + X_0(t)\cdot 1 + X_{-1}(t)e^{j\frac{2\pi}{T}t}+X_{-2}(t)e^{j2\cdot\frac{2\pi}{T}t}+\cdots$

$= X(t)$.

$BT > \frac{1}{2}$

☐ X(t), BL. $B > \frac{1}{2T}$

SOCS.



$1/T$

$R_{XX}(\tau) \overset{F}{\leftrightarrow} S_{XX}(f)$

$X(t)$. $\tilde{R}_{XX}(t_1,t_2)=R_{XX}(t_1-t_2)$

Vector SOS r.p.